TWMS J. App. and Eng. Math. V.15, N.5, 2025, pp. 1153-1163

ON FIBONACCI CORDIAL LABELING OF SOME PLANAR GRAPHS

S. MITRA[1*], A. PRITCHARD[1], S. BHOUMIK[2], §

Abstract. An injective function f from vertex set $V(G)$ of a graph G to the set $\{F_0, F_1, F_2, \cdots, F_n\}$, where F_i is the i^{th} Fibonacci number ($i = 0, 1, \cdots, n$), is said to be Fibonacci cordial labeling if the induced function f^* from the edge set $E(G)$ the set $\{0, 1\}$ defined by $f^*(uv) = (f(u) + f(v)) \pmod 2$ satisfies the condition $|e_f(0) - e_f(1)| \leq 1$, where $e_f(0)$ is the number of edges with label 0 and $e_f(1)$ is the number of edges with label 1. A graph that admits Fibonacci cordial labeling is called a Fibonacci cordial graph. In this paper we discuss Fibonacci cordial labeling of the families of planar graph (Comb graphs, Coconut trees, Jellyfish Graphs, $H-$graph and $W-$graph).

Keywords: Fibonacci Cordial labelling, Comb graph, Jellyfish, coconut tree, $H-$graph, $W-$graph.

AMS Subject Classification: 05C78

1. Introduction

Graph labeling focuses on the assignment of values to the vertices $V(G)$ and edges $E(G)$ of a graph G. In this paper we will consider graphs that are simple, finite, connected and undirected. In 1987, Cahit introduced Cordial Labeling as a variation of both graceful and harmonious labeling[1]. Till now many researchers worked on the various type of cordial labelings. A dynamic survey of graph labeling is published and updated every year by Gallian [2]. In 2013, Sridevi et. al [6] proved that Path, Cycle are Fibonacci divisor cordial graph. Fibonacci cordial labeling was introduced by Rokhad and Ghodasara [4]. This method of graph labeling assigns the vertices numbers from the Fibonacci sequence.

Definition 1.1. *A function $f : V(G) \to \{0, 1\}$ is said to be Cordial Labeling if the induced function $f^* : E(G) \to \{0, 1\}$ defined by*

$$f^*(uv) = |f(u) - f(v)|$$

[1] Fort Hays State University, Department of Mathematics, 600 Park St. Hays, KS, USA
e-mail: s_mitra@fhsu.edu; https://orcid.org/0009-0004-2730-9929.
e-mail: s_bhoumik@fhsu.edu; https://orcid.org/0000-0002-2039-5711.
[2] Kansas State University, Department of Mathematics, Manhattan, KS, USA.
e-mail: aepritchard@ksu.edu; https://orcid.org/0009-0001-2553-4215.
∗ Corresponding author.
§ Manuscript received: February 19, 2024; accepted: June 12, 2024.
TWMS Journal of Applied and Engineering Mathematics, Vol.15, No.5;

satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$, *as well as* $|e_f(0) - e_f(1)| \leq 1$, *where*
$v_f(0) :=$ *number of vertices with label* 0,
$v_f(1) :=$ *number of vertices with label* 1,
$e_f(0) :=$ *number of edges with label* 0,
$e_f(1) :=$ *number of edges with label* 1.

Fibonacci Cordial labeling is an extension of Cordial labeling, where we label the vertices with Fibonacci numbers instead of 0 and 1.

Definition 1.2. *The sequence* F_n *of Fibonacci numbers is defined by the recurrence relation:*

$$F_n = F_{n-1} + F_{n-2}; \quad F_0 = 0, F_1 = F_2 = 1,$$

Definition 1.3. *An injective function* $f : V(G) \to \{F_0, F_1, \cdots, F_n\}$ *is said to be Fibonacci cordial labeling if the induced function* $f^* : E(G) \to \{0, 1\}$ *defined by*

$$f^*(uv) = (f(u) + f(v)) \pmod 2$$

satisfies the condition $|e_f(0) - e_f(1)| \leq 1$.

Rokad and Ghodasara provided the result for Petersen graph, Wheel graph, Shell graph, Bistar, and some product families of graphs (corona etc.) [5]. Later in 2017, Fibonacci cordial labeling was explored for more families of graphs by Rokad [4]. In [3] Mitra and Bhoumik provided the Fibonacci cordial labeling for complete graphs, cycles, and corona products. We have consider the families of Comb graphs, Jellyfish graphs, Coconut trees, and finally Bipartite graph $K_{m,n}$. First let us define the families of graphs that we are considering in this paper.

Definition 1.4. *A Comb graph consists of a path of length* k *and a series of single pendant vertices attached to each vertex in the path.*

Definition 1.5. *A Jellyfish graph consists of four vertices* v_1, v_2, v_3, v_4 *joined in a cycle with an additional edge between* v_1 *and* v_3. *There are additionally* m *pendant vertices attached to* v_2 *and* n *pendant vertices attached to* v_4.

Definition 1.6. *A Coconut Tree consists of a path of length* n *with* m *pendant vertices attached to the final vertex in the path,* v_n.

2. Main Results

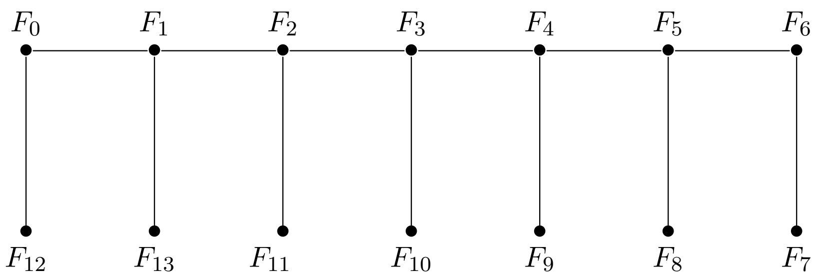


Figure 1. Fibonacci cordial labeling of a comb graph of length $n = 7$

2.1. **Comb graphs.** In this section we prove that comb graphs are Fibonacci cordial.

Theorem 2.1. *Comb graphs are Fibonacci cordial.*

Proof. Identify the vertices of a comb graph as $V(G) = \{u_0, u_2, \cdots, u_{n-1}\} \cup \{v_0, v_2, \cdots, v_{n-1}\}$ where $u_0, u_1, \cdots, u_{n-1}$ are the vertices of the path P_n and $v_0, v_2, \cdots, v_{n-1}$ are the attached pendant vertices (see Figure 1). As every Fibonacci number F_n is even if $3|n$, we will consider three different cases, $n = 3k, 3k+1$, and $3k+2$ for all $n \in \mathbb{Z}^+$. We denote the total number of odd edges by ε_1 (analogously ε_0 for even edges). We start will labeling $f(u_i) = F_i$, for $i \in \{0, 1, 2, \cdots, n-1\}$, and label the pendant vertices as follows:
Case 1. $n = 3k$

$$f(v_i) = \begin{cases} F_{6k-1}, & \text{if } i = 0 \\ F_{6k-i}, & \text{if } 1 \le i \le 3\lfloor k/2 \rfloor,\ i \equiv 0 \pmod 3 \\ F_{6k-i-1}, & \text{if } 1 \le i \le 3\lfloor k/2 \rfloor,\ i \equiv 1 \pmod 3 \\ F_{6k-i-2}, & \text{if } 1 \le i \le 3\lfloor k/2 \rfloor,\ i \equiv 2 \pmod 3 \\ F_{6k-i-1}, & \text{if } 1 + 3\lfloor k/2 \rfloor \le i \le n-1 \end{cases}$$

This produces $\varepsilon_0 = 3k + \lfloor k/2 \rfloor$ and $\varepsilon_1 = 4k - \lfloor k/2 \rfloor$. Thus, $\mid \varepsilon_0 - \varepsilon_1 \mid = \mid k - 2\lfloor k/2 \rfloor \mid$. Note that $k \equiv 0 \pmod 2$ implies $\mid \varepsilon_0 - \varepsilon_1 \mid = 0$ and $k \equiv 1 \pmod 2$ implies $\mid \varepsilon_0 - \varepsilon_1 \mid = 1$. Therefore, comb graph in this case admits Fibonacci cordial labeling.
Case 2. $n = 3k + 1$

$$f(v_i) = \begin{cases} F_{6k-i}, & \text{if } 0 \le i \le 3\lceil k/2 \rceil - 1,\ i \equiv 0 \pmod 3 \\ F_{6k-i+2}, & \text{if } 0 \le i \le 3\lceil k/2 \rceil - 1,\ i \equiv 1 \pmod 3 \\ F_{6k-i+1}, & \text{if } 0 \le i \le 3\lceil k/2 \rceil - 1,\ i \equiv 2 \pmod 3 \\ F_{6k-i+1}, & \text{if } 3\lceil k/2 \rceil \le i \le n-1 \end{cases}$$

This produces $\varepsilon_0 = 3k + \lfloor k/2 \rfloor$ and $\varepsilon_1 = 4k - \lfloor \frac{k}{2} \rfloor$. Thus, $\mid \varepsilon_0 - \varepsilon_1 \mid = \mid k - 2\lfloor k/2 \rfloor \mid$. Similar to the previous case, $k \equiv 0 \pmod 2$ implies to $\mid \varepsilon_0 - \varepsilon_1 \mid = 0$ and otherwise we have $\mid \varepsilon_0 - \varepsilon_1 \mid = 1$.
Case 3. $n = 3k + 2$

$$f(v_i) = \begin{cases} F_{6k-i+3}, & \text{if } 0 \le i \le 3\lfloor k/2 \rfloor + 1 \\ F_{6k-i+2}, & \text{if } 3\lfloor k/2 \rfloor + 2 \le i \le n-1,\ i \equiv 0 \pmod 3 \\ F_{6k-i+4}, & \text{if } 3\lfloor k/2 \rfloor + 2 \le i \le n-1,\ i \equiv 1 \pmod 3 \\ F_{6k-i+3}, & \text{if } 3\lfloor k/2 \rfloor + 2 \le i \le n-1,\ i \equiv 2 \pmod 3 \end{cases}$$

This above labeling generated $\varepsilon_0 = n + 3\lfloor k/2 \rfloor + \lceil k/2 \rceil + 2$ and $\varepsilon_1 = 2k + 2\lceil k/2 \rceil + 1$. Thus, $\mid \varepsilon_1 - \varepsilon_0 \mid = \mid k - 3\lfloor k/2 \rfloor + \lceil k/2 \rceil - 1 \mid$. It can be easily observed that $k \equiv 0, 1 \pmod 2$ both lead to $\mid \varepsilon_1 - \varepsilon_0 \mid = 1$. Therefore, comb graphs in this case admit Fibonacci cordial labeling.

□

2.2. **Coconut Tree Graphs.** A Coconut tree CT(n, m) is the graph,for all positive integer $m, n \ge 2$ is obtained from the path P_n by appending m many pendant edges at an end vertex of P_n. We identify the vertices of a coconut tree graph as $V(G) = \{u_1, u_2, \cdots, u_n\} \cup \{v_1, v_2, \cdots, v_m\}$, where $u_1, u_2, \cdots, u_n$ are the vertices of path P_n and $v_1, v_2, \cdots, v_m$ are the pendant vertices adjacent to u_n.

Theorem 2.2. CT(n, m) *graphs is not Fibonacci cordial if*

$$m - \left\lfloor \frac{m+n}{3} \right\rfloor \ge \begin{cases} 3, & \textit{if } n \equiv 0 \ (\text{mod}2) \\ 4, & \textit{otherwise} \end{cases}$$

Proof. This graph produces a total amount of usable even vertices $V_T(0) = \lfloor \frac{n+m}{3} \rfloor + 1$ and a total amount of usable odd vertices $V_T(1) = n + m - \lfloor \frac{n+m}{3} \rfloor - 1$. First we assume n is even. Now in order to achieve the optimum (minimum) value on $\varepsilon_0 - \varepsilon_1$ we assign t many even Fibonacci labels on $u_1, u_2, \cdots, u_{2t-1}$ of P_n. The rest of the even Fibonacci labels we can assign on k many pendant vertices. Clearly $k + t \leq \lfloor \frac{n+m}{3} \rfloor + 1$, and $t \leq \lfloor n/2 \rfloor$. Thus in order $\mathrm{CT}(n, m)$ be Fibonacci cordial, $|\varepsilon_0 - \varepsilon_1| \leq |(n - 4t + 1) + (m - 2k)| = |m + n - 2(k + t) - 2t + 1|$ must be less than 2. However $m + n - 2(k + t) - 2t + 1$ is less than 2, when $m + n \leq 2(\lfloor \frac{n+m}{3} \rfloor + 1) + 2(n/2) + 1$, or $m \leq 2\lfloor \frac{n+m}{3} \rfloor + 3$.

Similar argument shows that $\mathrm{CT}(n, m)$ is Fibonacci cordial only when $m \leq 2\lfloor \frac{n+m}{3} \rfloor + 4$ for n being odd. □

See Fibonacci cordial labeling for a coconut tree graph with $n = 6$ and $m = 4$ in Figure 2.

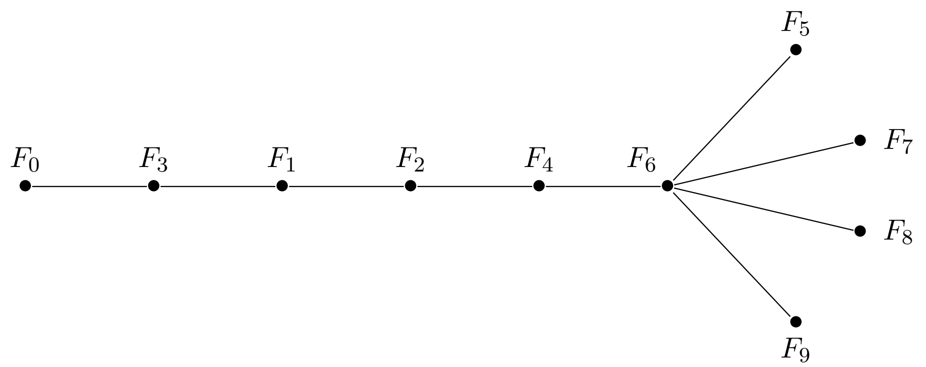


FIGURE 2. Fibonacci cordial labeling of a coconut tree graph

2.3. **Jellyfish Graphs.**

Theorem 2.3. *All Jellyfish graphs are Fibonacci cordial.*

Proof. Let us define the Jellyfish graph as follows: $v_1v_3v_2v_4v_1$ are the vertices forming C_4, where there is a chord between v_3 and v_4. $\{u_{ij} : 1 \leq j \leq m_i\}$ are pendant vertices connected with v_i for $i = 1, 2$ (see Figure 3). In order to show that this graph is Fibonacci cordial we label the vertices in the following manner.

- we label v_1 by even, and other v_i's (for $i = 2, 3, 4$) by odd Fibonacci numbers.
- Among the pendant vertices, for $i = 1, 2$, $\{u_{i1}, u_{i2}, \cdots, u_{ip_i}\}$ are labeled with odd and the rest $\{u_{ip_{i+1}}, u_{ip_{i+2}}, \cdots, u_{im_i}\}$ by even Fibonacci numbers

From the above labeling it is clear that the number of odd labels used is $p_1 + p_2 + p_3 + 3$ and even labels is $(m_1 - p_1) + (m_2 - p_2) + 1$. Note that

$$\varepsilon_1 - \varepsilon_0 = 2p_1 + m_2 - m_1 - 2p_2 - 1 \tag{1}$$

Let $f : V(G) \to S$, where $S = \{F_0, F_1, \cdots, F_{m_1+m_2+4}\}$, assign the Fibonacci labeling to the vertices of the graph. As the number of even Fibonacci numbers in S depends on the value of m_1 and m_2, we consider the following cases. Let $q_i = \lfloor m_i/3 \rfloor$, and $q_i = 4k_i + r_i$, for $i = 1, 2$.

Case 1. In this case we consider $m_1 = 3q_1$, $m_2 = 3q_2$. First we skip one even Fibonacci number, i.e., we use $q_1 + q_2 + 1$ many even Fibonacci numbers to assign on vertices. This leads to $p_1 + p_2 = 2(q_1 + q_2)$. Now with this relation Equation (1) simplifies to

$\varepsilon_1 - \varepsilon_0 = 2p_1 + 3q_2 - 3q_1 - 2(2(q_1+q_2) - p_1) - 1 = 4p_1 - q_2 - 7q_1 - 1$. In order to be Fibonacci cordial, $\varepsilon_1 - \varepsilon_0$ needs be either $1, 0$, or -1.

- Considering $\varepsilon_1 - \varepsilon_0 = 1$ implies $p_1 = (7q_1 + q_2 + 2)/4$ which is an integer only if $r_2 \equiv r_1 + 2 \pmod 4$.
- $\varepsilon_1 - \varepsilon_0 = 0$ implies $p_1 = (7q_1 + q_2 + 1)/4$ which is an integer only if $r_2 \equiv r_1 + 3 \pmod 4$.
- Finally $\varepsilon_1 - \varepsilon_0 = -1$ implies $p_1 = (7q_1 + q_2)/4$ which is an integer only if $r_2 \equiv r_1 \pmod 4$.

On the other hand, skipping one odd Fibonacci number gives $p_1 + p_2 = 2(q_1 + q_2) - 1$, and consequently (from Equation 1) $\varepsilon_1 - \varepsilon_0 = 4p_1 - q_2 - 7q_1 + 1$. As $\varepsilon_1 - \varepsilon_0 = 1$ and -1 lead to the previous result, the only conclusion we can make is $p_1 = (7q_1 + q_2 - 1)/4$, from $\varepsilon_1 - \varepsilon_0 = 0$. Clearly p_1 is an integer $r_2 \equiv r_1 + 1 \pmod 4$. We can summarize the result as follows:

$$p_1 = \begin{cases} (7q_1 + q_2)/4, & \text{if } r_2 \equiv r_1 \pmod 4 \\ (7q_1 + q_2 - 1)/4, & \text{if } r_2 \equiv r_1 + 1 \pmod 4 \\ (7q_1 + q_2 + 2)/4, & \text{if } r_2 \equiv r_1 + 2 \pmod 4 \\ (7q_1 + q_2 + 1)/4, & \text{if } r_2 \equiv r_1 + 3 \pmod 4 \end{cases}$$

It is easy to verify that

$$p_2 = \begin{cases} 2(q_1 + q_2) - 1 - p_1, & \text{if } r_2 \equiv r_1 + 1 \pmod 4 \\ 2(q_1 + q_2) - p_1, & \text{otherwise} \end{cases}$$

Now we consider other cases, and provide the values of p_1 and p_2. We skip the computation as it would be very similar to the previous case.

Case 2. Next we consider $m_1 = 3q_1$, $m_2 = 3q_2 + 1$

$$p_1 = \begin{cases} (7q_1 + q_2)/4, & \text{if } r_2 \equiv r_1 \pmod 4 \\ (7q_1 + q_2 + 3)/4, & \text{if } r_2 \equiv r_1 + 1 \pmod 4 \\ (7q_1 + q_2 + 2)/4, & \text{if } r_2 \equiv r_1 + 2 \pmod 4 \\ (7q_1 + q_2 + 1)/4, & \text{if } r_2 \equiv r_1 + 3 \pmod 4 \end{cases}$$

$$p_2 = \begin{cases} 2(q_1 + q_2) - p_1, & \text{if } r_2 \equiv r_1 + 1 \pmod 4 \\ 2(q_1 + q_2) + 1 - p_1, & \text{otherwise} \end{cases}$$

Case 3. Next we consider $m_1 = 3q_1$, $m_2 = 3q_2 + 2$

$$p_1 = \begin{cases} (7q_1 + q_2)/4, & \text{if } r_2 \equiv r_1 \pmod 4 \\ (7q_1 + q_2 - 1)/4, & \text{if } r_2 \equiv r_1 + 1 \pmod 4 \\ (7q_1 + q_2 + 2)/4, & \text{if } r_2 \equiv r_1 + 2 \pmod 4 \\ (7q_1 + q_2 + 1)/4, & \text{if } r_2 \equiv r_1 + 3 \pmod 4 \end{cases}$$

$$p_2 = \begin{cases} 2(q_1 + q_2) - p_1, & \text{if } r_2 \equiv r_1 + 1 \pmod 4 \\ 2(q_1 + q_2) - p_1 + 1, & \text{otherwise} \end{cases}$$

Case 4. Next we consider $m_1 = 3q_1 + 1$, $m_2 = 3q_2 + 1$

$$p_1 = \begin{cases} (7q_1 + q_2 + 4)/4, & \text{if } r_2 \equiv r_1 \pmod 4 \\ (7q_1 + q_2 + 3)/4, & \text{if } r_2 \equiv r_1 + 1 \pmod 4 \\ (7q_1 + q_2 + 2)/4, & \text{if } r_2 \equiv r_1 + 2 \pmod 4 \\ (7q_1 + q_2 + 1)/4, & \text{if } r_2 \equiv r_1 + 3 \pmod 4 \end{cases}$$

$$p_2 = \begin{cases} 2(q_1 + q_2) - p_1, & \text{if } r_2 \equiv r_1 + 3 \pmod 4 \\ 2(q_1 + q_2) + 1 - p_1, & \text{otherwise} \end{cases}$$

Case 5. Next we consider $m_1 = 3q_1 + 1$, $m_2 = 3q_2 + 2$

$$p_1 = \begin{cases} (7q_1 + q_2 + 4)/4, & \text{if } r_2 \equiv r_1 \pmod 4 \\ (7q_1 + q_2 - 1)/4, & \text{if } r_2 \equiv r_1 + 1 \pmod 4 \\ (7q_1 + q_2 + 2)/4, & \text{if } r_2 \equiv r_1 + 2 \pmod 4 \\ (7q_1 + q_2 + 5)/4, & \text{if } r_2 \equiv r_1 + 3 \pmod 4 \end{cases}$$

$$p_2 = \begin{cases} 2(q_1 + q_2) + 1 - p_1, & \text{if } r_2 \equiv r_1 + 2 \pmod 4 \\ 2(q_1 + q_2) + 2 - p_1, & \text{otherwise} \end{cases}$$

Case 6. Finally we consider $m_1 = 3q_1 + 2$, $m_2 = 3q_2 + 2$

$$p_1 = \begin{cases} (7q_1 + q_2 + 8)/4, & \text{if } r_2 \equiv r_1 \pmod 4 \\ (7q_1 + q_2 + 7)/4, & \text{if } r_2 \equiv r_1 + 1 \pmod 4 \\ (7q_1 + q_2 + 6)/4, & \text{if } r_2 \equiv r_1 + 2 \pmod 4 \\ (7q_1 + q_2 + 5)/4, & \text{if } r_2 \equiv r_1 + 3 \pmod 4 \end{cases}$$

$$p_2 = \begin{cases} 2(q_1 + q_2) + 2 - p_1, & \text{if } r_2 \equiv r_1 + 3 \pmod 4 \\ 2(q_1 + q_2) + 3 - p_1, & \text{otherwise} \end{cases}$$

□

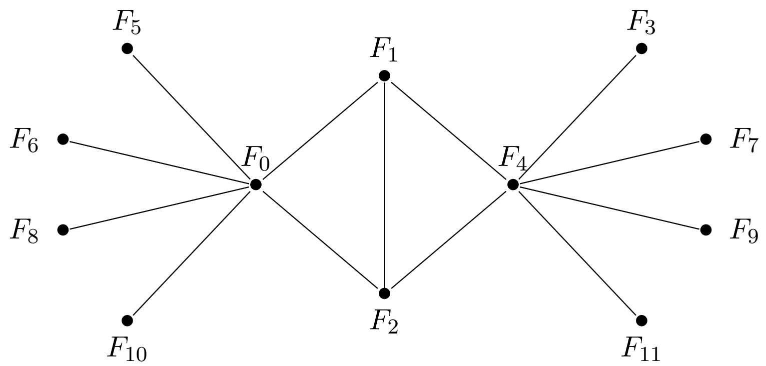


FIGURE 3. Jellyfish graph with $n = 4$

Fibonacci cordial labeling of a symmetrical Jellyfish graph with $n = 4$ has been shown in Figure 3.

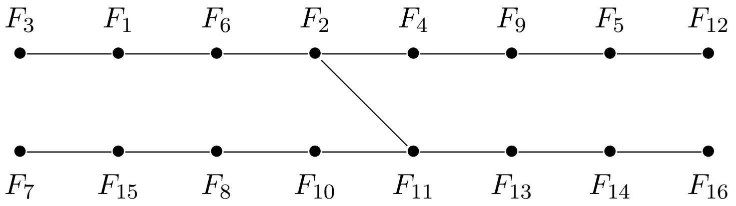


FIGURE 4. Fibonacci Cordial Labeling of $H-$graph of order 8

2.4. $H-$**Graph.** Let P_n^1 and P_n^2 be any two paths with n vertices. Let $V(P_n^1) = \{u_1, u_2, \cdots, u_n\}$ and $V(P_n^2) = \{v_1, v_2, \cdots, v_n\}$. We generate $H-$graph of order n by joining $u_{\lfloor (n+1)/2 \rfloor}$ and $v_{\lceil (n+1)/2 \rceil}$ by an edge (see Figure 4). Clearly the cardinality of the vertex set and edge set of $H-$graph is $2n$ and $2n-1$ respectively.

Theorem 2.4. *$H-$graph is Fibonacci cordial.*

Proof. We consider the following cases for the labeling of $H-$graph of order n. The vertex labeling of $f : V(H) \to \{F_0, F_1, \cdots, F_{2n}\}$ is defined as follows:
Case 1. $n = 6p$.

$$f(u_i) = \begin{cases} F_{\frac{3(i+1)}{2}}, & \text{for } 1 \le i \le p^* - 3, i \text{ odd} \\ F_{\frac{3i}{2}}, & \text{for } p^* \le i \le n, i \text{ even} \\ F_{\lfloor \frac{3i-1}{4} \rfloor}, & \text{for } 2 \le i \le p^* - 2, i \text{ even} \\ F_{\lceil \frac{3i-1}{4} \rceil}, & \text{for } p^* - 1 \le i \le n-1, i \text{ odd} \end{cases}$$

where

$$p^* = \begin{cases} 3p+1, & \text{if } p \text{ is odd} \\ 3p+2, & \text{if } p \text{ is even} \end{cases}$$

$$f(v_i) = \begin{cases} F_{\lfloor \frac{3n}{4} \rfloor + 1}, & \text{for } i = 1 \\ F_{3(3p+i-1)}, & \text{for } 2 \le i \le p+1 \\ F_{n+i-\lceil \frac{n+1-i}{2} \rceil}, & \text{for } p+2 \le i \le n \end{cases}$$

Case 2. $n = 6p+1$.

$$f(u_i) = \begin{cases} F_{\frac{3(i+1)}{2}}, & \text{for } 1 \le i \le p^* - 4, i \text{ odd} \\ F_{\frac{3i-3}{2}}, & \text{for } p^* \le i \le n, i \text{ odd} \\ F_{\lfloor \frac{3i-1}{4} \rfloor}, & \text{for } 2 \le i \le p^* - 3, i \text{ even} \\ F_{\lceil \frac{3i+2}{4} \rceil}, & \text{for } p^* - 1 \le i \le n-1, i \text{ even} \\ F_{\lceil \frac{3i-2}{4} \rceil}, & \text{for } i = p^* - 2, \end{cases}$$

where

$$p^* = \begin{cases} 3p+2, & \text{if } p \text{ is odd} \\ 3p+3, & \text{if } p \text{ is even} \end{cases}$$

$$f(v_i) = \begin{cases} F_{\lfloor \frac{3n}{4} \rfloor + 2}, & \text{for } i = 1 \\ F_{3(3p+i-1)}, & \text{for } 2 \le i \le p+1 \\ F_{n+i-\lceil \frac{n-i-1}{2} \rceil}, & \text{for } p+2 \le i \le n \end{cases}$$

Case 3. $n = 6p + 2$.

$$f(u_i) = \begin{cases} F_{\frac{3(i+1)}{2}}, & \text{for } 1 \leq i \leq p^* - 3, i \text{ odd} \\ F_{\frac{3i}{2}}, & \text{for } p^* \leq i \leq n, i \text{ even} \\ F_{\lfloor \frac{3i-1}{4} \rfloor}, & \text{for } 2 \leq i \leq p^* - 2, i \text{ even} \\ F_{\lceil \frac{3i-1}{4} \rceil}, & \text{for } p^* - 1 \leq i \leq n - 1, i \text{ odd} \end{cases}$$

where

$$p^* = \begin{cases} 3p + 3, & \text{if } p \text{ is odd} \\ 3p + 2, & \text{if } p \text{ is even} \end{cases}$$

$$f(v_i) = \begin{cases} F_{\lceil \frac{3n+1}{4} \rceil}, & \text{for } i = 1 \\ F_{3(3p+i)}, & \text{for } 2 \leq i \leq p + 1 \\ F_{n+i-\lceil \frac{n-i}{2} \rceil}, & \text{for } p + 2 \leq i \leq n \end{cases}$$

Case 4. $n = 6p + 3$.

$$f(u_i) = \begin{cases} F_{\frac{3(i+1)}{2}}, & \text{for } 1 \leq i \leq p^* - 4, i \text{ odd} \\ F_{\frac{3(i-1)}{2}}, & \text{for } p^* \leq i \leq n, i \text{ odd} \\ F_{\lfloor \frac{3i-1}{4} \rfloor}, & \text{for } 2 \leq i \leq p^* - 3, i \text{ even} \\ F_{\lceil \frac{3i+1}{4} \rceil}, & \text{for } p^* - 1 \leq i \leq n - 1, i \text{ even} \\ F_{\lceil \frac{3i-2}{4} \rceil}, & \text{for } i = p^* - 2, \end{cases}$$

where

$$p^* = \begin{cases} 3p + 4, & \text{if } p \text{ is odd} \\ 3p + 3, & \text{if } p \text{ is even} \end{cases}$$

$$f(v_i) = \begin{cases} F_{\lceil \frac{3n}{4} \rceil + 1}, & \text{for } i = 1 \\ F_{3(3p+i)}, & \text{for } 2 \leq i \leq p + 2 \\ F_{n+i-\lceil \frac{n-i+1}{2} \rceil}, & \text{for } p + 3 \leq i \leq n \end{cases}$$

Case 5. $n = 6p + 4$.

$$f(u_i) = \begin{cases} F_{\frac{3(i+1)}{2}}, & \text{for } 1 \leq i \leq p^* - 3, i \text{ odd} \\ F_{\frac{3i}{2}}, & \text{for } p^* \leq i \leq n, i \text{ even} \\ F_{\lfloor \frac{3i-1}{4} \rfloor}, & \text{for } 2 \leq i \leq p^* - 2, i \text{ even} \\ F_{\lceil \frac{3i-1}{4} \rceil}, & \text{for } p^* - 1 \leq i \leq n - 1, i \text{ odd} \end{cases}$$

where

$$p^* = \begin{cases} 3p + 3, & \text{if } p \text{ is odd} \\ 3p + 4, & \text{if } p \text{ is even} \end{cases}$$

$$f(v_i) = \begin{cases} F_{\lceil \frac{3n+2}{4} \rceil}, & \text{for } i = 1 \\ F_{3(3p+i+1)}, & \text{for } 2 \leq i \leq p + 1 \\ F_{n+i-\lceil \frac{n-i-1}{2} \rceil}, & \text{for } p + 2 \leq i \leq n \end{cases}$$

Case 6. $n = 6p + 5$.

$$f(u_i) = \begin{cases} F_{\frac{3(i+1)}{2}}, & \text{for } 1 \leq i \leq p^* - 4, i \text{ odd} \\ F_{\frac{3(i-1)}{2}}, & \text{for } p^* \leq i \leq n, i \text{ odd} \\ F_{\lfloor \frac{3i-1}{4} \rfloor}, & \text{for } 2 \leq i \leq p^* - 3, i \text{ even} \\ F_{\lceil \frac{3i+1}{4} \rceil}, & \text{for } p^* - 1 \leq i \leq n - 1, i \text{ even} \\ F_{\lceil \frac{3i-2}{4} \rceil}, & \text{for } i = p^* - 2, \end{cases}$$

where

$$p^* = \begin{cases} 3p + 4, & \text{if } p \text{ is odd} \\ 3p + 5, & \text{if } p \text{ is even} \end{cases}$$

$$f(v_i) = \begin{cases} F_{\lceil \frac{3n}{4} \rceil + 1}, & \text{for } i = 1 \\ F_{3(3p+i+1)}, & \text{for } 2 \leq i \leq p + 2 \\ F_{n+i-\lceil \frac{n-i}{2} \rceil}, & \text{for } p + 3 \leq i \leq n \end{cases}$$

□

2.5. W_{2n+1} **graph.**

Definition 2.5. *Ring sum $G_1 \oplus G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \oplus G_2 = (V_1 \cup V_2, (E1 \cup E_2) - (E_1 \cap E_2))$*

We now construct a family of graph $K_{1,n} \oplus K_{1,n}$ one pendant vertex in the one $K_{1,n}$ is also a pendant vertex in the other copy of $K_{1,n}$. This resulting graph is denoted as W_{2n+1} (see Figure 5). Note that $|V(W_{2n+1})| = 2n + 1$ and $|E(W_{2n+1})| = 2n$.

Theorem 2.6. *The graph W_{2n+1} is Fibonacci cordial.*

Proof. We only need to prove the result for $n \geq 3$, as W_3 and W_5 both graphs are Path, and hence Fibonacci cordial [3]. Let us consider $n = 3p + q$ for positive integer p, q such that $q \in \{0, 1, 2\}$. Now we first identify the vertices of W_{2n+1} as follow: u, v are the apex vertices of star graphs $K_{1,n}$ graphs, and $\{u_1, u_2, \cdots, u_n = v_1, v_2, \cdots, v_n\}$ are the pendant vertices. The edge set $E(W_{2n+1}) = \{uu_i, vv_i : 1 \leq i \leq n\}$. The vertex labeling of $f : V(G) \to \{F_0, F_1, \cdots, F_{2n+1}\}$ is defined as follows. $f(u) = F_0, f(v) = F_1$, and $f(u_n) = F_2$

$$f(u_i) = \begin{cases} F_{3i}, & \text{for } 1 \leq i \leq p \\ F_{3s_1+t_1+1}, & \text{for } p + 1 \leq i \leq n - 1 \end{cases}$$

where $s_1 = \lfloor (i - p + 1)/2 \rfloor$ and $t_1 = (i - p + 1)\%2$.

$$f(v_i) = \begin{cases} F_{3(p+i-1)}, & \text{for } 2 \leq i \leq p + 1 \\ F_{3p+3s_2+t_2+1}, & \text{for } p + 2 \leq i \leq n \end{cases}$$

where $s_2 = \lfloor (i - p + q - 1)/2 \rfloor$ and $t_2 = (i - p + q - 1)\%2$. In view of the above labeling, it is clear that $e_f(1) = e_f(0) = p + n - p - 1 + 1 = n$, which proves the result. □

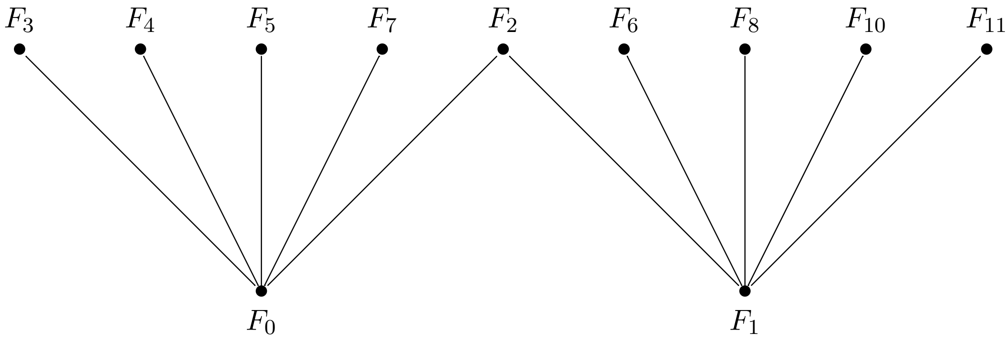


FIGURE 5. Fibonacci cordial labeling of W_{11}

3. Conclusion

Studying graphs that admit Fibonacci cordial labeling is indeed an intriguing area of research within graph theory. The concept of Fibonacci cordial labeling, which involves labeling the vertices of a graph such that certain properties related to Fibonacci numbers hold, has applications in various fields including computer science and combinatorics.

Investigating whether certain unusual graphs possess Fibonacci cordial labeling can lead to new insights into the structure and properties of these graphs. Additionally, exploring equivalent results for different families of graphs presents an open area of research, as it allows for the comparison and generalization of results across various graph classes.

Researchers in graph theory often seek to identify and understand patterns, properties, and relationships within different types of graphs, and the study of Fibonacci cordial labeling offers a unique perspective in this regard. By investigating the existence and properties of such labelings in various graph families, researchers can advance our understanding of graph theory and potentially uncover new connections between seemingly disparate graph structures.

References

[1] Cahit, Cordial graphs: A weaker version of graceful and Harmonic graphs, (1987), Ars Combinatoria, 23, pp. 201-207.

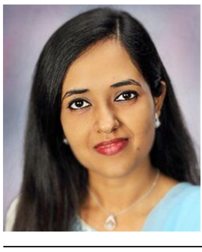
[2] Gallian J. A., (2017), A dynamic survey of graph labeling, The Electronics Journal of Combinatorics, 17 DS 6 pp. 1-535.

[3] Mitra S., Bhoumik S., (2020), Fibonacci Cordial Labeling of Some Special Families of Graphs, Annals of Pure and Applied Mathematics, 21-2 pp. 135-140.

[4] Rokad A. H., (2017), Fibonacci Cordial Labeling of Some Special Graphs, oriental Journal of Computer Science and Technology, 10-4 pp. 824-828.

[5] Rokad, A. H., Ghodasara G. V., (2016), Fibonacci Cordial Labeling of Some Special Graphs, Annals of Pure and Applied Mathematics, 11-1 pp. 133-144.

[6] Sridevi R., Nagarajun K., Nellaimurugan A., Navanaeethakrishnan S., (2013), Fibonacci divisor cordial graphs, International Journal of Mathematics and Soft Computing, 3(3) pp. 33-39.



Sarbari Mitra is currently an Associate professor in the Department of Mathematics, Fort Hays State University. She received her B.Sc. degree (in Mathematics Hons.) from University of Calcutta in 2005 and her M.Sc. degree in Mathematics, from Indian Institute of Technology Kharagpur in 2007. She received her PhD degree in the area of Cryptology from Indian Institute of Technology Kharagpur, India in 2014. Her research interests include Wireless Sensor Networking, Graph Coloring, and Domination in Graphs.

Soumya Bhoumik is currently a professor in the Department of Mathematics, at Fort Hays State University. He completed his B.Sc. in Mathematics from Calcutta University (India) in 2004, and his M.Sc. in Mathematics from the Indian Institute of Technology (Madras, India) in 2006. He received his Ph.D. Degree in Algebraic Graph Theory from Mississippi State University in 2013. His research interests include Algebraic Graph Theory, Graph Coloring, Domination in Graphs, and Isogeny based Cryptography.

Ashley Prichard is currently pursuing a B.S. degree in mathematics and statistics in statistics at Kansas State University. Her research has focused on graph theory and