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SOME RESULTS ON FERMATEAN FUZZY LINEAR SPACE

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Abstract. The Fermatean fuzzy set expands upon the Pythagorean fuzzy set, asserting that the combined value of the membership degree cube and non-membership degree cube falls within the unit interval $[0, 1]$. This paper introduce the notion of Fermatean fuzzy linear space **(FFLS)** as an extended interpretation encompassing both intuitionistic and Pythagorean fuzzy linear space. We provide evidence that the intersection of two Fermatean fuzzy linear spaces remains a Fermatean fuzzy linear space. But the union of two Fermatean fuzzy linear space need not necessarily be a Fermatean fuzzy linear space. To justify this statement we have a provided a counter example for it. Further, we define the concept of a Fermatean fuzzy level linear space and explore the cartesian product of a Fermatean fuzzy linear space. This structure investigates the image and inverse image of a Fermatean fuzzy linear space along with their associated properties.

Keywords: Fuzzy set, Intuitionistic fuzzy set, Pythagorean fuzzy set, Fermatean fuzzy set, Fermatean fuzzy linear space.

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1. Introduction

In 1965, L. A. Zadeh [28] introduced fuzzy sets as a revolutionary mathematical concept that aimed to represent uncertainty and vagueness within data. This introduction marked a significant milestone in the field of mathematics, fundamentally altering how uncertainty and imprecision were handled and modeled. Traditional sets in mathematics have clear boundaries (an element either belongs to the set or does not). However, real-world phenomena often exhibit degrees of membership that are not strictly binary but rather gradual or fuzzy. Fuzzy sets extend the concept of classical sets by allowing elements to have degrees of membership, ranging from 0 to 1, rather than strictly belonging or not belonging to the set. This flexibility enables the modeling of vague or imprecise

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information more effectively, exemplifying the intrinsic unpredictably that encompasses numerous real-world scenarios.

Presently there are many extensions of fuzzy sets. Among these, the generalization put forth by Atanassov [1], namely intuitionistic fuzzy set, is characterized by a membership function and a non-membership function for each element in the Universe. Intuitionistic fuzzy sets are a generalization of classical fuzzy sets. In addition to membership degrees, they introduce a concept called non-membership degree, which represents the degree to which an element does not belong to a set. This concept can be useful in modeling more complex and uncertain situations.

Smarandache [20] introduced the concept of a Neutrosophic set **(NS)** as a novel and thought-provoking idea in the field of philosophy. According to Smarandache's definition, a Neutrosophic set A, which exists within a universal set X, is characterized by three distinct membership functions namely truth-membership, indeterminacy-membership, and falsity-membership.

The truth-membership function of a Neutrosophic set A represents the degree to which an element of X belongs to A in a true sense. It quantifies the level of certainty or confidence in the membership of an element in A. Which provides a measure of the truthfulness of the statement that an element belongs to A.

On the other hand, the indeterminacy-membership function of A captures the degree of uncertainty or ambiguity in the membership of an element in A. It reflects the lack of precise information or the presence of contradictory evidence regarding the membership status of an element. Which allows for the representation of elements that are neither clearly in A nor clearly outside of it.

Lastly, the falsity-membership function of A denotes the degree to which an element of X does not belong to A. It quantifies the level of falseness or contradiction in the statement that an element is a member of A. It provides a measure of the extent to which an element is not part of A.

By incorporating these three membership functions, Smarandache's Neutrosophic set theory allows for a more nuanced and comprehensive representation of uncertainty, ambiguity and contradiction within a set. It acknowledges that in many real-world scenarios, the boundaries between membership and non-membership are not always clear-cut and there can exist varying degrees of truth, indeterminacy and falsity.

The introduction of Neutrosophic sets has opened up new avenues of research and application in various fields, including mathematics, computer science, artificial intelligence, decision-making, and information fusion. It has provided a powerful tool for dealing with complex and uncertain information, enabling a more flexible and realistic modeling of real-world phenomena. Recently, Sivaramakrishnan et al [14] have recently proposed the concept of Neutrosophic interval-valued anti fuzzy linear space **(NIVAFLS)**, incorporating Neutrosophic set principles and an interval-valued anti fuzzy linear space.

Pythagorean fuzzy sets, introduced by Yager [26] in 2013, are a variation of fuzzy sets that provide a structured approach to deal with uncertainty and ambiguity. While traditional fuzzy sets assign a single membership degree to each element, Pythagorean fuzzy sets extend this by assigning two values i.e., the membership degree and the non-membership degree. In Pythagorean fuzzy sets, the membership and non-membership degrees are represented as intervals rather than single values. These intervals are centered around a certain point, indicating the degree of truth and falsity of an element's membership, respectively. The width of these intervals reflects the uncertainty or fuzziness associated with the element's classification. This approach allows Pythagorean fuzzy sets to capture not only the degree of membership but also the degree of non-membership, providing a more comprehensive representation of uncertainty. Recently, Senapati and Yager [11] first introduced the Fermatean fuzzy set, which is a Pythagorean fuzzy set extension. Fermatean Fuzzy Sets **(FFSs)** are novel extension of conventional fuzzy sets, intuitionistic fuzzy sets and Pythagorean fuzzy sets. These sets have gained significant attention in various domains due to their ability to effectively handle complex and uncertain data. Unlike other fuzzy structures, **FFSs** introduce a unique approach to representing uncertainty by approximating a unit interval through the summation of cubes of membership grades.

Lubczonok and Muralli [7], introduced an interesting theory of flags and fuzzy subspaces of Linear space. This theory extends fuzzy sets into linear spaces. It deals with the concept of "flags" and explores the relationships between fuzzy subspaces of linear spaces. Flags are used to describe certain structures within linear spaces in a fuzzy way. Vijayabalji and Sivaramakrishnan [25] introduced the concept of cartesian Product and Homomorphism of Interval-Valued Fuzzy Linear Space. This concept involves the Cartesian product of interval-valued fuzzy linear spaces. It explores how linear spaces with interval-valued fuzzy elements can be combined, and it introduces the notion of homomorphism in this context, which is a mapping preserving certain properties between such spaces. Sivaramakrishnan and Vijayabaljai [15] introduced some interesting operations and theorems on interval-valued anti fuzzy linear space **(IVAFLS)**. Interval-valued anti fuzzy linear spaces are a mathematical framework that combines linear spaces with interval-valued anti fuzzy sets. It involves operations and theorems related to these structures and is used to handle uncertainty in linear algebraic contexts.

In this paper, we introduce the notion of Fermatean fuzzy linear space as an expansion of intuitionistic and Pythagorean fuzzy linear space. We demonstrate that the intersection of two Fermatean fuzzy linear spaces constitutes a Fermatean fuzzy linear space. However, the combination of two Fermatean fuzzy linear spaces may not yield a Fermatean fuzzy linear space, as evidenced by a counter example. Furthermore, we present the concept of Fermatean fuzzy level linear space, the cartesian product of Fermatean fuzzy linear space and explore the image and inverse image of a Fermatean fuzzy linear space.

2. PRELIMINARIES

This section recalls some basic definitions which will be needed for this paper.

Definition 2.1 (7). *Let* $\mathfrak{V}$ *be a crisp linear space over a field* $\mathbb{F}$. *Define a mapping* $\Psi : \mathfrak{V} \to [0,1]$ *is called as a* ***FLS*** *if* $\Psi(av_1 * bv_2) \geq min\{\Psi(v_1), \Psi(v_2)\}$, *for all* $v_1, v_2 \in \mathfrak{V}$, $a, b \in \mathbb{F}$ *and* $*$ *is any binary operation on* $\mathbb{F}$.

Definition 2.2 (11). *Let* X *be a universal set. A* ***FFS*** Ξ *on* X *is an object of the form:* $\Xi = \{(f, \boldsymbol{p}(f), \boldsymbol{q}(f)) | f \in X\}$, *where* $\boldsymbol{p}(f) \in [0,1]$, $\boldsymbol{q}(f) \in [0,1]$ *are the degree* *of membership and non-membership of* $f \in X$ *respectively, which satisfy the condition* $0 \leq (\boldsymbol{p}^3)(f) + (\boldsymbol{q}^3)(f) \leq 1$, *for all* $f \in X$. *For convenience we denote* Ξ *as* $\Xi = (\boldsymbol{p}, \boldsymbol{q})$.

Definition 2.3 (11). *Let* $\Xi_1 = (\boldsymbol{p}_1, \boldsymbol{q}_1)$ *and* $\Xi_2 = (\boldsymbol{p}_2, \boldsymbol{q}_2)$ *are two* ***FFSs*** *of* X. *Then*

(*i*) $\Xi_1 \cap \Xi_2 = (min\{\boldsymbol{p}_1(f), \boldsymbol{p}_2(f)\}, max\{\boldsymbol{q}_1(f), \boldsymbol{q}_2(f)\})$

(*ii*) $\Xi_1 \cup \Xi_2 = (max\{\boldsymbol{p}_1(f), \boldsymbol{p}_2(f)\}, min\{\boldsymbol{q}_1(f), \boldsymbol{q}_2(f)\})$, $\forall f \in X$.

Remark 2.1. *In simpler terms, Intuitionistic Fuzzy Sets* ***(IFS)*** *and Pythagorean Fuzzy Sets* ***(PFS)*** *are two different types of Fuzzy Sets that allow Fuzzy Logic Systems to handle uncertainties more effectively.*

To illustrate this concept, consider an example where we have values that do not meet the conditions of ***IFS*** *or* ***PFS****. When we add* 0.9 *and* 0.6 *together, i.e.,* 1.5 *we get a result greater than* 1, *which does not align with the principles of* ***IFS****.*

Similarly, when we square these values and add them together,

i.e., $(0.9)^2 + (0.6)^2 = 0.81 + 0.36 = 1.17 \geq 1$

we get a result greater than 1, *which violates the constraints of* ***PFS****.*

However, when we cube these values and add them together,

i.e., $(0.9)^3 + (0.6)^3 = 0.729 + 0.216 = 0.945 \leq 1$ *we get a result less than* 1, *which is suitable for applying Fermatean Fuzzy Sets* ***(FFSs)*** *for control purposes.*

This example demonstrates how different types of Fuzzy Sets can be used to handle uncertainties in Fuzzy Logic Systems.

3. Fermatean Fuzzy Linear Space

In this section, we construct a structure of **FFLS**.

Definition 3.1. *Let* $\mathfrak{V}$ *be a crisp linear space over a field* $\mathbb{F}$. *A* ***FFS*** $\Xi = (\boldsymbol{p}, \boldsymbol{q})$ *is said to be a* ***FFLS*** *if it meets the following conditions:*

(*i*) $\boldsymbol{p}(av_1 * bv_2) \geq min\{\boldsymbol{p}(v_1), \boldsymbol{p}(v_2)\}$

(*ii*) $\boldsymbol{q}(av_1 * bv_2) \leq max\{\boldsymbol{q}(v_1), \boldsymbol{q}(v_2)\}$, *for all* $v_1, v_2 \in \mathfrak{V}$ *and* $a, b \in \mathbb{F}$.

Example 3.1. *Over a field* $\boldsymbol{R}$, *let* $\mathfrak{V} = \boldsymbol{R}^2$ *be a linear space over a field* $\boldsymbol{R}$ *and let* $\Xi = (\boldsymbol{p}, \boldsymbol{q})$ *be a* ***FFLS*** *in* $\mathfrak{V}$. *For each* $v = (v_1, v_2) \in \boldsymbol{R}^2$, *mappings* $\boldsymbol{p} : \mathfrak{V} \to [0, 1]$ *and* $\boldsymbol{q} : \mathfrak{V} \to [0, 1]$ *are defined by*

$$\boldsymbol{p}(v) = \begin{cases} 0.42, & if\ v_1 = 0\ or\ v_2 = 0, \\ 0.81, & otherwise. \end{cases}$$

and

$$\boldsymbol{q}(v) = \begin{cases} \text{0.71}, & if \;\; v_1 = 0 \text{ or } v_2 = 0, \\ \text{0.56}, & otherwise. \end{cases}$$

Clearly, $\Xi = (\boldsymbol{p}, \boldsymbol{q})$ *is a* ***FFLS*** *of* $\mathfrak{V}$.

Example 3.2. *Let* $\mathfrak{V} = \{d_1, d_2, d_3, d_4\}$ *be the Klien 4-group defined by the binary operation* $*$ *as follows:*

$*$	d_1	d_2	d_3	d_4
d_1	d_1	d_2	d_3	d_4
d_2	d_2	d_1	d_4	d_3
d_3	d_3	d_4	d_1	d_2
d_4	d_4	d_3	d_2	d_1

Let F *be the field* $GF(2)$. *Let* $(0)w = e$, $(1)w = w$ *for all* $w \in \mathfrak{V}$.

So $\mathfrak{V}$ *is a linear space over* $\mathbb{F}$.

Define the mappings $\boldsymbol{p} : \mathfrak{V} \to [0,1]$ *and* $\boldsymbol{q} : \mathfrak{V} \to [0,1]$ *are defined by*

$\boldsymbol{p}(d_1) = 0.81,\ \boldsymbol{p}(d_2) = 0.23 = \boldsymbol{p}(d_3),\ \boldsymbol{p}(d_4) = 0.49$

and

$$\boldsymbol{q}(v) = \begin{cases} \text{0.3}, & if \;\; v = d_1, \\ \text{0.73}, & otherwise. \end{cases}$$

Note that $\Xi = (\boldsymbol{p}, \boldsymbol{q})$ *is a* ***FFLS*** *of* $\mathfrak{V}$.

Theorem 3.1. *Let* $\Xi_1 = (\boldsymbol{p}_1, \boldsymbol{q}_1)$ *and* $\Xi_2 = (\boldsymbol{p}_2, \boldsymbol{q}_2)$ *be two* ***FFLSs***. *Then their intersection,* $(\Xi_1 \cap \Xi_2)$=$(\boldsymbol{p}_1 \cap \boldsymbol{p}_2, \boldsymbol{q}_1 \cup \boldsymbol{q}_2)$ *is a* ***FFLS***.

Proof. Define $\mathbf{p}_1 \cap \mathbf{p}_2$ as follows

$(\mathbf{p}_1 \cap \mathbf{p}_2)(a_1v_1 * a_2v_2) = \min\{\mathbf{p}_1(a_1v_1 * a_2v_2), \mathbf{p}_2(a_1v_1 * a_2v_2)\}$

$$\begin{aligned} \text{Now } (\mathbf{p}_1 \cap \mathbf{p}_2)(a_1v_1 * a_2v_2) &= \min\{\mathbf{p}_1(a_1v_1 * a_2v_2), \mathbf{p}_2(a_1v_1 * a_2v_2)\} \\ &\geq \min\{\min[\mathbf{p}_1(v_1), \mathbf{p}_1(v_2)], \min[\mathbf{p}_2(v_1), \mathbf{p}_2(v_2)]\} \\ &= \min\{\min[\mathbf{p}_1(v_1), \mathbf{p}_2(v_1)], \min[\mathbf{p}_1(v_2), \mathbf{p}_2(v_2)]\} \\ &= \min\{(\mathbf{p}_1 \cap \mathbf{p}_2)(v_1), (\mathbf{p}_1 \cap \mathbf{p}_2)(v_2)\} \end{aligned}$$

$\Rightarrow (\mathbf{p}_1 \cap \mathbf{p}_2)(a_1v_1 * a_2v_2) \geq \min\{(\mathbf{p}_1 \cap \mathbf{p}_2)(v_1), (\mathbf{p}_1 \cap \mathbf{p}_2)(v_2)\}$

Also define $\mathbf{q}_1 \cup \mathbf{q}_2$ by

$(\mathbf{q}_1 \cup \mathbf{q}_2)(a_1v_1 * a_2v_2) = \max\{\mathbf{q}_1(a_1v_1 * a_2v_2), \mathbf{q}_2(a_1v_1 * a_2v_2)\}$.

$$\begin{aligned}
\text{So, } (\mathbf{q}_1 \cup \mathbf{q}_2)(a_1v_1 * a_2v_2) &= \max\{\mathbf{q}_1(a_1v_1 * a_2v_2), \mathbf{q}_2(a_1v_1 * a_2v_2)\} \\
&\leq \max\{\max[\mathbf{q}_1(v_1), \mathbf{q}_1(v_2)], \max[\mathbf{q}_2(v_1), \mathbf{q}_2(v_2)]\} \\
&= \max\{\max[\mathbf{q}_1(v_1), \mathbf{q}_2(v_1)], \max[\mathbf{q}_1(v_2), \mathbf{q}_2(v_2)]\} \\
&= \max\{(\mathbf{q}_1 \cup \mathbf{q}_2)(v_1), (\mathbf{q}_1 \cup \mathbf{q}_2)(v_2)\}
\end{aligned}$$

$\Rightarrow (\mathbf{q}_1 \cup \mathbf{q}_2)(a_1v_1 * a_2v_2) \leq \max\{(\mathbf{q}_1 \cup \mathbf{q}_2)(v_1), (\mathbf{q}_1 \cup \mathbf{q}_2)(v_2)\}$

Thus $(\Xi_1 \cap \Xi_2)$=$(\mathbf{p}_1 \cap \mathbf{p}_2, \mathbf{q}_1 \cup \mathbf{q}_2)$ is a **FFLS**. □

Example for intersection of two **FFLSs**.

Example 3.3. *Let $\mathfrak{V} = \{d_1, d_2, d_3, d_4\}$ be the Klien 4-group as in Example 3.2.*

Let $\mathbb{F}$ be the field GF(2). Let $(0)w = e$, $(1)w = w$ for all $w \in \mathfrak{V}$. Then $\mathfrak{V}$ is a linear space over $\mathbb{F}$.

Define $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ as follows:

$\boldsymbol{p}_1(d_1) = 0.61$, $\boldsymbol{p}_1(d_2) = 0.3 = \boldsymbol{p}_1(d_3)$, $\boldsymbol{p}_1(d_4) = 0.5$ *and*

$\boldsymbol{p}_2(d_1) = 0.5$, $\boldsymbol{p}_2(d_2) = 0.4$, $\boldsymbol{p}_2(d_3) = 0.2$=$\boldsymbol{p}_2(d_4)$.

Define $\boldsymbol{p}_1 \cap \boldsymbol{p}_2$ by $(\boldsymbol{p}_1 \cap \boldsymbol{p}_2)(v) = min\{\boldsymbol{p}_1(v), \boldsymbol{p}_2(v)\}$ for all $v \in \mathfrak{V}$.

So, $(\boldsymbol{p}_1 \cap \boldsymbol{p}_2)(d_1) = 0.5$, $(\boldsymbol{p}_1 \cap \boldsymbol{p}_2)(d_2) = 0.3$,

$(\boldsymbol{p}_1 \cap \boldsymbol{p}_2)(d_3) = 0.2$, $(\boldsymbol{p}_1 \cap \boldsymbol{p}_2)(d_4) = 0.2$.

When $a_1 = a_2 = 1$, then Definition 3.1 in (i) becomes

$(\boldsymbol{p}_1 \cap \boldsymbol{p}_2)(d_2 * d_4) \geq min\{(\boldsymbol{p}_1 \cap \boldsymbol{p}_2)(d_2), (\boldsymbol{p}_1 \cap \boldsymbol{p}_2)(d_4)\}$

$\Rightarrow (\boldsymbol{p}_1 \cap \boldsymbol{p}_2)(d_3) \geq min\{0.3, 0.2\} = 0.2$

But $(\boldsymbol{p}_1 \cap \boldsymbol{p}_2)(d_3) = 0.2 = 0.2$

Which satisfied the property.

Now define $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ in $\mathfrak{V}$ by

$\boldsymbol{q}_1(d_1) = 0.2,\ \boldsymbol{q}_1(d_2) = \boldsymbol{q}_1(d_3) = 0.7,\ \boldsymbol{q}_1(d_4) = 0.5$ *and*

$\boldsymbol{q}_2(d_1) = 0.3,\ \boldsymbol{q}_2(d_2) = 0.4,\ \boldsymbol{q}_2(d_3) = \boldsymbol{q}_2(d_4) = 0.6.$

Define $(\boldsymbol{q}_1 \cup \boldsymbol{q}_2)(v) = max\{\boldsymbol{q}_1(v), \boldsymbol{q}_2(v)\}$

Then $(\boldsymbol{q}_1 \cup \boldsymbol{q}_2)(d_1) = 0.3,\ (\boldsymbol{q}_1 \cup \boldsymbol{q}_2)(d_2) = 0.7,$

$(\boldsymbol{q}_1 \cup \boldsymbol{q}_2)(d_3) = 0.7$ *and* $(\boldsymbol{q}_1 \cup \boldsymbol{q}_2)(d_4) = 0.6.$

When $a_1 = a_2 = 1$, *then Definition 3.1 in* (ii) *becomes*

$(\boldsymbol{q}_1 \cup \boldsymbol{q}_2)(d_2 * d_4) \leq max\{(\boldsymbol{q}_1 \cup \boldsymbol{q}_2)(d_2), (\boldsymbol{q}_1 \cup \boldsymbol{q}_2)(d_4)\}$

$\Rightarrow (\boldsymbol{q}_1 \cup \boldsymbol{q}_2)(d_3) \leq max\{0.7, 0.6\} = 0.7$

But $(\boldsymbol{q}_1 \cup \boldsymbol{q}_2)(d_3) = 0.7 = 0.7$

So, the intersection of two ***FFLSs*** *is again a* ***FFLS****.*

Remark 3.1. *Let* $\Xi_1 = (\boldsymbol{p}_1, \boldsymbol{q}_1)$ *and* $\Xi_2 = (\boldsymbol{p}_2, \boldsymbol{q}_2)$ *be two* ***FFLSs****. Then their union,* $(\Xi_1 \cup \Xi_2)$=$(\boldsymbol{p}_1 \cup \boldsymbol{p}_2, \boldsymbol{q}_1 \cap \boldsymbol{q}_2)$ *need not be a* ***FFLS****.*

Proof. We shall prove the above statement by means of an example.

Let $\mathfrak{V} = \{d_1, d_2, d_3, d_4\}$ be the Klien 4-group as in Example 3.2.

Let $\mathbb{F}$ be the field GF(2). Let $(0)w = e$, $(1)w = w$ for all $w \in \mathfrak{V}$. Then $\mathfrak{V}$ is a linear space over $\mathbb{F}$.

Define $\mathbf{p}_1$ and $\mathbf{p}_2$ as follows:

$\mathbf{p}_1(d_1) = 0.61$, $\mathbf{p}_1(d_2) = 0.3 = \mathbf{p}_1(d_3)$, $\mathbf{p}_1(d_4) = 0.5$ and

$\mathbf{p}_2(d_1) = 0.5$, $\mathbf{p}_2(d_2) = 0.4$, $\mathbf{p}_2(d_3) = 0.2$=$\mathbf{p}_2(d_4)$.

Define $\mathbf{p}_1 \cup \mathbf{p}_2$ by $(\mathbf{p}_1 \cup \mathbf{p}_2)(v) = \max\{\mathbf{p}_1(v), \mathbf{p}_2(v)\}$ for all $v \in \mathfrak{V}$.

So, $(\mathbf{p}_1 \cup \mathbf{p}_2)(d_1) = 0.61$, $(\mathbf{p}_1 \cup \mathbf{p}_2)(d_2) = 0.4$,

$(\mathbf{p}_1 \cup \mathbf{p}_2)(d_3) = 0.3$, $(\mathbf{p}_1 \cup \mathbf{p}_2)(d_4) = 0.5$.

When $a_1 = a_2 = 1$, then Definition 3.1 in (i) becomes

$(\mathbf{p}_1 \cup \mathbf{p}_2)(d_2 * d_4) \geq \min\{(\mathbf{p}_1 \cup \mathbf{p}_2)(d_2), (\mathbf{p}_1 \cup \mathbf{p}_2)(d_4)\}$

$\Rightarrow (\mathbf{p}_1 \cup \mathbf{p}_2)(d_3) \geq \min\{0.4, 0.5\}$

But $(\mathbf{p}_1 \cup \mathbf{p}_2)(d_3) = 0.3 \geq 0.4$

This is absurd.

Now define $\mathbf{q}_1$ and $\mathbf{q}_2$ in $\mathfrak{V}$ by

$\mathbf{q}_1(d_1) = 0.2$, $\mathbf{q}_1(d_2) = \mathbf{q}_1(d_3) = 0.7$, $\mathbf{q}_1(d_4) = 0.5$ and

$\mathbf{q}_2(d_1) = 0.3$, $\mathbf{q}_2(d_2) = 0.4$, $\mathbf{q}_2(d_3) = \mathbf{q}_2(d_4) = 0.6$.

Define $(\mathbf{q}_1 \cap \mathbf{q}_2)(v) = \min\{\mathbf{q}_1(v), \mathbf{q}_2(v)\}$

Then $(\mathbf{q}_1 \cap \mathbf{q}_2)(d_1) = 0.2$, $(\mathbf{q}_1 \cap \mathbf{q}_2)(d_2) = 0.4$,

$(\mathbf{q}_1 \cap \mathbf{q}_2)(d_3) = 0.6$ and $(\mathbf{q}_1 \cap \mathbf{q}_2)(d_4) = 0.5$.

When $a_1 = a_2 = 1$, then Definition 3.1 in (ii) becomes

$(\mathbf{q}_1 \cap \mathbf{q}_2)(d_2 * d_4) \leq \max\{(\mathbf{q}_1 \cap \mathbf{q}_2)(d_2), (\mathbf{q}_1 \cap \mathbf{q}_2)(d_4)\}$

$\Rightarrow (\mathbf{q}_1 \cap \mathbf{q}_2)(d_3) \leq \max\{0.4, 0.5\} = 0.5$

But $(\mathbf{q}_1 \cap \mathbf{q}_2)(d_3) = 0.6 \leq 0.5$

This is also absurd. So, the union of two **FFLSs** need not be a **FFLS**. $\square$

Definition 3.2. *Let $\mathfrak{V}$ be a crisp linear space. Let $\Xi = (\boldsymbol{p}, \boldsymbol{q})$ be a **FFS** of $\mathfrak{V}$. For $j, k \in [0,1]$, the set $\Gamma(\Xi; \tilde{j}, \tilde{k}) = \{f \in \mathfrak{V} | \boldsymbol{p}(f) \geq \tilde{j}$ and $\boldsymbol{q}(f) \leq \tilde{k}\}$. Then $\Gamma(\Xi; \tilde{j}, \tilde{k})$ is called a Fermatean fuzzy level set of $\Xi = (\boldsymbol{p}, \boldsymbol{q})$.*

Theorem 3.2. *Let $\mathfrak{V}$ be a linear space over a field $\mathbb{F}$. A **FFLS** $\Xi = (\boldsymbol{p}, \boldsymbol{q})$ of $\mathfrak{V}$ if and only if for all $\tilde{j}, \tilde{k} \in [0,1]$, the set $\Gamma(\Xi; \tilde{j}, \tilde{k})$ is either empty or a crisp linear space of $\mathfrak{V}$ over a field $\mathbb{F}$.*

Proof. Let $\Xi = (\mathbf{p}, \mathbf{q})$ is a **FFLS** of $\mathfrak{V}$ over a field $\mathbb{F}$, let $\tilde{j}, \tilde{k} \in [0,1]$, be such that $\Gamma(\Xi; \tilde{j}, \tilde{k}) \neq \phi$.

Let $v_1, v_2 \in \mathfrak{V}$ be such that $v_1, v_2 \in \Gamma(\Xi; \tilde{j}, \tilde{k})$, then $\mathbf{p}(v_1) \geq \tilde{j}$, $\mathbf{p}(v_2) \geq \tilde{j}$ and $\mathbf{q}(v_1) \leq \tilde{k}$, $\mathbf{q}(v_2) \leq \tilde{k}$.

Therefore,

$$\begin{aligned} \mathbf{p}(a_1 v_1 * a_2 v_2) &\geq \min\{\mathbf{p}(v_1), \mathbf{p}(v_2)\}. \\ &\geq \min\{\tilde{j}, \tilde{j}\} = \tilde{j}. \end{aligned}$$

Moreover,

$\mathbf{q}(a_1 v_1 * a_2 v_2) \leq \max\{\mathbf{q}(v_1), \mathbf{q}(v_2)\}$.

$$\leq \max\{\tilde{k}, \tilde{k}\} = \tilde{k}.$$

So that $a_1v_1 * a_2v2 \in \Gamma(\Xi; \tilde{j}, \tilde{k})$.

Therefore, $\Gamma(\Xi; \tilde{j}, \tilde{k})$ is a linear space over a field $\mathbb{F}$.

Conversely, suppose that $\Gamma(\Xi; \tilde{j}, \tilde{k})$ is a crisp linear space $\mathfrak{V}$ over a field $\mathbb{F}$ and let $v_1, v_2 \in \mathfrak{V}$ and $a_1, a_2 \in \mathbb{F}$ be such that

$$\mathbf{p}(a_1v_1 * a_2v_2) < \min\{\mathbf{p}(v_1), \mathbf{p}(v_2)\},$$

$$\mathbf{q}(a_1v_1 * a_2v_2) > \max\{\mathbf{q}(v_1), \mathbf{q}(v_2)\}.$$

Taking $\tilde{\theta}_1 = \frac{1}{2}\{\mathbf{p}(a_1v_1 * a_2v_2) + \min\{\mathbf{p}(x_1), \mathbf{p}(v_2)\}\}$ and

$$\tilde{\theta}_2 = \frac{1}{2}\{\mathbf{q}(a_1v_1 * a_2v_2) + \max\{\mathbf{q}(v_1), \mathbf{q}(v_2)\}\}.$$

Also

$$\mathbf{p}(a_1v_1 * a_2v_2) < \tilde{\theta}_1 < \min\{\mathbf{p}(v_1), \mathbf{p}(v_2)\},$$

$$\mathbf{q}(a_1v_1 * a_2v_2) > \tilde{\theta}_2 > \max\{\mathbf{q}(v_1), \mathbf{p}(v_2)\}.$$

It follows that $v_1, v_2 \in \Gamma(\Xi; \tilde{\theta}_1, \tilde{\theta}_2)$ and $a_1v_1 * a_2v_2 \notin \Gamma(\Xi; \tilde{\theta}_1, \tilde{\theta}_2)$.

It's a contradiction here and therefore $\Xi = (\mathbf{p}, \mathbf{q})$ is a **FFLS** of $\mathfrak{V}$ over a field $\mathbb{F}$. □

Definition 3.3. *Let Ξ_1, Ξ_2 be two **FFSs** of $\mathfrak{V}_1$ and $\mathfrak{V}_2$ respectively. Then the cartesian product of Ξ_1 and Ξ_2 denoted by $\Xi_1 \times \Xi_2$ is defined by*

$$\Xi_1 \times \Xi_2 = \big\{\big((v_1 \times v_2), \boldsymbol{p}_{\Xi_1\times\Xi_2}(v_1, v_2), \boldsymbol{q}_{\Xi_1\times\Xi_2}(v_1, v_2)\big) : v_1 \in \mathfrak{V}_1, v_2 \in \mathfrak{V}_2\big\}$$

where $\boldsymbol{p}_{\Xi_1\times\Xi_2}(v_1, v_2) = \min\{\boldsymbol{p}_{\Xi_1}(v_1), \boldsymbol{p}_{\Xi_2}(v_2)\}$ and $\boldsymbol{q}_{\Xi_1\times\Xi_2}(v_1, v_2) = \max\{\boldsymbol{q}_{\Xi_1}(v_1), \boldsymbol{q}_{\Xi_2}(v_2)\}$.

Theorem 3.3. *If Ξ_1 and Ξ_2 are **FFLSs** of $\mathfrak{V}$, then $\Xi_1 \times \Xi_2$ is a **FFLS** of $\mathfrak{V}_1 \times \mathfrak{V}_2$.*

Proof. Let $v = (v_1, v_2)$, $w = (w_1, w_2) \in \mathfrak{V}_1 \times \mathfrak{V}_2$. Then

$$\begin{aligned}
\mathbf{p}_{\Xi_1\times\Xi_2}(a_1v * a_2w) &= \mathbf{p}_{\Xi_1\times\Xi_2}(a_1(v_1, v_2) * a_2(w_1, w_2)) \\
&= \mathbf{p}_{\Xi_1\times\Xi_2}((a_1v_1 * a_2w_1), (a_1v_2 * a_2w_2)) \\
&= \min\{\mathbf{p}_{\Xi_1}(a_1v_1 * a_2w_1), \mathbf{p}_{\Xi_2}(a_1v_2 * a_2w_2)\} \\
&\leq \min\{\min[\mathbf{p}_{\Xi_1}(v_1), \mathbf{p}_{\Xi_1}(w_1)], \min[\mathbf{p}_{\Xi_2}(v_2), \mathbf{p}_{\Xi_2}(w_2)]\} \\
&= \min\{\min[\mathbf{p}_{\Xi_1}(v_1), \mathbf{p}_{\Xi_2}(v_2)], \min[\mathbf{p}_{\Xi_1}(w_1), \mathbf{p}_{\Xi_2}(w_2)]\} \\
&= \min\{\mathbf{p}_{\Xi_1\times\Xi_2}(v_1, v_2), \mathbf{p}_{\Xi_1\times\Xi_2}(w_1, w_2)\}
\end{aligned}$$

$$= \min\{\mathbf{p}_{\Xi_1\times\Xi_2}(v), \mathbf{p}_{\Xi_1\times\Xi_2}(w)\}$$

$$\begin{aligned}
\mathbf{q}_{\Xi_1\times\Xi_2}(a_1v * a_2w) &= \mathbf{q}_{\Xi_1\times\Xi_2}(a_1(v_1,v_2) * a_2(w_1,w_2))\\
&= \mathbf{q}_{\Xi_1\times\Xi_2}((a_1v_1 * a_2w_1),(a_1v_2 * a_2w_2))\\
&= \max\{\mathbf{q}_{\Xi_1}(a_1v_1 * a_2w_1), \mathbf{q}_{\Xi_2}(a_1v_2 * a_2w_2)\}\\
&\le \max\{\max[\mathbf{q}_{\Xi_1}(v_1), \mathbf{q}_{\Xi_1}(w_1)], \max[\mathbf{q}_{\Xi_2}(v_2), \mathbf{q}_{\Xi_2}(w_2)]\}\\
&= \max\{\max[\mathbf{q}_{\Xi_1}(v_1), \mathbf{q}_{\Xi_2}(v_2)], \max[\mathbf{q}_{\Xi_1}(w_1), \mathbf{q}_{\Xi_2}(w_2)]\}\\
&= \max\{\mathbf{q}_{\Xi_1\times\Xi_2}(v_1,v_2), \mathbf{q}_{\Xi_1\times\Xi_2}(w_1,w_2)\}\\
&= \max\{\mathbf{q}_{\Xi_1\times\Xi_2}(v), \mathbf{q}_{\Xi_1\times\Xi_2}(w)\}
\end{aligned}$$

So, $(\Xi_1 \times \Xi_2)$ is a **FFLS** of $\mathfrak{V}_1 \times \mathfrak{V}_2$.

□

The image and inverse image of a **FFLS** are defined, and various results pertaining to them are studied.

Definition 3.4. *Let $g : \mathfrak{V}_1 \to \mathfrak{V}_2$ be a mapping of linear spaces of $\mathfrak{V}$ over $\mathbb{F}$. If $\Xi = (\boldsymbol{p}, \boldsymbol{q})$ is a **FFLS** of $\mathfrak{V}_2$ over $\mathbb{F}$, then the inverse image of $\Xi = (\boldsymbol{p}, \boldsymbol{q})$ under g, denoted by $g^{-1}(\Xi) = (\boldsymbol{g}^{-1}(\boldsymbol{p}_\Xi), g^{-1}(\boldsymbol{q}_\Xi))$, is a **FFLS** of $\mathfrak{V}_1$, defined by $g^{-1}(\Xi)(v) = \Xi(g(v)) = (\boldsymbol{p}_\Xi(g(v)), \boldsymbol{q}_\Xi(g(v)))$ for all $v \in \mathfrak{V}_1$.*

Theorem 3.4. *Let $g : \mathfrak{V}_1 \to \mathfrak{V}_2$ be homomorphism of linear spaces of $\mathfrak{V}$ over $\mathbb{F}$. If $\Xi = (\boldsymbol{p}, \boldsymbol{q})$ is a **FFLS** of $\mathfrak{V}_2$, then $g^{-1}(\Xi)(v) = \Xi(g(v)) = (\boldsymbol{p}_\Xi(g(v)), \boldsymbol{q}_\Xi(g(v)))$ for all $v \in \mathfrak{V}_1$.*

Proof. Assume that $\Xi = (\mathbf{p}, \mathbf{q})$ is a **FFLS** of $\mathfrak{V}_2$ and $v, w \in \mathfrak{V}_1$ and $a, b \in \mathbb{F}$.
Then we have

$$\begin{aligned}
(i)\ g^{-1}(\mathbf{p}_\Xi)(av * bw) &= \mathbf{p}_\Xi(g(av * bw))\\
&= \mathbf{p}_\Xi(g(v)g(w)) \quad \text{(since } g \text{ is a homomorphism)}\\
&\ge \min\{\mathbf{p}_\Xi(g(v)), \mathbf{p}_\Xi(g(w))\}\\
&= \min\{g^{-1}(\mathbf{p}_\Xi(v)),\ g^{-1}(\mathbf{p}_\Xi(w))\}\\
\Rightarrow g^{-1}(\mathbf{p}_\Xi)(av * bw) &\ge \min\{g^{-1}(\mathbf{p}_\Xi(v)),\ g^{-1}(\mathbf{p}_\Xi(w))\}
\end{aligned}$$

Therefore $g^{-1}(\mathbf{p}_\Xi)$ is a **FFLS** of $\mathfrak{V}_1$.

$$\begin{aligned}
(ii)\ g^{-1}(\mathbf{q}_\Xi)(av * bw) &= \mathbf{q}_\Xi(g(av * bw))\\
&= \mathbf{q}_\Xi(g(v)g(w)) \quad \text{(since } g \text{ is a homomorphism)}\\
&\le \max\{\mathbf{q}_\Xi(g(v)), \mathbf{q}_\Xi(g(w))\}\\
&= \max\{g^{-1}(\mathbf{q}_\Xi(v)),\ g^{-1}(\mathbf{q}_\Xi(w))\}\\
\Rightarrow g^{-1}(\mathbf{q}_\Xi)(av * bw) &\le \max\{g^{-1}(\mathbf{q}_\Xi(v)),\ g^{-1}(\mathbf{q}_\Xi(w))\}
\end{aligned}$$

Therefore $g^{-1}(\mathbf{q}_\Xi)$ is a **FFLS** of $\mathfrak{V}_1$. □

Theorem 3.5. *Let $\Xi = (\boldsymbol{p}, \boldsymbol{q})$ be a **FFLS** of $\mathfrak{V}$ and let $g : \mathfrak{V} \to \mathfrak{V}$ be an onto homomorphism. Then the mapping $\Xi^{\ g} : \mathfrak{V} \to [0,1]$, defined by $\Xi^{\ g}(v) = \Xi(g(v))$ for all $v \in \mathfrak{V}$, is a **FFLS** of $\mathfrak{V}$.*

Proof. For any $u, v \in \mathfrak{V}$ and $a, b \in \mathbf{F}$.

$$\begin{aligned}(i)\ \mathbf{p}^{\ g}(av * bw) &= \mathbf{p}(\ g(av * bw))\\ &= \mathbf{p}(\ g(v)\ g(w)) \quad \text{(since } g \text{ is a homomorphism)}\\ &\geq \min\{\mathbf{p}(\ g(v)), \mathbf{p}(\ g(w))\}\\ &= \min\{\mathbf{p}_{\Xi}^{g}(v), \mathbf{p}_{\Xi}^{g}(w)\}\\ \Rightarrow \mathbf{p}^{\ g}(av * bw) &\geq \min\{\mathbf{p}_{\Xi}^{g}(v), \mathbf{p}_{\Xi}^{g}(w)\}\end{aligned}$$

$$\begin{aligned}(ii)\ \mathbf{q}^{\ g}(av * bw) &= \mathbf{q}(\ g(av * bw))\\ &= \mathbf{q}(\ g(v)\ g(w)) \quad \text{(since } g \text{ is a homomorphism)}\\ &\leq \max\{\mathbf{q}(\ g(v)), \mathbf{q}(\ g(w))\}\\ &= \max\{\mathbf{q}_{\Xi}^{g}(v), \mathbf{q}_{\Xi}^{g}(w)\}\\ \Rightarrow \mathbf{q}^{\ g}(av * bw) &\leq \max\{\mathbf{q}_{\Xi}^{g}(v), \mathbf{q}_{\Xi}^{g}(w)\}\end{aligned}$$

So, $\Xi^{\ g}$ is a **FFLS** of $\mathfrak{V}$. □

Theorem 3.6. *Let $g : \mathfrak{V}_1 \to \mathfrak{V}_2$ be an epimorhism of **FFLS** of $\mathfrak{V}$ over $\mathbb{F}$. Let $\Xi = (\boldsymbol{p}, \boldsymbol{q})$ be a g-invariant **FFLS** of $\mathfrak{V}_1$. Then $g(\Xi)$ is a **FFLS** of $\mathfrak{V}_2$.*

Proof. Let $v^{'}, w^{'} \in \mathfrak{V}_2$ and $a, b \in \mathbb{F}$. Then there exist $v, w \in \mathfrak{V}_1$ such that $g(v) = v^{'}$ and $g(w) = w^{'}$.

Also $av^{'} * bw^{'} = g(av * bw)$. Since Ξ is g-invariant.

$$\begin{aligned}(i)\ g(\mathbf{p}_{\Xi})(av * bw) &= \mathbf{p}_{\Xi}(av^{'} * bw^{'}) \geq \min\{\mathbf{p}_{\Xi}(v^{'}), \mathbf{p}_{\Xi}(w^{'})\}\\ &= \min\{g(\mathbf{p}_{\Xi})(v), g(\mathbf{p}_{\Xi})(w)\}\\ \Rightarrow g(\mathbf{p}_{\Xi})(av * bw) &\geq \min\{g(\mathbf{p}_{\Xi})(v), g(\mathbf{p}_{\Xi})(w)\}\end{aligned}$$

Therefore $g(\mathbf{p}_{\Xi})$ is a **FFLS** of $\mathfrak{V}_2$.

$$\begin{aligned}(ii)\ g(\mathbf{q}_{\Xi})(av * bw) &= \mathbf{q}_{\Xi}(av^{'} * bw^{'}) \leq \max\{\mathbf{q}_{\Xi}(v^{'}), \mathbf{q}_{\Xi}(w^{'})\}\\ &= \max\{g(\mathbf{q}_{\Xi})(v), g(\mathbf{q}_{\Xi})(w)\}\\ \Rightarrow g(\mathbf{q}_{\Xi})(av * bw) &\leq \max\{g(\mathbf{q}_{\Xi})(v), g(\mathbf{q}_{\Xi})(w)\}\end{aligned}$$

Therefore $g(\mathbf{q}_{\Xi})$ is a **FFLS** of $\mathfrak{V}_2$. □

Theorem 3.7. *If $\Xi = (\boldsymbol{p}, \boldsymbol{q})$ is a **FFLS** then $\Xi^c = (\boldsymbol{p}^c, \boldsymbol{q}^c)$ is a **FFLS**.*

Proof. Let $\Xi = (\mathbf{p}, \mathbf{q})$ be a **FFLS** of $\mathfrak{V}$ over $\mathbb{F}$. We have for all $v, w \in \mathfrak{V}$ and $a, b \in \mathbb{F}$.

$$\begin{aligned}(i)\ \mathbf{p}^{\ c}(av * bw) &= 1 - \mathbf{p}(av * bw)\\ &\leq 1 - \min\{\mathbf{p}(v), \mathbf{p}(w)\}\\ &= \max\{1 - \mathbf{p}(v), 1 - \mathbf{p}(w)\}\\ &= \max\{\mathbf{p}^{\ c}(v), \mathbf{p}^{\ c}(w)\}\\ \Rightarrow \mathbf{p}^{\ c}(av * bw) &\leq \max\{\mathbf{p}^{\ c}(v), \mathbf{p}^{\ c}(w)\}\end{aligned}$$

$\Rightarrow \mathbf{p}^{\,c}$ is a **FFLS** of $\mathfrak{V}$ over $\mathbb{F}$.

$$\begin{aligned}(ii)\ \mathbf{q}^{\,c}(av*bw) &= 1-\mathbf{q}(av*bw)\\ &\geq 1-\max\{\mathbf{q}(v),\mathbf{q}(w)\}\\ &= \min\{1-\mathbf{q}(v),1-\mathbf{q}(w)\}\\ &= \min\{\mathbf{q}^{\,c}(v),\mathbf{q}^{\,c}(w)\}\\ \Rightarrow \mathbf{q}^{\,c}(av*bw) &\leq \min\{\mathbf{q}^{\,c}(v),\mathbf{q}^{\,c}(w)\}\end{aligned}$$

$\Rightarrow \mathbf{q}^{\,c}$ is a **FFLS** of $\mathfrak{V}$ over $\mathbb{F}$. □

4. Conclusion

This paper introduced a new notion called Fermatean fuzzy linear space. This concept is a generalization of Fermatean fuzzy set and linear space. We ascertained that the intersection of two **FFLSs** is also **FFLS**. We investigate few properties of **FFLSs**. Based upon the investigation the present work explored the union of two **FFLSs**. Using a counter example, union of two **FFLSs** need not always constructs **FFLSs**. In addition to that results, the work delves into various properties of Fermatean fuzzy linear spaces.

5. Direction for further research

In the future, researchers can incorporate **FFLS** with other extensions of **FSs** and then apply it to the decision-making scenarios. This leads to advancements in decision-making methodologies, particularly in contexts where uncertainty and imprecision play significant roles. Further, the concept of Fermatean fuzzy linear spaces can be applied to different algebraic structures. To mention few

- semigroups,
- M-semigroups,
- rings,
- soft sets, rough sets and
- neutrosophic sets etc.,

By extending the application of **FFLSs** to these algebraic structures, the work opens up new avenues for research and potential applications in various fields.

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