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CONNECTED CERTIFIED DOMINATION STABLE AND CRITICAL GRAPHS UPON EDGE ADDITION

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ABSTRACT. A set of vertices D_c in a connected graph $\Gamma = (V_\Gamma, E_\Gamma)$ is called a certified dominating set if $|N_\Gamma(u) \cap (V_\Gamma - D_c)|$ is either 0 or at least 2, $\forall u \in D_c$. The set D_c is called as connected certified dominating set if $|N(u) \cap (V_\Gamma - D_c)|$ is either 0 or at least 2, $\forall u \in D_c$ and the subgraph $\Gamma[D_c]$ induced by D_c is connected. The cardinality of the smallest connected certified dominating set is called connected certified domination number of the graph Γ denoted by $\gamma^c_{cer}(\Gamma)$. In this article, we examine and characterize those graphs that exhibit both connected certified domination stable and critical behavior when an edge is added to them. Also we will discuss characterization of connected certified domination stable trees.

Keywords: Connected certified dominating set, Connected certified domination edge stable graphs, Connected certified domination edge critical graphs.

AMS Subject Classification: 05CXX, 05C69.

1. INTRODUCTION

For general definitions and notations used in the article we refer the readers [1], [2] and [3].

In the context of this article, when we refer to a graph $\Gamma = (V_\Gamma, E_\Gamma)$, we are specifically describing a connected, undirected, and unweighted simple graph. We will denote the set of leaves, weak support and strong support vertices of a graph Γ by L_Γ, $S_1(\Gamma)$ and $S_2(\Gamma)$, respectively. Furthermore, $pn(u, D)$ and $epn(u, D)$ will be used to denote the $D-$private neighborhood of u and $D-$external private neighborhood of u, where $u \in D$ and $D \subseteq V_\Gamma$.

Connected domination is an interesting domination parameter which is in the literature for over more than four decades. Sampathkumar and Walikar first proposed the concept of connected domination in 1979 in response to a suggestion from S.T. Hedetniemi [4]. A set of nodes that dominates an isolate-free graph Γ and whose induced subgraph is connected is referred to as a connected dominating set (CDS). The cardinality of such a set is known

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as the connected domination number of graph Γ and is denoted by $\gamma_c(\Gamma)$. The theory of connected domination is widely used in wireless sensor networks (WSNs), where a CDS is considered as the virtual backbone (VB) of the WSN. This concept is essential in many applications of sensor networks. The WSN is modeled by a graph Γ, and a VB is modeled by a connected dominating set of Γ. To make the transmission easier and use less energy, it is interesting to find a minor VB i.e., finding a smallest CDS.

Detlaff et al [5] introduced certified domination as a new parameter in graph domination theory in 2020, which has various applications in social networks. Since then, certified domination has become a well-studied domination parameter, as evidenced by recent literature on the topic (e.g.,[6, 7, 8, 9, 10]). In a connected graph $\Gamma = (V_\Gamma, E_\Gamma)$ a set $D \subseteq V_\Gamma$ is called as certified dominating set if $|N(u) \cap (V_\Gamma - D)|$ is either 0 or at least 2, $\forall u \in D$. A set of vertices $D_c \subseteq V_\Gamma$ in a connected graph $\Gamma = (V_\Gamma, E_\Gamma)$ is a connected certified dominating set, abbrevaited CCDS, if:

(1) Every node in the graph Γ either belongs to $\mathcal{D}_c$ or is adjacent to at least one node in the set $\mathcal{D}_c$,
(2) For every node $u \in \mathcal{D}_c$, $|N(u) \cap (V_\Gamma - \mathcal{D}_c)|$ is either 0 or at least 2, and
(3) The subgraph induced by $\mathcal{D}_c$, i.e., $\Gamma[D_c]$ is connected.

The cardinality of the smallest CCDS of Γ, is the connected certified domination number of the graph Γ denoted by $\gamma^c_{cer}(\Gamma)$ and is abbrevaited CCDN. In this paper, we continue our study on connected certified domination which is introduced in [3, 11, 12]. We shall further attempt to evaluate the criticality and stability parameters of a graph considering its importance in evaluating the CCDN.

In applications that utilize graphical parameters, it is crucial to comprehend how these parameters react when a graph is modified. Graphs that experience changes in parameters such as domination number or chromatic number due to the removal or addition of edges or vertices have been extensively researched. Walikar and Acharya [13] studied graphs where the domination number changes with the removal of one edge, while Dutton and Brigham [14] investigated graphs where the domination number remains the same. These problems have been used to study different types of domination, such as global domination, total domination, connected domination, and certified domination. Chen, Sun, and Ma [15] in 2004 initiated the study of γ_c-critical graphs, and Desormeaux, Haynes, and van der Merwe [16] began the study of γ_c-stable graphs in 2015. In 2020, Detlaff et al. [5] explored the impact of edge addition and deletion on the certified domination number of graphs. The graph's criticality and stability have been investigated by numerous researchers for various other domination parameters such as [17, 18, 19], and the influence of edge deletion on CCDN of graphs has been recently studied in [3]. This research focuses on examining graphs where adding an edge $e \in E_{\overline{\Gamma}}$ to graph Γ leads to a change in CCDN, as well as graphs where the addition of an edge leaves the CCDN unchanged.

Definition 1.1. *"Let D be a dominating set of a graph Γ. An element of D that has all neighbors in D is said to be shadowed with respect to D (shadowed for short), an element of D that has exactly one neighbor in $V_\Gamma(u)\backslash D$ is said to be half-shadowed (HS) with respect to D (half-shadowed for short), while an element of D having at least two neighbors in $V_\Gamma(u)\backslash D$ is said to be illuminated with respect to D (illuminated for short)"*[5].

Definition 1.2. *A graph Γ is said to be connected certified domination stable graph upon edge addition denoted by $[\gamma^c_{cer}]^{e^+}-$ stable, if the addition of any edge $e \in E_{\overline{\Gamma}}$ does not alter its CCDN, that is, for any edge $e \in E_{\overline{\Gamma}}, \gamma^c_{cer}(\Gamma + e) = \gamma^c_{cer}(\Gamma)$. If $\gamma^c_{cer}(\Gamma) = k$, and Γ is $[\gamma^c_{cer}]^{e^+}-$stable, we say that Γ is $[k^c_{cer}]^{e^+}-$stable.*

Definition 1.3. *A graph Γ is connected certified domination critical graph upon edge addition denote by $[\gamma^c_{cer}]^{e^+}-$critical, if the addition of any edge $e \in E_{\overline{\Gamma}}$ changes the CCDN of Γ. We note that addition of an edge to a graph Γ cannot increase its CCDN. Hence if Γ is $[\gamma^c_{cer}]^{e^+}-$ critical, then $\gamma^c_{cer}(\Gamma + e) < \gamma^c_{cer}(\Gamma)$ for every edge $e \in E_{\overline{\Gamma}}$. If $\gamma^c_{cer}(\Gamma) = k$, and Γ is $[\gamma^c_{cer}]^{e^+}-$ critical, we say that Γ is $[k^c_{cer}]^{e^+}-$critical.*

Definition 1.4. *"The $C-$private neighborhood of u is denoted by $pn(u, C)$, and is defined by $pn(u, C) = N_\Gamma[u] - N_\Gamma[C - u]$. Thus if $w \in pn(u, C)$, then $N_\Gamma(w) \cap C = \{u\}$. We refer to a vertex $w \in pn(u, C)$ as a $C-$ private neighborhood of u"*[3].

Definition 1.5. *"The $C-$external private neighbor of $u \in C$ is a vertex $v \in V - \Gamma \setminus C$ which is adjacent to u but to no other vertex of C. The set of $C-$external private neighbor of u is denoted by $epn(u, C)$"*[3].

Proposition 1.1. *Every vertex in $S_2(\Gamma)$ of graph Γ belongs to every $\gamma^c_{cer} - set$ of Γ.*

Proof. Let D_c be a $\gamma^c_{cer} - set$ of Γ, let $s_1 \in S_2(\Gamma)$ be a strong support vertex of Γ, and let $l_1 \in L_\Gamma$ is such that $l_1 \subseteq N_\Gamma(s_1)$. If $s_1 \notin D_c$, then $l_1 \in D_c$. But then l_1 would have only one neighbor in $V_\Gamma \setminus D_c$, and D_c would not be a $\gamma^c_{cer}-$set. □

Observation 1.1. *$S_1(\Gamma) + L_1(\Gamma) \in \gamma^c_{cer}(\Gamma)-$set, where $L_1(\Gamma)$ is the set of leaves adjacent to weak supports.*

2. $[\gamma^c_{cer}]^{e^+}-$Stable Graphs

In this part of the paper we discuss characterization of $[\gamma^c_{cer}]^{e^+}-$stable graphs and $[\gamma^c_{cer}]^{e^+}-$stable trees. We start this section with the following proposition.

Proposition 2.1. *Let Γ be a connected graph of order n and C be a $\gamma^c_{cer}(\Gamma)-$ set then:*
(1) *C has no half shadowed vertex.*
(2) *Every vertex in C is either a strong support, or a shadowed vertex, or an illuminated vertex.*
(3) *If all the vertices of C are shadowed then $C = V_\Gamma$.*
(4) *If $u \in V_\Gamma$ is a strong support vertex then $|epn(u, C)| \geq 2$.*
(5) *If $u \in V_\Gamma$ is a shadowed vertex with respect to C then $|epn(u, C)| = 0$.*
(6) *If $u \in V_\Gamma$ is an illuminated vertex with respect to C then $|epn(u, C)| \geq 0$.*

Proof of this preposition follows directly from the definition of $\gamma^c_{cer}(\Gamma)-$set.

Lemma 2.1. *Let C be a $\gamma^c_{cer}(\Gamma)-$set of a connected graph Γ of order $n \geq 3$, then:*
(1) *In a $\gamma^c_{cer}-$set C, every shadowed vertex is either a weak support or a leaf, or of same degree in Γ and $\Gamma[C]$.*
(2) *Every non leaf neighbor of a shadowed weak support is either an illuminated or weak support vertex.*

Proof of this lemma directly follows from the proof of lemma 6.1 in [5].

Observation 2.1. *Let Γ be a connected graph of order n and C be a $\gamma^c_{cer}(\Gamma)-$set. If all the vertices of C are strong supports then the graph Γ is always $[\gamma^c_{cer}]^{e^+}-$stable.*

Next we present the characterization of $[\gamma^c_{cer}]^{e^+}-$stable graph. We start with the following observation.

Observation 2.2. *If C is the $\gamma^c_{cer}-$set of a graph Γ then $|epn(u, C)| \geq 0$, $\forall u \in C$.*

Addition of an edge from the complement $\overline{\Gamma}$ can decrease the $CCDN$ of the graph Γ by as much as four. We then have the following theorem.

Theorem 2.1. *Let Γ be a connected graph of order n. Then, for any edge $e = uv \in E_{\overline{\Gamma}}$, $\gamma^c_{cer}(\Gamma) - 4 \leq \gamma^c_{cer}(\Gamma + e) \leq \gamma^c_{cer}(\Gamma)$.*

Proof. Let Γ be a connected graph of order n, then it is clear that $\gamma^c_{cer}(\Gamma + e) \leq \gamma^c_{cer}(\Gamma)$ where $e = uv \in E_{\overline{\Gamma}}$. Now we only show that $\gamma^c_{cer}(\Gamma) - 4 \leq \gamma^c_{cer}(\Gamma + e)$ for any edge say $xy = e \in E_{\overline{\Gamma}}$, where $x, y \in V_\Gamma$. Let C be the $\gamma^c_{cer}(\Gamma + e)-$set of $\Gamma + e$.

Case 1. If both $x, y \notin C$, then C is also $\gamma^c_{cer}-$ set of the modified graph $\Gamma + e$. Thus, $\gamma^c_{cer}(\Gamma + e) \leq \gamma^c_{cer}(\Gamma)$.

Case 2. If $y \notin C$ and $x \in C$. Then x can be either a strong support, or shadowed, or an illuminated vertex by proposition 2.1 (the case $x \notin C$ and $y \in C$ is either a strong support , or shadowed, or an illuminated vertex can be analyzed in a similar way). Assume that $y \notin C$ and $x \in C$ is a strong support vertex. Since C is $\gamma^c_{cer}-$ set of Γ, the vertex y will be adjacent to atleast one vertex in $C - x$, that is, $y \in N(C - x)$. Now addition of the edge $e = xy$ in Γ, where $x \in C$ is a strong support vertex and $y \in N(C - x)$, does not change the CCDN of the modified graph $\Gamma + e$, that is, $\gamma^c_{cer}(\Gamma + e) = \gamma^c_{cer}(\Gamma)$. The case when x is an illuminated vertex with respect to C can be analysed in a similar way. Afterwards, suppose that $y \notin C$ and $x \in C$ is a shadowed vertex with respect to C, i.e., $|N(x) \cap V_\Gamma - C| = 0$ and $y \in N(C - x)$, then the addition of edge $e = xy$ in graph Γ does not change the CCDN of the modified graph $\Gamma + e$ implying that $\gamma^c_{cer}(\Gamma + e) \leq \gamma^c_{cer}(\Gamma)$.

Case 3. If $x \in C$ and $y \in C$, then x and y are either shadowed or strong supports, or illuminated with respect to C by proposition 2.1. We then have the following subcases:

Subcase 1. If x and y are both shadowed vertices with respect to $\gamma^c_{cer}(\Gamma)-$set C, then by lemma 2.1, x and y are either leaves, or weak support vertices, or have same degree in Γ and $\Gamma[C]$. Now if both x and y are leaves then clearly $\gamma^c_{cer}(\Gamma + e) \leq \gamma^c_{cer}(\Gamma)$. The case when both x and y are weak supports is illustrated in figure 1 below. In situation (a) of figure 1, the CCDN of the

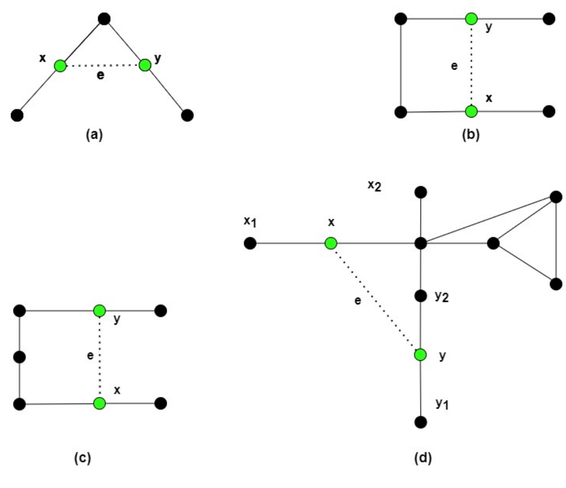

FIGURE 1. Illustration of the case when both x and y are weak support

modified graph $\Gamma + e$ is reduced by 3, that is, $\gamma^c_{cer}(\Gamma) - 3 = \gamma^c_{cer}(\Gamma + e)$. In situation (b), the CCDN of the modified graph $\Gamma + e$ is reduced by 4, that is, $\gamma^c_{cer}(\Gamma) - 4 = \gamma^c_{cer}(\Gamma + e)$ and in last situation (c) it remains the same, that is, $\gamma^c_{cer}(\Gamma) = \gamma^c_{cer}(\Gamma + e)$. Similary, assume that $N_\Gamma\{x\} = \{x_1, x_2\}$ such that $x_1 \in L_\Gamma$ and x_2 is illuminated with respect to C, and $N_\Gamma\{y\} = \{y_1, y_2\}$ such that $y_1 \in L_\Gamma$, y_2 is shadowed, and $y_2 \sim x_2$. Then the $\gamma^c_{cer}-$set of the modified graph $\Gamma + e$ as illustrated in situation (d) of figure 1 will be $C - \{x_1, y_1, y_2\}$ implying that $\gamma^c_{cer}(\Gamma) - 3 = \gamma^c_{cer}(\Gamma + e)$. Finally suppose that x is a leaf and y is a weak support. Then we have the following possible situations as shown in Figure 2 below. Clearly, the CCDN of the modified graph $\Gamma + e$ remains

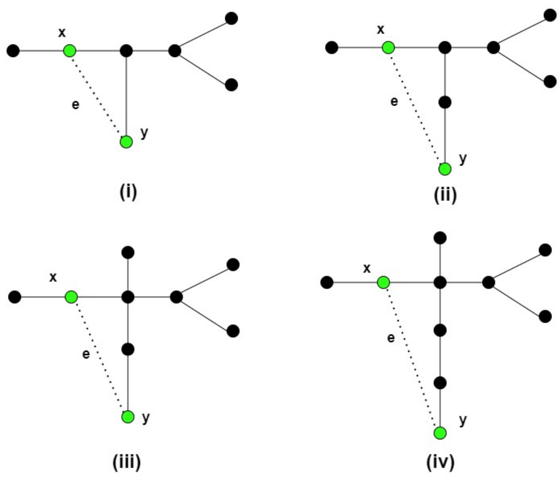

FIGURE 2. Illustration of the case when x is a weak support and y is a leaf.

the same in situation (i), (ii), and (iv) in Figure 2, and in situation (iii), it is reduced by 2, that is $\gamma^c_{cer}(\Gamma) - 2 = \gamma^c_{cer}(\Gamma + e)$.

Subcase 2. Suppose that x and y are strong support (illuminated) vertices in graph Γ. If there exist a strong support (illuminated) vertex $z \in C$ such that $x \sim z$ and $y \sim z$ then clearly $\gamma^c_{cer}(\Gamma + e) \leq \gamma^c_{cer}(\Gamma)$. Now, assume that the vertex $z \in C$ is a shadowed vertex such that $x \sim z$ and $y \sim z$ then the addition of the edge $e = xy$ in graph Γ will reduce the CCDN of the modified gaph $\Gamma + e$ by 1, that is, $\gamma^c_{cer}(\Gamma) - 1 \leq \gamma^c_{cer}(\Gamma + e)$.

Subcase 3. Suppose x is shadowed and y is a strong support vertex, then clearly CCDN of the modified graph $\Gamma + e$ will not change if x is a weak support or x has same degree in Γ and $\Gamma[C]$. Assume that x is a leaf, since $x \in C$, therefore x will be the only leaf neigbor of a weak support vertex say u and if $u \sim y$ then the size of the $\gamma^c_{cer}(\Gamma + e)-$set will be reduced by 2. Suppose that $u \nsim y$, then u will be adjacent to either a shadowed vertex other then x or strong support (illuminated) vertex. In this particular scenario, the following possible situations are illustrated in Figure 3 below. In situation (i) of Figure 3, $u \sim v$ and $v \sim y$, where v is shadowed, addition of the edge $e = xy$ in graph Γ does not change the CCDN of the modified graph $\Gamma + e$. In situation (ii), the vertex v is weak support (it can be strong support or illuminated), and the addition of the edge $e = xy$ reduces the CCDN of the modified graph $\Gamma + e$ by 3.

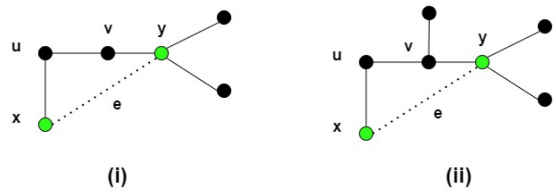

FIGURE 3. Illustration of the case when x is a leaf and y is a strong support.

Hence, from the above three cases we conclude that $\gamma^c_{cer}(\Gamma) - 4 \leq \gamma^c_{cer}(\Gamma + e) \leq \gamma^c_{cer}(\Gamma)$ for any edge say $e \in E_{\overline{\Gamma}}$. □

As a consequence of theorem 2.1 we have the following result.

Corollary 2.1. *If $\gamma^c_{cer}(\Gamma+e) \leq \gamma^c_{cer}(\Gamma)$ for any edge $e = uv \in E_{\overline{\Gamma}}$, then every $\gamma^c_{cer}(\Gamma+e)-$set C contains at least one of u and v or both u and v.*

Theorem 2.2. *A connected graph Γ is $[\gamma^c_{cer}]^{e^+}-$stable graph with $2 \leq \gamma^c_{cer}(\Gamma) \leq n-2$ if and only if for any $\gamma^c_{cer}-$set C and a vertex $u \in C$, the following holds.*

1. *If u is not a cut vertex in $\Gamma[C]$, then $|epn(u, C)| \geq 2$.*
2. *If $u \in C$ is such that $deg_{\Gamma[C]}(u) = deg_{\Gamma}(u)$ and $\Gamma[C \backslash u]$ has exactly two components and then $|epn(u, C)| = 0$.*

Proof. Let Γ be a $[\gamma^c_{cer}]^{e^+}-$stable graph upon edge addition with $2 \leq \gamma^c_{cer}(\Gamma) \leq n-2$, and let C be a $\gamma^c_{cer}(\Gamma)-$set and $u \in C$. Now, if u is not a cut vertex in $\Gamma[C]$ then u will be an end vertex in $\Gamma[C]$. Since Γ is $[\gamma^c_{cer}]^{e^+}-$stable graph, then u is either an illuminated or a strong support vertex, and by Proposition 2.2 $|epn(u, C)| \geq 2$.

Suppose that $u \in C$ is a cut vertex in $\Gamma[C]$ such that u has exactly two components and $deg_{\Gamma[C]}(u) = deg_{\Gamma}(u)$. If $|epn(u, C)| \geq 2$, then u is either an illuminated or strong support vertex and $\Gamma[C]$ has more than two components, and also $deg_{\Gamma}[C](u) \neq deg_{\Gamma}(u)$, a contradiction to our assumption. Therefore, $|epn(u, C)| \leq 2$. Now, since $u \in C$ is such that $deg_{\Gamma[C]}(u) = deg_{\Gamma}(u)$ which means that u is shadowed with respect to C and by Proposition 2.1, $|epn(u, C)| = 0$. Hence, $|epn(u, C)| = 0$ whenever $\Gamma[C \backslash u]$ has exactly two components and $deg_{\Gamma[C]}(u) = deg_{\Gamma}(u)$.

Conversely, suppose that Γ is an isolated free graph that is not $[\gamma^c_{cer}]^{e^+}-$stable. Therefore there exists some edge $uv = e \in E_{\overline{\Gamma}}$ such that $\gamma^c_{cer}(\Gamma+e) < \gamma^c_{cer}(\Gamma)$. Let $\Gamma' = \Gamma+uv$ and D be any $\gamma^c_{cer}(\Gamma')-$set. By Theorem 2.1, $\gamma^c_{cer}(\Gamma') \in \{\gamma^c_{cer}(\Gamma) - 4, \gamma^c_{cer}(\Gamma) - 3, \gamma^c_{cer}(\Gamma) - 2, \gamma^c_{cer}(\Gamma) - 1\}$. If $D \cap \{u, v\} = \phi$, then D is a $\gamma^c_{cer}-$set of Γ with cardinality less then $\gamma^c_{cer}(\Gamma)$, a contradiction. Therefore, either u or v, or both, must belong to D. We then have the following cases:

Case 1. Assume that $D \cap \{u, v\} = \{u, v\}$. Clearly, D is certified dominating set of Γ. If $\Gamma'[D] - uv$ is connected then D is $\gamma^c_{cer}-$set with cardinality less than $\gamma^c_{cer}(\Gamma)$ a contradiction. Therefore, the edge uv is a bridge in $\Gamma'[D]$, implying that $\Gamma[D]$ has precisely two components. Define D_u as the component of $\Gamma[D]$ that contains vertex u, and let D_v be the component of $\Gamma[D]$ that contains vertex v. Now, if $\gamma^c_{cer}(\Gamma') = \gamma^c_{cer}(\Gamma)-3$ (or $\gamma^c_{cer}(\Gamma)-4$), then there exists a vertex $w \in D$ such that $\Gamma[D \backslash w]$ has at least two components, implying that w has at least two $D-$external private neighbors, that is, $|epn(w, D)| \geq 2$, negating Condition 2; or $\gamma^c_{cer}(\Gamma') = \gamma^c_{cer}(\Gamma) - 2$. If $\gamma^c_{cer}(\Gamma') = \gamma^c_{cer}(\Gamma) - 2$, then there exists a vertex $x \in V_{\Gamma} \backslash D$ such that $x \in N(D_u) \cap N(D_v)$, then $D \cup x$ is a $\gamma^c_{cer}(\Gamma)-$set with cardinality at most $\gamma^c_{cer}(\Gamma) - 1$ a contradiction.

Case 2. Now, assume that $D \cap \{u, v\} = \{u\}$. It is obvious that $\Gamma[D]$ is connected, and D dominates $\Gamma - v$. Let $y \in V_\Gamma \backslash D$ be a neighbor of v in Γ. Since D is $\gamma^c_{cer}-$set, y has neighbor in D. Assume that u is neighbor oy y in D and $u \in S_2(|gamma)$, then $C = D \cup \{v, y\}$will be a $\gamma^c_{cer}-$set for the graph Γ, implying that $\gamma^c_{cer}(\Gamma) \leq \gamma^c_{cer}(\Gamma^{'}) + 2$, so $\gamma^c_{cer}(\Gamma^{'}) = \gamma^c_{cer}(\Gamma) - 2$. But then C is a $\gamma^c_{cer}(\Gamma)-$set in which y is not a cut vertex of $\Gamma[C]$ and $|epn(y, C)| = 0$, a negation of Condition (1).

□

As a consequence of the above result we have the following corollary.

Corollary 2.2. *In any isolate free graph Γ if every vertex in $\gamma^c_{cer}(\Gamma)-$set is illuminated or strong support then Γ is $[\gamma^c_{cer}]^{e^+}-$stable.*

Proposition 2.2. *Let Γ be a connected graph of order n. If Γ has a universal vertex then the graph Γ is always $[\gamma^c_{cer}]^{e^+}-$stable.*

Proof. In isolate free graph Γ let $v \in V_\Gamma$ is such that $deg_\Gamma(v) = n - 1$ and let C be the $\gamma^c_{cer}(\Gamma)-$set. Then $\gamma^c_{cer}(\Gamma) = 1$, since Γ has a universal vertex [11]. And we know that if $\gamma^c_{cer}(\Gamma) = 1$ then Γ is always $[\gamma^c_{cer}]^{e^+}-$stable. □

Proposition 2.3. (a) *If Γ is a star graph $\mathcal{S}_{(1,n)}$, then Γ is $[\gamma^c_{cer}]^{e^+}-$stable $\forall n \geq 2$.*
(b) *If Γ is a wheel graph $\mathcal{W}_n$, then Γ is $[\gamma^c_{cer}]^{e^+}-$stable $\forall n$.*
(c) *If Γ is a complete graph $\mathcal{K}_n$, then Γ is $[\gamma^c_{cer}]^{e^+}-$stable.*
(d) *If Γ is a fan graph $\mathcal{F}_{(p.q)}$, then Γ is $[\gamma^c_{cer}]^{e^+}-$stable for*

$$\begin{cases} p = 1, q \geq 2 \\ p \geq 1, q = 2, 3 \end{cases}$$

(e) *If Γ is a double star graph $\mathcal{DS}(m, n)$, then Γ is $[\gamma^c_{cer}]^{e^+}-$stable $\forall m \geq 2$.*
(f) *If Γ is complete bipartite graph $\mathcal{K}_{(m,n)}$, then Γ is $[\gamma^c_{cer}]^{e^+}-$stable for $max(m, n) \geq 3$.*

Proof. Proof of $(a) - (d)$ are obvious by Proposition 2.2.

(e) Let Γ be a double star $\mathcal{DS}(m, n)$ and $\mathcal{C}$ be the $\gamma^c_{cer}-$set of Γ. Let u and v be the non leaf vertices of the double star graph $\mathcal{DS}(m, n)$, then $\gamma^c_{cer}-$set $\mathcal{C}$ of $\mathcal{DS}(m, n)$ is $\mathcal{C} = \{u, v\}$. Therefore, it implies that, every vertex of $\gamma^c_{cer}-$set $\mathcal{C}$ is illuminated and hence the double star graph $\mathcal{DS}(m, n)$ is $[\gamma^c_{cer}]^{e^+}-$stable by corollary 2.2.
(f) Let Γ be a complete bipartite graph $\mathcal{K}_{(m,n)}$ with $\max(m, n) \geq 3$ and $\mathcal{C}$ be the $\gamma^c_{cer}-$set. We know that the connected certified domination of complete bipartite graphs is 2 [11]. So, let $\mathcal{C} = \{u, v\}$ and since $\max(m, n) \geq 3$ implying that both the vertices in $\mathcal{C}$ are illuminated and hence by corollary 2.2 graph Γ is $[\gamma^c_{cer}]^{e^+}-$stable.

□

Afterward, we will provide a description of trees that are $[\gamma^c_{cer}]^{e^+}-$stable, We will begin by making the following observation.

Observation 2.3. *If $\mathcal{T}$ is a tree with n vertices where n is greater than or equal to 3, then the set of vertices that contains non-leaves and all the leaves that are adjacent to the $S_1(\mathcal{T})$ is the only $\gamma^c_{cer}-$set of $\mathcal{T}$.*

Theorem 2.3. *A tree $\mathcal{T}$ of order $n \geq 3$ is $[\gamma^c_{cer}]^{e^+}-$stable if and only if $\gamma^c_{cer}(\mathcal{T}) = 1$ or $\mathcal{T}$ has no weak support vertices.*

Proof. As stated above, any graph Γ with $\gamma^c_{cer}(\Gamma) = 1$ is $[\gamma^c_{cer}]^{e^+}-$stable. Henceforward, assume that $\gamma^c_{cer}(\mathcal{T}) \geq 2$. Let $\mathcal{T}$ be a $[\gamma^c_{cer}]^{e^+}-$stable tree, and assume, to the contrary that there exists $u \in \mathcal{T}$ such that $u \in S_1(\mathcal{T})$. Let $\mathcal{C}$ be a $\gamma^c_{cer}(\mathcal{T})-$set. Let $l \in L_{\mathcal{T}}$ be the only leaf adjacent to u and $v \in V_{\mathcal{T}}$ be any non leaf vertex adjacent to u, since $deg(u) \geq 2$ as $u \in S_1(\mathcal{T})$. By Observation 1.1, $\{u, l\} \in \mathcal{C}$. Let $v \in C$ such that $v \sim u$ and $e \in E_{\overline{\mathcal{T}}}$ be an edge in $\overline{\mathcal{T}}$ such that $e = vl$. Then, the set $\mathcal{C}\backslash \{v, l\}$ is a $\gamma^c_{cer}(\mathcal{T} + vl)-$set, which is a contradiction.

For the necessary condition, suppose that $\gamma^c_{cer}(\mathcal{T}) \geq 2$ and $\mathcal{T}$ has no weak support vertices. Demonstrating the fulfillment of the two requirements specified in Theorem 2.2 is sufficient. According to Observation 2.3, the set $\mathcal{C} \subseteq V_\Gamma$ comprised of the non-leaf vertices and adjacent leaves connected to weak support vertices of $\mathcal{T}$ is the only $\gamma^c_{cer}(\mathcal{T})-$set. Therefore, based on our assumption, every vertex present in $\mathcal{C}$ has a minimum degree of 3.

Now suppose that u is not a cut vertex in $\mathcal{T}[\mathcal{C}]$. Since $\mathcal{T}[\mathcal{C}]$ is a tree, which implies that u is leaf of $\mathcal{T}[\mathcal{C}]$. Furthermore, since $deg_{\mathcal{T}}(u) \geq 3$, it follows that u has at least leaf two neighbors in $\mathcal{T}$, that implies $|epn(u, C)| \geq 2$, satisfying the condition (2) of Theorem 2.2.

For the last condition, let $u \in \mathcal{C}$ is such that $\mathcal{T}[\mathcal{C}\backslash u]$ has exactly two components and $deg_{\mathcal{T}[\mathcal{C}]}(u) = deg_{\mathcal{T}}(u)$. Since u have equal degree in $\mathcal{T}$ and the induced sub graph $\mathcal{T}[\mathcal{C}]$, which implies there exist no vertex in $V_{\mathcal{T}}\backslash\mathcal{C}$ which is adjacent to u implying that $|epn(u, C)| = 0$. Hence we conclude that $\mathcal{T}$ is $[\gamma^c_{cer}]^{e^+}-$stable. □

Above result is not satisfied in case of simple trees that is path graphs. For example, path graph P_6 contains weak support vertices but P_6 is $[\gamma^c_{cer}]^{e^+}-$critical as shown in Figure 1 below.

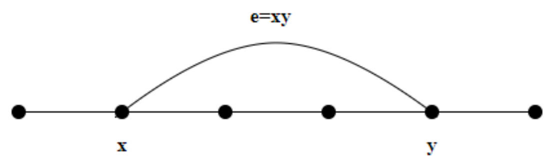

FIGURE 4. Path graph P_6 is $[\gamma^c_{cer}]^{e^+}-$critical as the CCDN of P_6 is 6 and $\gamma^c_{cer}(P_6 + xy) = 2$, but P_6 has weak support vertices x and y.

We conclude this section with the following observation.

Observation 2.4. *Path graph P_n is $[\gamma^c_{cer}]^{e^+}-$stable $\forall n \neq 4, 5, 6$.*

3. $[\gamma^c_{cer}]^{e^+}-$Critical Graphs

For any isolate free graph Γ, $\gamma^c_{cer}(\Gamma) \geq 1$. We have observed that for any graph Γ if $\gamma^c_{cer}(\Gamma) = 1$ then Γ is always $[\gamma^c_{cer}]^{e^+}-$ stable. So we will consider graphs for which $\gamma^c_{cer}(\Gamma) \geq 2$. We start this section with the following Proposition.

Proposition 3.1. *Let Γ be a connected graph of order n. If Γ has at least one vertex of degree $n - 2$, then Γ is $[\gamma^c_{cer}]^{e^+}-$critical.*

Proof. Let be Γ an isolate free graph of order n and $\mathcal{C}$ be a $\gamma^c_{cer}(\Gamma)-$set. Assume that in graph Γ there esist a vertex $v \in V_\Gamma$ such that $deg_\Gamma(v) = n - 2$ and let $e = uv \in E_{\overline{\Gamma}}$ be the added edge in Γ. Let $w \in N_\Gamma(v)$ such that w is adjacent to u, then by properties of $\gamma^c_{cer}-$set it implies that $C = \{u, v, w\}$. Now in the modified graph $\Gamma + uv$, $deg_{\Gamma+uv}(v) = n - 1$, since

$deg_\Gamma(v) = n-2$ which clearly implies that $\gamma^c_{cer}(\Gamma) < \gamma^c_{cer}(\Gamma+uv)$ and hence graph Γ is $[\gamma^c_{cer}]^{e^+}-$critical. □

We have the following observations as a consequence of the above Proposition 3.1.

Observation 3.1. *Cycle graph $\mathcal{C}_n$ is $[\gamma^c_{cer}]^{e^+}-$critical graph for $n = 4, 5, 6$.*

One can easily verify that $\mathcal{C}_3$ is $[\gamma^c_{cer}]^{e^+}-$stable. For $n \geq 5 \neq 6$ the $\gamma^c_{cer}(\mathcal{C}_n)-$set $\mathcal{C} = V_{\mathcal{C}_n}$ and $deg_{\mathcal{C}_n[C]}(u) = deg_{\mathcal{C}_n}(u)\ \forall u \in \mathcal{C}$ which implies $|epn(u, \mathcal{C}| = 0$ and by Theorem 7, $\mathcal{C}_n$ for $n \geq 5 \neq 6$ is $[\gamma^c_{cer}]^{e^+}-$stable. For $n = 4$, $\mathcal{C}_n$ is $[\gamma^c_{cer}]^{e^+}-$critical by Proposition 3.1.

Observation 3.2. *If Γ is a fan graph $\mathcal{F}_{(p,q)}$, then Γ is $[\gamma^c_{cer}]^{e^+}-$critical for*

$$\begin{cases} p = 2, q \geq 4 \\ p \geq 2, q = 4 \end{cases}$$

For $2-[\gamma^c_{cer}]^{e^+}-$critical graph we have the following theorem.

Theorem 3.1. *An isolate free graph Γ is $2-[\gamma^c_{cer}]^{e^+}-$critical if $\overline{\Gamma} = \bigcup_{i=1}^{j} S_{1,n_i}$ for $n_i \geq 1$ and $j \geq 2$.*

Proof. Let Γ be an isolate free $2-[\gamma^c_{cer}]^{e^+}-$critical graph and let $e = vw \in E_{\overline{\Gamma}}$ be any edge in $\overline{\Gamma}$. Then $\gamma^c_{cer}(\Gamma+vw) = 1$.Consequently, it implies, without loss of generality, that $\{w\}$ dominates $\Gamma+vw$ and so w is an isolate vertex of $\overline{\Gamma}-vw$. Therefore, we have proved that every edge of $\overline{\Gamma}$ is incident with an end vertex of $\overline{\Gamma}$. Since Γ is an isolate free graph, it follows that $\overline{\Gamma} = \bigcup_{i=1}^{j} S_{1,n_i}$ for $n_i \geq 1$ and $j \geq 2$. □

Next we will provide an upper bound on the diameter of $k-[\gamma^c_{cer}]^{e^+}-$critical graphs.

Theorem 3.2. *Let Γ be a $k-[\gamma^c_{cer}]^{e^+}-$critical graph then $dia(\Gamma) \leq k$.*

Proof. Let Γ be a $k-[\gamma^c_{cer}]^{e^+}-$critical graph. Assume that $dia(\Gamma) = l \geq k+1$. Let $u, v \in V_\Gamma$ such that $d(u,v) = l$ and let $\mathcal{C}$ be the $\gamma^c_{cer}-$set of the modified graph $\Gamma+uv$. Then $|\mathcal{C}| \leq k-1$, Since Γ is $k-[\gamma^c_{cer}]^{e^+}-$critical and by Corollary 3.2 either $u \in \mathcal{C}$ or $v \in \mathcal{C}$. Without loss of generality assume that $u \in \mathcal{C}$. Let $D_i = \{w \in \mathcal{C} | d(u,w) = i\}$ for $0 \leq i \leq l$. Clearly $D_i \neq \phi$. Further, $D_0 = \{u\}$ and $v \in D_l$. Let m be a largest integer in which $\mathcal{C} \cap D_i \neq \phi$ for each $0 \leq i \leq m$ and $\Gamma[\bigcup_{i=0}^{m}(\mathcal{C} \cap D_i)]$ is connected. Since $m+1 \leq |\mathcal{C}| \leq k-1$ and $l \geq k+1$, it consequently implies that, $m \leq l-3$. Consider D_{m+2}. Clearly no vertex of $\bigcup_{i=0}^{m}(\mathcal{C} \cap D_i)$ dominates D_{m+2}. Thus $\mathcal{C} \cap (D_{m+2} \cup D_{m+3} \neq \phi$. It follows that, $\mathcal{C} \cap D_j \neq \phi$ for each $m+3 \leq j \leq l$. Then $v \in \mathcal{C}$, because $\mathcal{C}$ is connected. Thus,$|\bigcup_{j=m+3}^{l}(\mathcal{C} \cap D_i)| \geq l-(m+3)+1 = l-m-2 \geq k+1-m-2 = k-m-1$. Therefore, $|\mathcal{C}| = |\bigcup_{i=0}^{m}(\mathcal{C} \cap D_i) \cup (\bigcup_{j=m+3}^{l}(\mathcal{C} \cap D_i))| \geq 1+m+(k-m-1) = k$, a contradiction to our assumption. Hence $dia(\Gamma) \leq k$ if Γ is $k-[\gamma^c_{cer}]^{e^+}-$critical graph. □

4. Conclusions

In this article, we have studied the influence of edge addition on CCDN of any arbitrary graph. We have proved that addition of an edge in a graph Γ from its complement $\overline{\Gamma}$ can decrease the CCDN of the graph Γ by as much as four. We then have provided the necessary and sufficient condition for a graph to be $[\gamma^c_{cer}]^{e^+}-$ stable and characterization of $[\gamma^c_{cer}]^{e^+}-$ stable graphs. In addition to it, we have proved that a tree $\mathcal{T}$ of order $n \geq 3$ is $[\gamma^c_{cer}]^{e^+}-$stable if and only if $\gamma^c_{cer}(\mathcal{T}) = 1$ or $\mathcal{T}$ has no weak support vertices.

In the last section, we have discussed the characterization of $[\gamma^c_{cer}]^{e^+}-$ critical graphs. We have proved that an isolate free graph Γ is $2-[\gamma^c_{cer}]^{e^+}-$critical if $\overline{\Gamma} = \bigcup_{i=1}^{j} S_{1,n_i}$ for $n_i \geq 1$ and $j \geq 2$. And in addition to it, we have provided an upper bound for $k-[\gamma^c_{cer}]^{e^+}-$critical graphs.

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