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PAIR MEAN CORDIAL LABELING OF DIAMOND SNAKE GRAPH, BANANA TREE AND TORTOISE GRAPH

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ABSTRACT. Let a graph $G = (V, E)$ be a (p, q) graph. Define

$$\rho = \begin{cases} \frac{p}{2} & p \text{ is even} \\ \frac{p-1}{2} & p \text{ is odd,} \end{cases}$$

and $M = \{\pm 1, \pm 2, \cdots \pm \rho\}$ called the set of labels. Consider a mapping $\lambda : V \to M$ by assigning different labels in M to the different elements of V when p is even and different labels in M to $p-1$ elements of V and repeating a label for the remaining one vertex when p is odd. The labeling as defined above is said to be a pair mean cordial labeling if for each edge uv of G, there exists a labeling $\frac{\lambda(u)+\lambda(v)}{2}$ if $\lambda(u)+\lambda(v)$ is even and $\frac{\lambda(u)+\lambda(v)+1}{2}$ if $\lambda(u)+\lambda(v)$ is odd such that $|\bar{\mathbb{S}}_{\lambda_1} - \bar{\mathbb{S}}_{\lambda_1^c}| \leq 1$ where $\bar{\mathbb{S}}_{\lambda_1}$ and $\bar{\mathbb{S}}_{\lambda_1^c}$ respectively denote the number of edges labeled with 1 and the number of edges not labeled with 1. A graph G with a pair mean cordial labeling is called a pair mean cordial graph. In this paper, we investigate the pair mean cordial labeling behavior of the diamond snake graph, banana tree, tortoise graph and generalized web graph without the central vertex.

Keywords: diamond snake graph, banana tree, tortoise graph and generalized web graph without the central vertex, pair mean cordial labeling.

AMS Subject Classification: 05C78

1. Introduction

Throughout this paper, a finite, undirected, and simple graph is referred to as graph G=(V,E) with $|V(G)| = p$ vertices and $|E(G)| = q$ edges. Rosa [21] was the one who initially initiated the study of graceful labeling techniques. Cahit[1] originally proposed the idea of cordial labeling, and [4-10,16-20, 22-24] examined cordial related graphs. Many kinds of graph labeling exist, and a thorough analysis may be found in [2]. We refer to Harray[3] for various notations and terminology related to graph theory. The notion of pair mean labeling was initially presented in [11], and corresponding pair mean graphs were examined in [12-15]. At the present work, we investigate the pair mean cordial labeling

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behavior of diamond snake graph, banana tree, tortoise graph and generalized web graph without the central vertex.

2. Preliminaries

Definition 2.1. *Graph labeling is the assignment of integers to vertices, edges, or both under certain conditions.*

Definition 2.2. *The graph G consists of collection of n cycles C_4, these cycles are connected in such a way that any two disjoint cycles sharing a common vertex, the resulting graph is called the diamond snake graph and it is denoted by D_n. A snake is a Eulerian path that has no chords.*

Definition 2.3. *The banana tree $B_{m,n}$ is a graph obtained by connecting one leaf of each of $m-$ copies of the star $K_{1,n}$ with a single root vertex that is distinct from all the stars.*

Definition 2.4. *A tortoise graph $G(T_n)$, $n \geq 3$ is a graph obtained from the path $v_1v_2\dots v_n$, where n is odd by attaching an edge between v_i and $v_{(n-i+1)}$ for $i = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$.*

Definition 2.5. *A web graph is a graph obtained by joining the pendent vertices of a helm graph to form a cycle and then adding a single pendent edge at each vertex of this outer cycle. In 1996, Yang[24] has extended the notion of a web by iterating the process of joining the pendant vertices to form a cycle and then adding a pendant edge to the new cycle. this graph is called as a generalized web graph denoted by W(t, n) with t number of n cycles. A generalized web graph without a central vertex is a graph obtained from the generalized web graph by deleting the cental vertex and denoted by $W_0(m, n)$.*

3. Main Theorem

Theorem 3.1. *The diamond snake graph D_n is pair mean cordial for all $n \geq 2$*

Proof. The vertex set and edge set of the diamond snake graph D_n respectively are defined by $V(D_n) = \{u_i, v_j, w_j \backslash 1 \leq i \leq n+1,\ 1 \leq j \leq n\}$ and $E(D_n) = \{u_iv_i, u_iw_i \backslash 1 \leq i \leq n\} cup \{v_iu_{i+1}, v_iu_{i+1} \backslash 1 \leq i \leq n\}$. Then the diamond graph has $3n+1$ vertices and $4n$ edges. Define $\lambda(u_1) = -1$, $\lambda(u_2) = -2$, $\lambda(v_1) = 2$ and $\lambda(w_1) = 3$.

Case (i) : n **is odd**

We now assign the labels $-3, -6, \dots, \frac{-3n+3}{2}$ to the vertices $u_3, u_5, \dots, u_n$ respectively and $6, 9, \dots, \frac{3n-3}{2}$ respectively to the vertices $u_4, u_6, \dots, u_{n-1}$. Fix the label 1 to the vertex u_{n+1}. Also assign the labels $4, 7, \dots, \frac{3n-1}{2}$ to the vertices $v_2, v_4, \dots, v_{n-1}$ respectively and $-4, -7, \dots, \frac{-3n+1}{2}$ respectively to the vertices $v_3, v_5, \dots, v_n$. Next assign the labels $5, 8, \dots, \frac{3n+1}{2}$ to the vertices $w_2, w_4, \dots, w_{n-1}$ respectively and $-5, -8, \dots, \frac{-3n-1}{2}$ respectively to the vertices $w_3, w_5, \dots, w_n$.

Case (ii) : n **is even**

If $n = 2$, $\lambda(u_3) = 1$, $\lambda(v_2) = 1$ and $\lambda(w_2) = -3$. Let $n \geq 4$. Then we give the labels $-3, -6, \dots, \frac{-3n}{2}$ to the vertices $u_3, u_5, \dots, u_{n+1}$ respectively and $6, 9, \dots, \frac{3n}{2}$ respectively to the vertices $u_4, u_6, \dots, u_n$. Thus give the labels $4, 7, \dots, \frac{3n-4}{2}$ to the vertices $v_2, v_4, \dots, v_{n-2}$ respectively and $-4, -7, \dots, \frac{-3n+4}{2}$ respectively to the vertices $v_3, v_5, \dots, v_{n-1}$. Fix the label 1 to the vertex v_n. Further we give the labels $5, 8, \dots, \frac{3n-2}{2}$ to the vertices $w_2, w_4, \dots,$ w_{n-2} respectively and $-5, -8, \dots, \frac{-3n+2}{2}$ respectively to the vertices $w_3, w_5, \dots, w_{n-1}$. Finally assign the label 1 to the vertex w_n. In both cases, the edges u_1v_1, u_1w_1, w_1u_2, u_2v_2, v_iu_{i+1} and w_iu_{i+1}, for $2 \leq i \leq n-1$ are labeled with 1 and all other edges are labeled by the integers other than 1. Hence $\bar{\mathbb{S}}_{\lambda_1} = \bar{\mathbb{S}}_{\lambda_1^c} = 2n$.

□

Example 3.1. *The pair mean cordial labeling of the diamond snake graph D_5 is shown in Figure* 1.

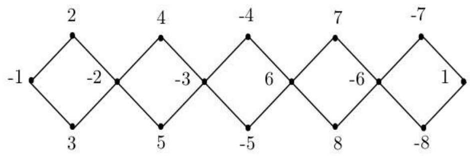

FIGURE 1

Theorem 3.2. *The banana tree $B_{m,4}$ is pair mean cordial for all $m \geq 1$*

Proof. Define $V(B_{m,4}) = \{u, u_i, v_i, x_i, y_i \backslash 1 \leq i \leq m\}$ and $E(B_{m,4}) = \{uu_i, u_iv_i, v_ix_i, v_iy_i \backslash 1 \leq i \leq m\}$. Then it has $4m+1$ vertices and $4m$ edges. Let $\lambda(u) = 1$.
Case(i) : m **is odd**
We now assign the labels $\frac{-3m-1}{2}, \frac{-3m-3}{2}, \ldots, -2m$ to the vertices $u_1, u_2, \ldots, u_{\frac{m+1}{2}}$ respectively and $\frac{3m+5}{2}, \frac{3m+7}{2}, \ldots, 2m$ respectively to the vertices $u_{\frac{m+3}{2}}, u_{\frac{m+5}{2}}, \ldots, u_{m-1}$. Also assign the label $-2m$ to the vertex u_m. Then we assign the labels $-1, -4, \ldots, \frac{-3m+1}{2}$ to the vertices $v_1, v_3, \ldots, v_m$ respectively and $4, 7, \ldots, \frac{3m-1}{2}$ respectively to the vertices $v_2, v_4, \ldots, v_{m-1}$. Next we assign the labels $2, 5, \ldots, \frac{3m+1}{2}$ to the vertices $x_1, x_3, \ldots, x_m$ respectively and $-2, -5, \ldots, \frac{-3m-1}{2}$ respectively to the vertices $x_2, x_4, \ldots, x_{m-1}$. Assign the labels $3, 6, \ldots, \frac{3m+3}{2}$ to the vertices $y_1, y_3, \ldots, y_m$ respectively and $-3, -6, \ldots, \frac{-3m-3}{2}$ respectively to the vertices $y_2, y_4, \ldots, y_{m-1}$. If $m = 1$, take $\lambda(y_1) = 2$.
Case(ii) : m **is even**
In this case, we give the labels $\frac{-3m-2}{2}, \frac{-3m-4}{2}, \ldots, -2m$ to the vertices $u_1, u_2, \ldots, u_{\frac{m}{2}}$ respectively and $\frac{3m+4}{2}, \frac{3m+6}{2}, \ldots, 2m$ respectively to the vertices $u_{\frac{m+2}{2}}, u_{\frac{m+4}{2}}, \ldots, u_{m-1}$. Fix the label $-2m$ to the vertex u_m. Then we give the labels $-1, -4, \ldots, \frac{-3m+4}{2}$ to the vertices $v_1, v_3, \ldots, v_{m-1}$ respectively and $4, 7, \ldots, \frac{3m+2}{2}$ respectively to the vertices $v_2, v_4, \ldots, v_m$. Now we give the labels $2, 5, \ldots, \frac{3m-2}{2}$ to the vertices $x_1, x_3, \ldots, x_{m-1}$ respectively and $-2, -5, \ldots, \frac{-3m+2}{2}$ respectively to the vertices $x_2, x_4, \ldots, x_m$. Finally we give the labels $3, 6, \ldots, \frac{3m}{2}$ to the vertices $y_1, y_3, \ldots, y_{m-1}$ respectively and $-3, -6, \ldots, \frac{-3m}{2}$ respectively to the vertices $y_2, y_4, \ldots, y_m$. In both cases, the edges v_ix_i and v_iy_i, for $1 \leq i \leq m$ are labeled with 1 and all other edges are labeled by the integers other than 1. Hence $\bar{\mathbb{S}}_{\lambda_1} = \bar{\mathbb{S}}_{\lambda_1^c} = 2m$. □

Example 3.2. *The pair mean cordial labeling of banana tree $B_{5,4}$ is shown in Figure* 2.

Theorem 3.3. *The banana tree $B_{m,5}$ is pair mean cordial iff $m \leq 6$*

Proof. Let us now define $V(B_{m,5}) = \{u, u_i, v_i, w_i, x_i, y_i \backslash 1 \leq i \leq m\}$ and $E(B_{m,5}) = \{uu_i, u_iv_i, v_iw_i, v_ix_i, v_iy_i \backslash 1 \leq i \leq m\}$. Thus the banana tree $B_{m,5}$ has $5m+1$ vertices and $5m$ edges.
Case(i) : $m \leq 6$

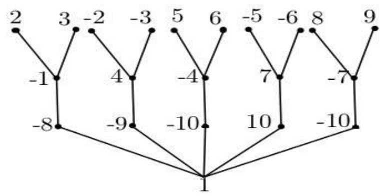

FIGURE 2

Define

$$\lambda(v_i) = \begin{cases} -2i+1 & 1 \le i \le m \ \& \ i \text{ is odd} \\ 2i+1 & 1 \le i \le m \ \& \ i \text{ is even} \end{cases}$$

$$\lambda(w_i) = \begin{cases} 2i & 1 \le i \le m \ \& \ i \text{ is odd} \\ -2i+2 & 1 \le i \le m \ \& \ i \text{ is even} \end{cases}$$

$$\lambda(x_i) = \begin{cases} 2i+1 & 1 \le i \le m \ \& \ i \text{ is odd} \\ -2i+1 & 1 \le i \le m \ \& \ i \text{ is even} \end{cases}$$

$$\lambda(y_i) = \begin{cases} 2i+2 & 1 \le i \le m \ \& \ i \text{ is odd} \\ -2m & 1 \le i \le m \ \& \ i \text{ is even} \end{cases}$$

Subcase $(i):\ m=6$
Define $\lambda(u)=15$ and assign the labels $1,-13,-14,-15,14,-12$ to the vertices $u_1,u_2,\dots,u_6$ respectively.
Subcase $(ii):\ m=5$
Let $\lambda(u)=13$. Then we give the labels $1,-10,-11,-12,-13$ to the vertices $u_1,u_2,\dots,u_5$ respectively.
Subcase $(iii):\ m=4$
Define $\lambda(u)=10$. Next assign the labels $1,-8,-9,-10$ to the vertices $u_1,u_2,\dots,u_4$ respectively.
Subcase $(iv):\ m=3$
Let $\lambda(u)=5$. Also give the labels $1,-7,-8$ to the vertices u_1,u_2,u_3 respectively.
Subcase $(v):\ m=2$
Let $\lambda(u)=1$. Now we assign the labels $1,-5$ to the vertices u_1,u_2 respectively.
Subcase $(v):\ m=1$
Let $\lambda(u)=1$. Now we assign the labels $-3,-2$ to the vertices u_1,v_1 respectively. The given table shows that this vertex labeling λ is pair mean cordial labeling of $B_{m,5}$ for all $m \le 6$.

m	$\mathbb{S}_{\lambda_1}$	$\mathbb{S}_{\lambda_1^c}$
m is odd	$\frac{5m-1}{2}$	$\frac{5m+1}{2}$
m is even	$\frac{5m}{2}$	$\frac{5m}{2}$

Table 1

Case $(ii):\ m>6$
Suppose that $B_{m,5}$ is pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u)+\lambda(v)=1$ or $\lambda(u)+\lambda(v)=2$.

Subcase (i) : m **is odd**
Hence the maximum number of edges label with 1 is $2m+2$. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 2m+2$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q-(2m+2) = 3m-2$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 3m-2-(2m+2) = m-4 \geq 3 > 1$, a contradiction.
Subcase (ii) : m **is even**
Thus the maximum number of edges label with 1 is $2m+3$. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 2m+3$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q-(2m+3) = 3m-3$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 3m-3-(2m+3) = m-6 \geq 2 > 1$, a contradiction.

□

Example 3.3. *The pair mean cordial labeling of banana tree $B_{5,5}$ is shown in Figure 3.*

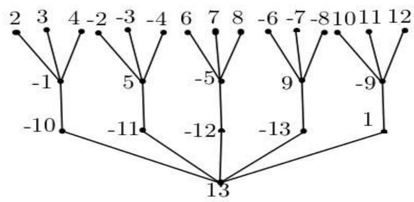

FIGURE 3

Theorem 3.4. *The banana tree $B_{m,6}$ is pair mean cordial iff $m \leq 3$*

Proof. Let us now define $V(B_{m,6}) = \{u, u_i, v_i, w_i, x_i, y_i, z_i \backslash 1 \leq i \leq m\}$ and $E(B_{m,6}) = \{uu_i, u_iv_i, v_iw_i, v_ix_i, v_iy_i, v_iz_i \backslash 1 \leq i \leq m\}$. Then the banana tree $B_{m,6}$ has $6m+1$ vertices and $6m$ edges.
Case (i) : $m \leq 3$
Subcase (i) : $m = 3$
Let $\lambda(u) = 9$. Now assign the labels $1, -1, 2, 3, 4, 5$ to the vertices $u_1, v_1, w_1, x_1, y_1, z_1$ respectively and $-7, 6, -2, -3, -4, -5$ respectively to the vertices $u_2, v_2, w_2, x_2, y_2, z_2$. Finally assign the labels $-8, -6, 7, 8, -9, 7$ to the vertices $u_3, v_3, w_3, x_3, y_3, z_3$ respectively.
Subcase (ii) : $m = 2$
Let $\lambda(u) = -3$. Then we assign the labels $4, -1, 1, 2, 3, -2$ to the vertices $u_1, v_1, w_1, x_1, y_1, z_1$ respectively and $5, 6, -3, -4, -5, -6$ respectively to the vertices $u_2, v_2, w_2, x_2, y_2, z_2$. **Subcase** (iii) : $m = 1$
Let $\lambda(u) = 2$. Then we assign the labels $-1, 3, -2, -3, 1, -3$ to the vertices $u_1, v_1, w_1, x_1, y_1, z_1$ respectively. In all cases, $\bar{\mathbb{S}}_{\lambda_1} = \bar{\mathbb{S}}_{\lambda_1^c} = 3m$.
In both cases, $\bar{\mathbb{S}}_{\lambda_1} = \bar{\mathbb{S}}_{\lambda_1^c} = 3m$.
Case (ii) : $m > 3$
Suppose that $B_{m,6}$ is pair mean cordial. Now if the edge uv get the label 1, the possibilities are $\lambda(u)+\lambda(v) = 1$ or $\lambda(u)+\lambda(v) = 2$. Hence the maximum number of edges label with 1 is $2m+3$. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 2m+3$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q-(2m+3) = 4m-3$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 4m-3-(2m+3) = 2m-6 \geq 2 > 1$, a contradiction.

□

Theorem 3.5. *The banana tree $B_{m,7}$ is pair mean cordial iff $m \leq 3$*

Proof. Now define $V(B_{m,7}) = \{u, u_i, v_i, w_i, x_i, y_i, z_i, r_i \backslash 1 \leq i \leq m\}$ and $E(B_{m,7}) = \{uu_i, u_iv_i, v_iw_i, v_ix_i, v_iy_i, v_iz_i, v_ir_i \backslash 1 \leq i \leq m\}$. Then the banana tree $B_{m,7}$ has $7m+1$

vertices and $7m$ edges.

Case $(i):\ m=1$

Let $\lambda(u)=2$. Assign the labels $-1,3,-2,-3,4,-4,1$ to the vertices $u_1,v_1,w_1,x_1,y_1,z_1,r_1$ respectively. Hence $\bar{\mathbb{S}}_{\lambda_1}=3$ and $\bar{\mathbb{S}}_{\lambda_1^c}=4$.

Case $(ii):\ m=2$

Let $\lambda(u)=-4$. Assign the labels $5,-1,-2,-3,2,3,4$ to the vertices $u_1,v_1,w_1,x_1,y_1,z_1,r_1$ respectively and $6,7,-5,-6,-7,1,-5$ to the vertices $u_2,v_2,w_2,x_2,y_2,z_2,r_2$ respectively. Hence $\bar{\mathbb{S}}_{\lambda_1}=\bar{\mathbb{S}}_{\lambda_1^c}=7$.

Case $(iii):\ m\geq 3$

Suppose $B_{m,7}$ is pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u)+\lambda(v)=1$ or $\lambda(u)+\lambda(v)=2$.

Subcase $(i):$ m **is odd**

Hence the maximum number of edges label with 1 is $2m+2$. That is $\bar{\mathbb{S}}_{\lambda_1}\leq 2m+2$. Then $\bar{\mathbb{S}}_{\lambda_1^c}\geq q-(2m+2)=5m-2$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c}-\bar{\mathbb{S}}_{\lambda_1}\geq 5m-2-(2m+2)=3m-4\geq 5>1$, a contradiction.

Subcase $(ii):$ m **is even**

Thus the maximum number of edges label with 1 is $2m+3$. That is $\bar{\mathbb{S}}_{\lambda_1}\leq 2m+3$. Then $\bar{\mathbb{S}}_{\lambda_1^c}\geq q-(2m+3)=5m-3$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c}-\bar{\mathbb{S}}_{\lambda_1}\geq 5m-3-(2m+3)=3m-6\geq 6>1$, a contradiction.

□

Theorem 3.6. *The banana tree $B_{m,n}$ is not a pair mean cordial for all $m\geq 2$ and $n\geq 8$.*

Proof. The vertex and edge set of the banana tree $B_{m,n}$ are respectively defined by $V(B_{m,n})=\{u,u_{i,j\backslash 1\leq i\leq m \text{ and } 1\leq j\leq n}\}$. Clearly the banana tree $B_{m,n}$ has $mn+1$ vertices and mn edges.

Suppose $B_{m,n}$ is pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u)+\lambda(v)=1$ or $\lambda(u)+\lambda(v)=2$.

Case $(i):$ m **is odd**

Subcase $(i):$ n **is odd**

Hence the maximum number of edges label with 1 is $2m+2$. That is $\bar{\mathbb{S}}_{\lambda_1}\leq 2m+2$. Then $\bar{\mathbb{S}}_{\lambda_1^c}\geq q-(2m+2)=mn-2m-2$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c}-\bar{\mathbb{S}}_{\lambda_1}\geq mn-2m-2-(2m+2)=mn-4m-4=m(n-4)-4\geq 5>1$, a contradiction.

Subcase $(ii):$ n **is even**

Thus the maximum number of edges label with 1 is $2m+3$. That is $\bar{\mathbb{S}}_{\lambda_1}\leq 2m+3$. Then $\bar{\mathbb{S}}_{\lambda_1^c}\geq q-(2m+3)=mn-2m-3$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c}-\bar{\mathbb{S}}_{\lambda_1}\geq mn-2m-3-(2m+3)=mn-4m-6\geq 6>1$, a contradiction.

Case $(ii):$ m **is even**

Hence the maximum number of edges label with 1 is $2m+3$. That is $\bar{\mathbb{S}}_{\lambda_1}\leq 2m+3$. Then $\bar{\mathbb{S}}_{\lambda_1^c}\geq q-(2m+3)=mn-2m-3$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c}-\bar{\mathbb{S}}_{\lambda_1}\geq mn-2m-3-(2m+3)=mn-4m-6=m(n-4)-6\geq 2>1$, a contradiction.

□

Theorem 3.7. *The generalized web graph $W_0(m,n)$ without central vertex is pair mean cordial for all $m,n\geq 3$.*

Proof. Let $V(W_0(m,n))=\{v_{i,j}\backslash 1\leq i\leq m+1 \text{ and } 1\leq j\leq n\}$ and $E(W_0(m,n))=\{v_{i,j}v_{i+1,j}\backslash 1\leq i\leq m \text{ and } 1\leq j\leq n\}\cup\{v_{i,j}v_{i,j+1}\backslash 2\leq i\leq m+1 \text{ and } 1\leq j\leq n-1\}\cup\{v_{i,n}v_{i,1}\backslash 1\leq i\leq m\}$. Then the generalized web graph without central vertex $W_0(m,n)$ has $mn+n$ vertices and $2mn$ edges.

Case $(i):$ m **is odd**

Define a map $\lambda : V \to \{\pm 1, \pm 2, \ldots, \pm(mn+n)\}$ as follows:
Subcase (i) : n is odd

$$\lambda(v_{i,j}) = \begin{cases} \frac{-jm+i-2}{2} & 1 \le i \le m,\ 1 \le j \le n,\ i \text{ is odd and } j \text{ is odd} \\ \frac{jm-i+5}{2} & 1 \le i \le m,\ 1 \le j \le n,\ i \text{ is odd and } j \text{ is even,} \end{cases}$$

$$\lambda(v_{i,j}) = \begin{cases} \frac{(j-1)m+i+4}{2} & 1 \le i \le m,\ 1 \le j \le n,\ i \text{ is even and } j \text{ is odd} \\ \frac{(-j+1)m-i-1}{2} & 1 \le i \le m,\ 1 \le j \le n,\ i \text{ is even and } j \text{ is even,} \end{cases}$$

$$\lambda(v_{m+1,1}) = 2,$$

$$\lambda(v_{m+1,j}) = \begin{cases} \frac{-mn-2j+1}{2} & 2 \le j \le \frac{n+1}{2} \\ \frac{(m-1)n+2j+2}{2} & \frac{n+3}{2} \le j \le n-1, \end{cases}$$

$$\lambda(v_{m+1,n}) = 1.$$

Subcase (ii) : n is even
Assign the labels to the vertices $v_{i,j}$ for $1 \le i \le m$, $1 \le j \le n$, $v_{m+1,1}$ and $v_{m+1,n}$ as in Case (i) of Subcase (i). Then

$$\lambda(v_{m+1,j}) = \begin{cases} \frac{-mn-2j+2}{2} & 2 \le j \le \frac{n+2}{2}, \\ \frac{(m-1)n+2j+2}{2} & \frac{n+4}{2} \le j \le n-1, \end{cases}$$

Case (ii) : m is even
Subcase (i) : n is odd
Define a map $\lambda : V \to \{\pm 1, \pm 2, \ldots, \pm(mn+n-1)\}$ as follows:

$$\lambda(v_{i,j}) = \begin{cases} \frac{jm-i+5}{2} & 1 \le i \le m,\ 1 \le j \le n,\ i \text{ is odd and } j \text{ is odd} \\ \frac{-jm+i-2}{2} & 1 \le i \le m,\ 1 \le j \le n,\ i \text{ is odd and } j \text{ is even,} \end{cases}$$

$$\lambda(v_{i,j}) = \begin{cases} \frac{(-j+1)m-i-1}{2} & 1 \le i \le m,\ 1 \le j \le n,\ i \text{ is even and } j \text{ is odd} \\ \frac{(j-1)m+i+4}{2} & 1 \le i \le m,\ 1 \le j \le n,\ i \text{ is even and } j \text{ is even,} \end{cases}$$

$$\lambda(v_{m+1,1}) = 2,$$

$$\lambda(v_{m+1,j}) = \begin{cases} \frac{-mn-2j+2}{2} & 2 \le j \le \frac{n+1}{2} \\ \frac{(m-1)n+2j+3}{2} & \frac{n+3}{2} \le j \le n-2, \end{cases}$$

$$\lambda(v_{m+1,n-1}) = 1,$$
$$\lambda(v_{m+1,n}) = 1.$$

Subcase (ii) : n is even
Define a map $\lambda : V \to \{\pm 1, \pm 2, \ldots, \pm(mn+n)\}$ as follows:
Now assign the labels to the vertices $v_{i,j}$ for $1 \le i \le m$, $1 \le j \le n$, $v_{m+1,1}$ and $v_{m+1,n}$ as in Case (ii) of Subcase (i). Then

$$\lambda(v_{m+1,j}) = \begin{cases} \frac{-mn-2j+2}{2} & 2 \le j \le \frac{n+2}{2} \\ \frac{(m-1)n+2j+4}{2} & \frac{n+4}{2} \le j \le n-1, \end{cases}$$

In all cases, the edges $v_{i,j}v_{i+1,j}$, $1 \le i \le m-1$ and $1 \le j \le n$, $v_{1,j}v_{1,j+1}$, $1 \le j \le n-1$ and j is odd and $v_{m,j}v_{m,j+1}$, $1 \le j \le n-1$ and j is even are labeled with 1 and all other edges are labeled by the integers other than 1. Thus $\bar{\mathbb{S}}_{\lambda_1} = \bar{\mathbb{S}}_{\lambda_1^c} = mn$. $\square$

Example 3.4. *The pair mean cordial labeling of generalized web graph without central vertex $W_0(4,7)$ is shown in Figure 4.*

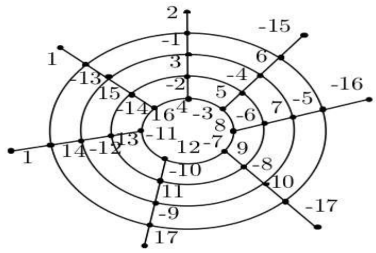

FIGURE 4

Theorem 3.8. *Tortoise graph $G(T_n)$ is pair mean cordial for all $n \geq 3$.*

Proof. Let us now define $V(G(T_n)) = \{v_i : 1 \leq i \leq n\}$ and $E(G(T_n)) = \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{v_iv_{n-i+1} : i = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. Then the tortoise graph $G(T_n)$ has n vertices and $\frac{3n-2}{2}$ edges. Let $\lambda(v_{n-1}) = 1$ and $\lambda(v_n) = 1$.

Case (i) : $n \equiv 1 \pmod 8$

In this case, we assign the labels $2, 3, \ldots, \frac{3n+5}{8}$ to the vertices $v_1, v_3, \ldots, v_{\frac{3n-7}{4}}$ respectively and $-1, -2, \ldots, \frac{-3n+3}{8}$ respectively to the vertices $v_2, v_4, \ldots, v_{\frac{3n-3}{4}}$. Then we give the labels $\frac{-3n-5}{8}, \frac{-3n-13}{8}, \ldots, \frac{-n+1}{2}$ to the vertices $v_{\frac{3n+1}{4}}, v_{\frac{3n+5}{4}}, \ldots, v_{\frac{7n-7}{8}}$ respectively and assign the labels $\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{3n+13}{8}$ respectively to the vertices $v_{\frac{7n+1}{8}}, v_{\frac{7n+9}{8}}, \ldots, v_{n-2}$. Hence the edges v_iv_{i+1}, $1 \leq i \leq \frac{3n-7}{4}$ and v_nv_{n-1} are labeled with 1 and all other edges are labeled by the integers other than 1.

Case (ii) : $n \equiv 3 \pmod 8$

Furthermore we assign the labels $2, 3, \ldots, \frac{3n+7}{8}$ to the vertices $v_1, v_3, \ldots, v_{\frac{3n-5}{4}}$ respectively and $-1, -2, \ldots, \frac{-3n+9}{8}$ respectively to the vertices $v_2, v_4, \ldots, v_{\frac{3n-9}{4}}$. Next we give the labels $\frac{3n+15}{8}, \frac{3n+23}{8}, \ldots, \frac{n-1}{2}$ to the vertices $v_{\frac{3n-1}{4}}, v_{\frac{3n+3}{4}}, \ldots, v_{\frac{7n-21}{8}}$ respectively and assign the labels $\frac{-n+1}{2}, \frac{-n+3}{2}, \ldots, \frac{-3n+1}{8}$ respectively to the vertices $v_{\frac{7n-13}{8}}, v_{\frac{7n-5}{8}}, \ldots, v_{n-2}$. Thus the edges v_iv_{i+1}, $1 \leq i \leq \frac{3n-5}{4}$ and v_nv_{n-1} are labeled with 1 and all other edges are labeled by the integers other than 1.

Case (iii) : $n \equiv 5 \pmod 8$

Let us now we assign the labels $2, 3, \ldots, \frac{3n+9}{8}$ to the vertices $v_1, v_3, \ldots, v_{\frac{3n-3}{4}}$ respectively and $-1, -2, \ldots, \frac{-3n+7}{8}$ respectively to the vertices $v_2, v_4, \ldots, v_{\frac{3n-7}{4}}$. Also we give the labels $\frac{3n+17}{8}, \frac{3n+25}{8}, \ldots, \frac{n-1}{2}$ to the vertices $v_{\frac{3n+1}{4}}, v_{\frac{3n+5}{4}}, \ldots, v_{\frac{7n-19}{8}}$ respectively and assign the labels $\frac{-n+1}{2}, \frac{-n+3}{2}, \ldots, \frac{-3n-1}{8}$ respectively to the vertices $v_{\frac{7n-11}{8}}, v_{\frac{7n-3}{8}}, \ldots, v_{n-2}$. Note that the edges v_iv_{i+1}, $1 \leq i \leq \frac{3n-7}{4}$ and v_nv_{n-1} are labeled with 1 and all other edges are labeled by the integers other than 1.

Case (iv) : $n \equiv 7 \pmod 8$

In this case, assign the labels $2, 3, \ldots, \frac{3n+3}{8}$ to the vertices $v_1, v_3, \ldots, v_{\frac{3n-5}{4}}$ respectively and $-1, -2, \ldots, \frac{-3n+5}{8}$ respectively to the vertices $v_2, v_4, \ldots, v_{\frac{3n-9}{4}}$. We also give the labels $\frac{-3n-3}{8}, \frac{-3n-11}{8}, \ldots, \frac{-n+1}{2}$ to the vertices $v_{\frac{3n-1}{4}}, v_{\frac{3n+3}{4}}, \ldots, v_{\frac{7n-9}{8}}$ respectively and assign the labels $\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{3n+11}{8}$ respectively to the vertices $v_{\frac{7n-1}{8}}, v_{\frac{7n+7}{8}}, \ldots, v_{n-2}$. Therefore the edges v_iv_{i+1}, $1 \leq i \leq \frac{3n-5}{4}$ and v_nv_{n-1} are labeled with 1 and all other edges are labeled by the integers other than 1. The given table shows that this vertex labeling λ is pair mean cordial labeling of $G(T_n)$ for $n \geq 3$. $\square$

n	$\mathbb{S}_{\lambda_1}$	$\mathbb{S}_{\lambda_1^c}$
$n \equiv 1 \pmod 8$	$\frac{3n-3}{4}$	$\frac{3n-3}{4}$
$n \equiv 3 \pmod 8$	$\frac{3n-5}{4}$	$\frac{3n-1}{4}$
$n \equiv 5 \pmod 8$	$\frac{3n-3}{4}$	$\frac{3n-3}{4}$
$n \equiv 7 \pmod 8$	$\frac{3n-5}{4}$	$\frac{3n-1}{4}$

Table 2

Example 3.5. *The pair mean cordial labeling of the tortoise $G(T_9)$ is shown in Figure 5.*

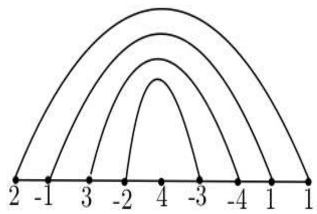

FIGURE 5

4. CONCLUSION

In this paper, we look into the pair mean cordial labeling behaviour of diamond snake graph, banana tree, tortoise graph and generalized web graph without the central vertex. It may be further continued to explore some other graphs.

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R. Ponraj for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.15, N.2.

S. Prabhu for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.15, N.2.