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THE SUBORDINATION THEOREM FOR γ-μ-SPIRAL-LIKE FUNCTIONS OF ORDER λ

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Abstract. In this paper, we introduce a novel subclass $\mathcal{S}^{\gamma}(\lambda,\mu)$ of normalized analytic functions in the open unit disk $\mathbb{U}$. First we obtain several sufficient conditions that a function belongs to the subclass $\mathcal{S}^{\gamma}(\lambda,\mu)$. Furthermore, we derive the upper bounds for the initial Taylor-Maclaurin coefficients $|a_2|, |a_3|$ and $|a_4|$ as well as the Fekete-Szegö type inequalities for functions belong to this subclass. The results presented in this paper would generalize and improve some recent works of several earlier authors.

Keywords: γ-spiral-like functions, γ-spiral-like functions of order λ, Subordination, Hadamard product, Coefficient estimates.

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1. Introduction

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S} \subset \mathcal{A}$ consist of univalent functions in $\mathbb{U}$. (see details [3]).

Since univalent functions are one-to-one, they are invertible and inverse functions need not be defined on the entire unit disk $\mathbb{U}$. The Koebe one-quarter theorem [3] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$.

Definition 1.1. [3] *For two functions f and g, which are analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$ and write*

$$f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

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if there exists a Schwarz function ω *which, by definition, is analytic in* $\mathbb{U}$ *with*

$$\omega(0) = 0 \quad \textit{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g(\omega(z)).$$

Remark 1.1. *In particular, if the function* g *is univalent in* $\mathbb{U}$*, then*

$$f \prec g \Longleftrightarrow f(0) = g(0) \quad \textit{and} \quad f(\mathbb{U}) \subseteq g(\mathbb{U}).$$

Špaček [13] introduced the concept of spirallikeness which is a natural generalization of starlikeness. Spirallike functions can be characterized by the following analytic condition.

A function f in $\mathcal{A}$ is said to be in the class of γ-spiral-like functions in $\mathbb{U}$, denoted by $\mathcal{S}^*(\gamma)$, if

$$\Re\left(e^{i\gamma}\frac{zf'(z)}{f(z)}\right) > 0, \ (|\gamma| < \frac{\pi}{2}).$$

A function f in $\mathcal{A}$ is said to be in the class of γ-spiral-like functions of order λ in $\mathbb{U}$, denoted by $\mathcal{S}^*(\gamma;\lambda)$, if

$$\Re\left(e^{i\gamma}\frac{zf'(z)}{f(z)}\right) > \lambda\cos\gamma, \ (|\gamma| < \frac{\pi}{2}, 0 \leq \lambda < 1).$$

The class $\mathcal{S}^*(\gamma;\lambda)$ was studied by Libera [5] and Keogh and Merkes [4].

In this paper, we introduce the following subclass $\mathcal{S}^\gamma(\lambda,\mu)$ of analytic and spiral-like functions. First we obtain several sufficient conditions that a function belongs to the subclass $\mathcal{S}^\gamma(\lambda,\mu)$. Moreover, for functions of this class, we investigate convolution properties and also find coefficient estimates of the Taylor-Maclaurin coefficients $|a_2|$, $|a_3|, |a_4|$ and Fekete-Szegö inequality.

Definition 1.2. *For* $0 \leq \mu \leq 1$, $|\gamma| < \frac{\pi}{2}$ *and* $0 \leq \lambda < 1$, *a function* $f \in \mathcal{A}$ *given by* (1) *is said to be in the class* $\mathcal{S}^\gamma(\lambda,\mu)$ *of* γ-μ*-spiral-like functions of order* λ*, if the following condition is satisfied:*

$$\Re\left(e^{i\gamma}\frac{zf'(z)+\mu z^2 f''(z)}{(1-\mu)f(z)+\mu zf'(z)}\right) > \lambda\cos\gamma, \tag{2}$$

where $z \in \mathbb{U}$.

Remark 1.2. *There are many options of the parameters* λ, μ *and* γ *which would provide interesting subclasses of analytic functions. For example:*

(A) *By putting* $\mu = 0$*, we obtain the class* $\mathcal{S}^\gamma(\lambda,0) = \mathcal{S}^*(\gamma;\lambda)$ *of functions* $f \in \mathcal{A}$ *which are satisfying the following condition*

$$\Re\left(e^{i\gamma}\frac{zf'(z)}{f(z)}\right) > \lambda\cos\gamma.$$

Indeed, the class $\mathcal{S}^*(\gamma;\lambda)$ *introduced and studied by Libera* [5] *and Keogh and Merkes* [4].

(B) *By putting* $\mu = \lambda = 0$*, we obtain the class* $\mathcal{S}^\gamma(0,0) = \mathcal{S}^*(\gamma)$ *of functions* $f \in \mathcal{A}$ *which are satisfying the following condition*

$$\Re\left(e^{i\gamma}\frac{zf'(z)}{f(z)}\right) > 0.$$

Indeed, the class $\mathcal{S}^*(\gamma)$ *introduced and studied by Špaček* [13].

(C) *By putting $\mu = \gamma = 0$, we obtain the class $\mathcal{S}^0(\lambda, 0) = \mathcal{S}^*_\lambda$ of functions $f \in \mathcal{A}$ which are satisfying the following condition*

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \lambda.$$

Indeed, the class $\mathcal{S}^*_\lambda$ introduced by Robertson [10].

(D) *By putting $\mu = 1$, we obtain the class $\mathcal{S}^\gamma(\lambda, 1) = \mathcal{K}(\gamma, \lambda)$ of functions $f \in \mathcal{A}$ which are satisfying the following condition*

$$\Re\left(e^{i\gamma}\left[1 + \frac{zf''(z)}{f'(z)}\right]\right) > \lambda \cos\gamma.$$

Indeed, the class $\mathcal{K}(\gamma, \lambda)$ introduced by Murugusundaramoorthy [8].

(E) *By putting $\mu = 1$ and $\gamma = 0$, we obtain the class $\mathcal{S}^0(\lambda, 1) = \mathcal{K}(\lambda)$ of functions $f \in \mathcal{A}$ which are satisfying the following condition*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \lambda.$$

Indeed, the class $\mathcal{K}(\lambda)$ introduced by Robertson [10]. *In special case, $\mathcal{K}(0) = \mathcal{K}$, is the well-known standard class of convex functions.*

2. Membership characterizations

In this section, we obtain several sufficient conditions for a function $f \in \mathcal{A}$ to be in the class $\mathcal{S}^\gamma(\lambda, \mu)$.

Theorem 2.1. *Let $f \in \mathcal{A}$ given by* (1) *and let σ be a real number with $0 \le \lambda \le \sigma < 1$. If*

$$\left|\frac{zf'(z) + \mu z^2 f''(z)}{(1-\mu)f(z) + \mu z f'(z)} - 1\right| < 1 - \sigma, \tag{3}$$

then $f \in \mathcal{S}^\gamma(\lambda, \mu)$ provided that

$$|\gamma| \le cos^{-1}\left(\frac{1-\sigma}{1-\lambda}\right).$$

Proof. In view of (3), it follows that

$$\frac{zf'(z) + \mu z^2 f''(z)}{(1-\mu)f(z) + \mu z f'(z)} - 1 = (1-\sigma)\omega(z), \tag{4}$$

where $\omega(z)$ is a Schwarz function and $|\omega(z)| < 1$, for $z \in \mathbb{U}$. Therefore

$$\begin{aligned}
\Re\left(e^{i\gamma}\frac{zf'(z) + \mu z^2 f''(z)}{(1-\mu)f(z) + \mu z f'(z)}\right) &= \Re\left(e^{i\gamma}[1 + (1-\sigma)\omega(z)]\right) \\
&= \cos\gamma + (1-\sigma)\Re\left(e^{i\gamma}\omega(z)\right) \\
&\ge \cos\gamma - (1-\sigma)|e^{i\gamma}\omega(z)| \\
&> \cos\gamma - (1-\sigma) \\
&\ge \lambda\cos\gamma,
\end{aligned}$$

for $|\gamma| \le cos^{-1}\left(\frac{1-\sigma}{1-\lambda}\right)$, which completes the proof. □

By setting $\sigma = 1 - (1-\lambda)\cos\gamma$ in Theorem 2.1, we get the following result.

Corollary 2.1. *Let $f \in \mathcal{A}$. If*

$$\left|\frac{zf'(z)+\mu z^2 f''(z)}{(1-\mu)f(z)+\mu z f'(z)}-1\right|<(1-\lambda)\cos\gamma,$$

then $f \in \mathcal{S}^\gamma(\lambda,\mu)$.

Now we obtain the another sufficient condition in the next theorem, for a function $f \in \mathcal{A}$ to be in the class $\mathcal{S}^\gamma(\lambda,\mu)$ in terms of coefficients inequality.

Theorem 2.2. *Let $f \in \mathcal{A}$ given by (1), if*

$$\sum_{n=2}^{\infty}(\mu(n-1)+1)\left[(n-1)\sec\gamma+1-\lambda\right]|a_n|<1-\lambda, \tag{5}$$

then $f \in \mathcal{S}^\gamma(\lambda,\mu)$.

Proof. By Corollary 2.1, it suffices to show that $\left|\frac{zf'(z)+\mu z^2 f''(z)}{(1-\mu)f(z)+\mu z f'(z)}-1\right|<(1-\lambda)\cos\gamma$. For $|z|<1$, we have

$$\begin{aligned}\left|\frac{zf'(z)+\mu z^2 f''(z)}{(1-\mu)f(z)+\mu z f'(z)}-1\right| &= \left|\frac{\sum\limits_{n=2}^{\infty}(n-1)\left(\mu(n-1)+1\right)a_n z^{n-1}}{1+\sum\limits_{n=2}^{\infty}\left(\mu(n-1)+1\right)a_n z^{n-1}}\right| \\ &< \frac{\sum\limits_{n=2}^{\infty}(n-1)\left(\mu(n-1)+1\right)|a_n|}{1-\sum\limits_{n=2}^{\infty}\left(\mu(n-1)+1\right)|a_n|}.\end{aligned}$$

Thus last expression is bounded above by $(1-\lambda)\cos\gamma$, if

$$\sum_{n=2}^{\infty}(n-1)\left(\mu(n-1)+1\right)|a_n|<(1-\lambda)\cos\gamma\left(1-\sum_{n=2}^{\infty}\left(\mu(n-1)+1\right)|a_n|\right),$$

which is equivalent to

$$\sum_{n=2}^{\infty}(\mu(n-1)+1)\left[(n-1)\sec\gamma+1-\lambda\right]|a_n|<1-\lambda.$$

□

By putting $\mu=0$ in Theorem 2.2, we obtain the following result which proved by Murugusundaramoorthy [8, Lemma 1.3].

Corollary 2.2. *Let $f \in \mathcal{A}$ given by (1), if*

$$\sum_{n=2}^{\infty}\left[1+\frac{(n-1)\sec\gamma}{1-\lambda}\right]|a_n|<1,$$

then $f \in \mathcal{S}^*(\gamma;\lambda)$.

By putting $\mu=\lambda=0$ in Theorem 2.2, we obtain the following result.

Corollary 2.3. *Let $f \in \mathcal{A}$ given by (1), if*

$$\sum_{n=2}^{\infty}\left[1+(n-1)\sec\gamma\right]|a_n|<1,$$

then $f \in \mathcal{S}^*(\gamma)$.

By putting $\mu = 1$ in Theorem 2.2, we obtain the following result which proved by Murugusundaramoorthy [8, Lemma 1.4].

Corollary 2.4. *Let $f \in \mathcal{A}$ given by* (1), *if*

$$\sum_{n=2}^{\infty} n\left[(n-1)\sec\gamma + 1 - \lambda\right]|a_n| < 1 - \lambda,$$

then $f \in \mathcal{K}(\gamma, \lambda)$.

3. Subordination Result

In this section, we obtain subordination results for the class $\mathcal{S}^{\gamma}(\lambda, \mu)$. To prove our results we need the following definition and lemma.

Definition 3.1 ([16])**.** *A sequence $\{c_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is regular, univalent and convex in $\mathbb{U}$, we have*

$$\sum_{n=1}^{\infty} c_n b_n z^n \prec h(z) \ (z \in \mathbb{U}). \tag{6}$$

Lemma 3.1 ([16])**.** *The sequence $\{c_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if,*

$$\Re\left(1 + 2\sum_{n=1}^{\infty} c_n z^n\right) > 0 \ (z \in \mathbb{U}).$$

Theorem 3.1. *Let $f \in \mathcal{S}^{\gamma}(\lambda, \mu)$ satisfy the coefficient inequality* (5). *Then*

$$\frac{(\mu+1)\sec\gamma + (1-\lambda)(\mu+1)}{2\left[(\mu+1)\sec\gamma + (1-\lambda)(\mu+2)\right]}(f * h)(z) \prec h(z), \tag{7}$$

for every function $h \in \mathcal{K}$. In particular

$$\Re f(z) > -\frac{(\mu+1)\sec\gamma + (1-\lambda)(\mu+2)}{(\mu+1)\sec\gamma + (1-\lambda)(\mu+1)}. \tag{8}$$

The constant $\frac{(\mu+1)\sec\gamma+(1-\lambda)(\mu+1)}{2[(\mu+1)\sec\gamma+(1-\lambda)(\mu+2)]}$ cannot be replaced by any larger one.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}^{\gamma}(\lambda, \mu)$ satisfy the coefficient inequality (5) and suppose that $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$ be any function in $\mathcal{K}$. Then

$$\frac{(\mu+1)\sec\gamma + (1-\lambda)(\mu+1)}{2\left[(\mu+1)\sec\gamma + (1-\lambda)(\mu+2)\right]}(f * h)(z) =$$
$$\frac{(\mu+1)\sec\gamma + (1-\lambda)(\mu+1)}{2\left[(\mu+1)\sec\gamma + (1-\lambda)(\mu+2)\right]}\left(z + \sum_{n=2}^{\infty} a_n b_n z^n\right). \tag{9}$$

Thus, by Definition 3.1, the subordination result holds true if

$$\left\{\frac{(\mu+1)\sec\gamma + (1-\lambda)(\mu+1)}{2\left[(\mu+1)\sec\gamma + (1-\lambda)(\mu+2)\right]}a_n\right\}_{n=1}^{\infty} \tag{10}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 3.1, this is equivalent to the following inequality

$$\Re\left(1+\sum_{n=1}^{\infty}\frac{(\mu+1)\sec\gamma+(1-\lambda)(\mu+1)}{(\mu+1)\sec\gamma+(1-\lambda)(\mu+2)}a_n z^n\right)>0 \quad (z\in\mathbb{U}). \tag{11}$$

Which implies from following inequalities

$$\begin{aligned}
&\Re\left(1+\sum_{n=1}^{\infty}\frac{(\mu+1)\sec\gamma+(1-\lambda)(\mu+1)}{(\mu+1)\sec\gamma+(1-\lambda)(\mu+2)}a_n z^n\right)\\
&=\Re\left(1+\frac{(\mu+1)\sec\gamma+(1-\lambda)(\mu+1)}{(\mu+1)\sec\gamma+(1-\lambda)(\mu+2)}z+\frac{\sum_{n=2}^{\infty}\{(\mu+1)\sec\gamma+(1-\lambda)(\mu+1)\}}{(\mu+1)\sec\gamma+(1-\lambda)(\mu+2)}a_n z^n\right)\\
&\geq\left[1-\frac{(\mu+1)\sec\gamma+(1-\lambda)(\mu+1)}{(\mu+1)\sec\gamma+(1-\lambda)(\mu+2)}r-\frac{\sum_{n=2}^{\infty}\{(\mu+1)\sec\gamma+(1-\lambda)(\mu+1)\}}{(\mu+1)\sec\gamma+(1-\lambda)(\mu+2)}|a_n|r^n\right],
\end{aligned}$$

(because $(\mu+1)\sec\gamma+(1-\lambda)(\mu+1)\leq(\mu(n-1)+1)\left[(n-1)\sec\gamma+1-\lambda\right]$ for $n\geq 2$, $|\gamma|<\frac{\pi}{2}$)

$$\begin{aligned}
&\geq 1-\frac{(\mu+1)\sec\gamma+(1-\lambda)(\mu+1)}{(\mu+1)\sec\gamma+(1-\lambda)(\mu+2)}r-\frac{\sum_{n=2}^{\infty}(\mu(n-1)+1)\left[(n-1)\sec\gamma+1-\lambda\right]}{(\mu+1)\sec\gamma+(1-\lambda)(\mu+2)}|a_n|r^n\\
&>1-\frac{(\mu+1)\sec\gamma+(1-\lambda)(\mu+1)}{(\mu+1)\sec\gamma+(1-\lambda)(\mu+2)}r-\frac{1-\lambda}{(\mu+1)\sec\gamma+(1-\lambda)(\mu+2)}r\\
&=1-r>0,\ (|z|=r<1).
\end{aligned}$$

Thus (11) holds true in $\mathbb{U}$. It means by using Definition 3.1 and Lemma 3.1 the subordinition (7) is obtained.
The inequality (8) follows from (7) by taking

$$h(z)=\frac{z}{1-z}=z+\sum_{n=2}^{\infty}z^n\in\mathcal{K}.$$

To prove the sharpness of the constant $\frac{(\mu+1)\sec\gamma+(1-\lambda)(\mu+1)}{2[(\mu+1)\sec\gamma+(1-\lambda)(\mu+2)]}$, we consider the function

$$f_0(z)=z-\frac{1-\lambda}{(\mu+1)[\sec\gamma+1-\lambda]}z^2.$$

Clearly $f_0\in\mathcal{S}^{\gamma}(\lambda,\mu)$ because it satisfies in (5). Using (7), we get

$$\frac{(\mu+1)\sec\gamma+(1-\lambda)(\mu+1)}{2\left[(\mu+1)\sec\gamma+(1-\lambda)(\mu+2)\right]}f_0(z)\prec\frac{z}{1-z}.$$

It is easily verified that

$$\min_{|z|\leq r}\Re\left\{\frac{(\mu+1)\sec\gamma+(1-\lambda)(\mu+1)}{2\left[(\mu+1)\sec\gamma+(1-\lambda)(\mu+2)\right]}f_0(z)\right\}=\frac{-1}{2}.$$

This shows that the constant $\frac{(\mu+1)\sec\gamma+(1-\lambda)(\mu+1)}{2[(\mu+1)\sec\gamma+(1-\lambda)(\mu+2)]}$ cannot be replaced by any larger one. $\square$

4. The Fekete-Szegö problem

In this part, we find the coefficient bounds for the Taylor-Maclaurin coefficients $|a_2|$, $|a_3|$ and $|a_4|$. In the following, we derive the upper bounds for the Fekete-Szegö type inequalities for functions belonging to the function class $\mathcal{S}^\gamma(\lambda, \mu)$. In recent years, this problem has been investigated by many researchers including [1, 2, 6, 7, 12, 14, 15]. In order to obtain sharp upper-bounds for the Fekete-Szegö functional for the class $\mathcal{S}^\gamma(\lambda, \mu)$ the following lemma is required.

Lemma 4.1. [9] *Let the function $\omega(z)$ is analytic in the unit disk $\mathbb{U}$ with $\omega(0) = 0$, $|\omega(z)| < 1$, and suppose that*

$$\omega(z) = \sum_{n=1}^{\infty} \omega_n z^n.$$

Then

$$|\omega_n| \le 1 \ , \ |\omega_n| \le 1 - |\omega_1|^n \text{ for all } n = 1, 2, 3, \cdots .$$

Remark 4.1. [11] *Let λ and γ be real numbers with $0 \le \lambda < 1$ and $|\gamma| < \frac{\pi}{2}$. Then the function $\mathcal{P}_{\lambda,\gamma}$ defined by*

$$\mathcal{P}_{\lambda,\gamma}(z) = \frac{1 + e^{-i\gamma}\left(e^{-i\gamma} - 2\lambda\cos\gamma\right) z}{1 - z} \ (z \in \mathbb{U})$$

is analytic and univalent in $\mathbb{U}$ with $\mathcal{P}_{\lambda,\gamma}(0) = 1$. In addition, the function $\mathcal{P}_{\lambda,\gamma}(z)$ maps $\mathbb{U}$ onto the half-plane $H_{\lambda,\gamma} = \left\{w \in \mathbb{C} : \ \Re\{e^{i\gamma} w\} > \lambda\cos\gamma\right\}$.
On the other hand

$$\mathcal{P}_{\lambda,\gamma}(z) = \frac{1 + e^{-i\gamma}\left(e^{-i\gamma} - 2\lambda\cos\gamma\right) z}{1 - z} = 1 + \sum_{n=1}^{\infty} p_n z^n$$

where

$$p_n = 2e^{-i\gamma}(1 - \lambda)\cos\gamma \ (n = 1, 2, 3, \cdots). \tag{12}$$

Theorem 4.1. *Let $f \in \mathcal{A}$ given by (1) be in the class $\mathcal{S}^\gamma(\lambda, \mu)$. Then*

$$|a_2| \le \frac{2(1-\lambda)\cos\gamma}{1+\mu},$$
$$|a_3| \le \frac{2(1-\lambda)\left[1 + 2(1-\lambda)\cos\gamma\right]\cos\gamma}{1+2\mu}$$

and

$$|a_4| \le \frac{2(1-\lambda)\cos\gamma\left[2(1-\lambda)^2\cos^2\gamma + 3(1-\lambda)\cos\gamma + 3\right]}{3(1+3\mu)}.$$

Proof. Let $f \in \mathcal{S}^\gamma(\lambda, \mu)$. By taking $F(z) = \frac{zf'(z) + \mu z^2 f''(z)}{(1-\mu)f(z) + \mu z f'(z)}$, we get $F(z)$ is analytic in $\mathbb{U}$, $F(0) = 1$ and $\Re\{e^{i\gamma} F(z)\} > \lambda\cos\gamma$. Then by using Remark 1.1 and Remark 4.1, we have $F(z) \prec \mathcal{P}_{\lambda,\gamma}(z)$, that means

$$\frac{zf'(z) + \mu z^2 f''(z)}{(1-\mu)f(z) + \mu z f'(z)} \prec \mathcal{P}_{\lambda,\gamma}(z).$$

Hence, there exist a function $\omega(z) = \sum\limits_{n=1}^{\infty} \omega_n z^n$ analytic in $\mathbb{U}$ with $\omega(0) = 0$, $|\omega(z)| < 1$, $z \in \mathbb{U}$, such that

$$\frac{zf'(z) + \mu z^2 f''(z)}{(1-\mu)f(z) + \mu z f'(z)} = \mathcal{P}_{\lambda,\gamma}(\omega(z)). \tag{13}$$

Then

$$\frac{zf'(z)+\mu z^2 f''(z)}{(1-\mu)f(z)+\mu z f'(z)} = 1+(1+\mu)a_2 z+\left(2(1+2\mu)a_3-(1+\mu)^2 a_2^2\right)z^2 + \left((1+\mu)^3 a_2^3-3(1+\mu)(1+2\mu)a_2a_3+3(1+3\mu)a_4\right)z^3+\cdots. \quad (14)$$

By comparing the corresponding coefficients of (13) and (14), we get

$$(1+\mu)a_2 = p_1\omega_1, \quad (15)$$

$$2(1+2\mu)a_3-(1+\mu)^2a_2^2 = p_1\omega_2+p_2\omega_1^2 \quad (16)$$

and

$$(1+\mu)^3a_2^3-3(1+\mu)(1+2\mu)a_2a_3+3(1+3\mu)a_4 = p_1\omega_3+2p_2\omega_1\omega_2+p_3\omega_1^3. \quad (17)$$

From (15), we get

$$a_2=\frac{p_1\omega_1}{1+\mu}. \quad (18)$$

By using Lemma 4.1, we have

$$|a_2|\le \frac{|p_1|}{1+\mu}=\frac{2(1-\lambda)\cos\gamma}{1+\mu}. \quad (19)$$

From (16) and (18), we get

$$2(1+2\mu)a_3 = p_1^2\omega_1^2+p_1\omega_2+p_2\omega_1^2. \quad (20)$$

By using Lemma 4.1 and (20), we have

$$\begin{aligned} 2(1+2\mu)|a_3| &\le |p_1|^2|\omega_1|^2+|p_1||\omega_2|+|p_2||\omega_1|^2 \\ &\le |p_1|^2|\omega_1|^2+|p_1|(1-|\omega_1|^2)+|p_2||\omega_1|^2 \ (p_1=p_2) \\ &\le |p_1|^2|\omega_1|^2+|p_1| \\ &\le |p_1|(|p_1|+1) \\ &= 2(1-\lambda)\left(1+2(1-\lambda)\cos\gamma\right)\cos\gamma. \end{aligned}$$

Now, we obtain the upper bound of $|a_4|$, from (17), (18) and (20), we have

$$3(1+3\mu)a_4=\left(\frac{1}{2}p_1^3+\frac{3}{2}p_1p_2+p_3\right)\omega_1^3+\left(\frac{3}{2}p_1^2+2p_2\right)\omega_1\omega_2+p_1\omega_3. \quad (21)$$

By applying Lemma 4.1 again, we have

$$\begin{aligned} 3(1+3\mu)|a_4| \le\ & \left(\frac{1}{2}|p_1|^3+\frac{3}{2}|p_1||p_2|+|p_3|\right)|\omega_1|^3+\left(\frac{3}{2}|p_1|^2+2|p_2|\right)|\omega_1|(1-|\omega_1|^2) \\ & +|p_1|(1-|\omega_1|^3). \end{aligned}$$

Since $|p_1|=|p_2|=|p_3|$ and $|\omega(z)|<1$, we have

$$\begin{aligned} 3(1+3\mu)|a_4| &\le |p_1|\Big[\left(\frac{1}{2}|p_1|^2+\frac{3}{2}|p_1|+1\right)|\omega_1|^3+\left(\frac{3}{2}|p_1|+2\right)|\omega_1|(1-|\omega_1|^2) \\ &\quad +1-|\omega_1|^3\Big] \\ &\le |p_1|\Big[\left(\frac{1}{2}|p_1|^2-2\right)|\omega_1|^3+\left(\frac{3}{2}|p_1|+2\right)|\omega_1|+1\Big] \\ &\le |p_1|\left[\frac{1}{2}|p_1|^2+\frac{3}{2}|p_1|+3\right] \\ &= 2(1-\lambda)\cos\gamma\left[2(1-\lambda)^2\cos^2\gamma+3(1-\lambda)\cos\gamma+3\right]. \end{aligned}$$

□

By putting $\mu = 0$ in Theorem 4.1, we have the following corollary.

Corollary 4.1. *Let $f \in \mathcal{A}$ given by (1) be in the class $\mathcal{S}^*(\gamma;\lambda)$. Then*

$$|a_2| \leq 2(1-\lambda)\cos\gamma,$$
$$|a_3| \leq 2(1-\lambda)\left[1+2(1-\lambda)\cos\gamma\right]\cos\gamma$$

and

$$|a_4| \leq \frac{2(1-\lambda)\cos\gamma\left[2(1-\lambda)^2\cos^2\gamma+3(1-\lambda)\cos\gamma+3\right]}{3}.$$

By putting $\mu = \lambda = 0$ in Theorem 4.1, we have the following corollary.

Corollary 4.2. *Let $f \in \mathcal{A}$ given by (1) be in the class $\mathcal{S}^*(\gamma)$. Then*

$$|a_2| \leq 2\cos\gamma,$$
$$|a_3| \leq 2\left[1+2\cos\gamma\right]\cos\gamma$$

and

$$|a_4| \leq \frac{2\cos\gamma\left[2\cos^2\gamma+3\cos\gamma+3\right]}{3}.$$

By putting $\mu = 1$ in Theorem 4.1, we have the following corollary.

Corollary 4.3. *Let $f \in \mathcal{A}$ given by (1) be in the class $\mathcal{K}(\gamma,\lambda)$. Then*

$$|a_2| \leq (1-\lambda)\cos\gamma,$$
$$|a_3| \leq \frac{2}{3}(1-\lambda)\left[1+2(1-\lambda)\cos\gamma\right]\cos\gamma$$

and

$$|a_4| \leq \frac{1}{6}(1-\lambda)\cos\gamma\left[2(1-\lambda)^2\cos^2\gamma+3(1-\lambda)\cos\gamma+3\right].$$

Finally in the following theorem we obtain bound for the Fekete-Szegö functional for the class $\mathcal{S}^\gamma(\lambda,\mu)$.

Theorem 4.2. *Let $f \in \mathcal{A}$ given by (1) be in the class $\mathcal{S}^\gamma(\lambda,\mu)$ and let η be a real number. Then*

$$|a_3 - \eta a_2^2| \leq \frac{(1-\lambda)\cos\gamma}{(1+2\mu)} \begin{cases} 1+2(1-\lambda)\left[1-2\eta\frac{(1+2\mu)}{(1+\mu)^2}\right] \ ; & \eta \leq \Theta_1 \\ 1 \ ; & \Theta_1 < \eta < \Theta_2 \\ -\left(1+2(1-\lambda)\left[1-2\eta\frac{(1+2\mu)}{(1+\mu)^2}\right]\right) \ ; & \eta \geq \Theta_2. \end{cases}$$

where

$$\Theta_1 = \frac{(1+\mu)^2}{2(1+2\mu)}$$

and

$$\Theta_2 = \frac{(1+\mu)^2(2-\lambda)}{2(1+2\mu)(1-\lambda)}.$$

Proof. From (18) and (20), we get

$$\begin{aligned} a_3-\eta a_2^2 &= \frac{1}{2(1+2\mu)}\left[p_1^2\omega_1^2+p_1\omega_2+p_2\omega_1^2\right]-\eta\frac{p_1^2\omega_1^2}{(1+\mu)^2} \\ &= \frac{p}{2(1+2\mu)}\left[p\omega_1^2+\omega_2+\omega_1^2-2p\eta\frac{(1+2\mu)\omega_1^2}{(1+\mu)^2}\right] \quad (p_1=p_2=p) \\ &= \frac{p}{2(1+2\mu)}\left[\omega_2+\left(1+p-2p\eta\frac{(1+2\mu)}{(1+\mu)^2}\right)\omega_1^2\right]. \end{aligned}$$

It follows that

$$|a_3-\eta a_2^2|\leq\frac{(1-\lambda)\cos\gamma}{(1+2\mu)}\left[|\omega_2|+\left|1+p-2p\eta\frac{(1+2\mu)}{(1+\mu)^2}\right||\omega_1|^2\right].$$

By applying Lemma 4.1, we get

$$\begin{aligned} |a_3-\eta a_2^2| &\leq \frac{(1-\lambda)\cos\gamma}{(1+2\mu)}\left[1+\left\{\left|1+p-2p\eta\frac{(1+2\mu)}{(1+\mu)^2}\right|-1\right\}|\omega_1|^2\right] \\ &\leq \frac{(1-\lambda)\cos\gamma}{(1+2\mu)}\left[1+\left\{\left|1+p\left(1-2\eta\frac{(1+2\mu)}{(1+\mu)^2}\right)\right|-1\right\}|\omega_1|^2\right] \\ &\leq \frac{(1-\lambda)\cos\gamma}{(1+2\mu)}\left[1+\left\{\left|1+2e^{-i\gamma}(1-\lambda)\cos\gamma\left(1-2\eta\frac{(1+2\mu)}{(1+\mu)^2}\right)\right|-1\right\}|\omega_1|^2\right] \\ &\leq \frac{(1-\lambda)\cos\gamma}{(1+2\mu)}\left[1+\left\{\left|1+Me^{-i\gamma}\cos\gamma\right|-1\right\}|\omega_1|^2\right] \\ &\leq \frac{(1-\lambda)\cos\gamma}{(1+2\mu)}\left[1+\left\{\sqrt{1+M(M+2)\cos^2\gamma}-1\right\}|\omega_1|^2\right], \end{aligned}$$

where

$$M=2(1-\lambda)\left(1-2\eta\frac{(1+2\mu)}{(1+\mu)^2}\right). \tag{22}$$

Now, let $x=\cos\gamma\in[0,1]$ and $y=|\omega_1|\in[0,1]$, then we get

$$|a_3-\eta a_2^2|\leq\frac{(1-\lambda)\cos\gamma}{(1+2\mu)}H(x,y), \tag{23}$$

where

$$H(x,y)=1+\left[\sqrt{1+M(M+2)x^2}-1\right]y^2.$$

Now, we need to maximize $H(x,y)$ in the closed square $S=[0,1]\times[0,1]$. The function $H(x,y)$ does not have a local maximum at any intrior point of the open rectangle $(0,1)\times(0,1)$. Thus, we investigate the maximum of H on the boundary of the square S.

For $x=0$ and $0\leq y\leq 1$ (similarly $y=0$ and $0\leq x\leq 1$), we obtain

$$H(0,y)=1. \tag{24}$$

For $x=1$ and $0\leq y\leq 1$ (similarly $y=1$ and $0\leq x\leq 1$), we obtain

$$H(1,y)=1+(|1+M|-1)y^2. \tag{25}$$

Hence

$$\max H(1,y)=\begin{cases} 1\ ;\ |1+M|\leq 1 \\ \\ |1+M|\ ;\ |1+M|>1. \end{cases}$$

From (24) and (25), we have

$$\max\left\{H(x,y):\ (x,y)\in[0,1]\times[0,1]\right\}=\max\left\{1,|1+M|\right\}. \tag{26}$$

Therefore, from (23) and (26) we obtain

$$|a_3 - \eta a_2^2| \leq \frac{(1-\lambda)\cos\gamma}{(1+2\mu)} \max\{1, |1+M|\}.$$

Hence

$$|a_3 - \eta a_2^2| \leq \frac{(1-\lambda)\cos\gamma}{(1+2\mu)} \begin{cases} 1+M\ ; & M \geq 0 \\ 1\ ; & -2 < M < 0 \\ -(1+M)\ ; & M \leq -2. \end{cases}$$

This completes the proof. □

5. Conclusions

In this investigation, we defined new subclass of analytic functions $\mathcal{S}^{\gamma}(\lambda, \mu)$. For this subclass we investigated some useful results such as sufficient conditions that a function belongs to the subclass $\mathcal{S}^{\gamma}(\lambda, \mu)$, coefficient estimates and Fekete-Szegö problem. There are some problems open for researchers such as distortion theorems, closure theorems, convolution properties and radii problems. Moreover, these results can be extended to multivalent functions and meromorphic functions.

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