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CONCERNING THE ROUGH IDEAL CONVERGENCE OF DOUBLE SEQUENCES WITHIN THE TOPOLOGY INDUCED BY A FUZZY 2-NORM

NIFEEN H. ALTAWEEL[1], M.H.M. RASHID[2], NIDAL H.E. ELJANEID[1] AND RAZAN ALBALAWI[1], §

Abstract. This research paper presents a thorough exploration of rough $\mathcal{I}_2$-convergence, rough $\mathcal{I}_2^*$-convergence, rough $\mathcal{I}_2$-limit points, and rough $\mathcal{I}_2$-cluster points for double sequences within a fuzzy 2-normed linear space. A key contribution is the proof of a specific decomposition theorem related to rough $\mathcal{I}_2$-convergence of double sequences. Additionally, we introduce the concepts of rough $\mathcal{I}_2^{\mathfrak{C}}$-double Cauchy sequences and $\mathcal{I}_2^{*,\mathfrak{C}}$-double Cauchy sequences, alongside an exploration of their properties. Notably, our investigation establishes connections between the notion of rough ideal cluster points in a fuzzy 2-normed space and conventional criteria for ideal convergence, highlighting the interplay between these two seemingly distinct mathematical ideas. This study provides a comprehensive analysis of various aspects of rough convergence, the set of rough limit points, and rough cluster points in the context of sequences within fuzzy 2-normed spaces.

Keywords: rough $\mathcal{I}_2$-convergence, rough $\mathcal{I}_2$-Cauchy, rough $\mathcal{I}_2$-limit, rough $\mathcal{I}_2$-cluster, fuzzy 2-normed space

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1. Introduction

In 1951, both Fast [14] and Steinhaus [33] independently introduced the concept of statistical convergence for sequences of real numbers. Since then, various researchers, including those mentioned in [25], have further expanded upon and explored this idea. Additionally, one of its intriguing generalizations, known as $\mathcal{I}$-convergence, as described by Kostyrko et al., is referenced in [23]. Furthermore, Balcerzak et al. [8], who recently

[1] Department of Mathematics-Faculty of Science, University of Tabuk, P.O.Box 741-Tabuk 71491, Saudi Arabia.
e-mail: naltawil@ut.ed.sa; ORCID: https://orcid.org/0000-0002-0349-0012
e-mail: neljaneid@ut.edu.sa; ORCID: https://orcid.org/0000-0002-9092-3065
e-mail: razan20191ra@gmail.com; ORCID: https://orcid.org/0009-0002-7795-8738
[2] Department of Mathematics, Faculty of Science P.O. Box(7), Mutah University, Al-Karak-Jordan.
e-mail:mrash@mutah.edu.jo; ORCID: https://orcid.org/0000-0002-3816-5287
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investigated $\mathcal{I}$-convergence in the context of sequences of functions, are cited in this context.

Statistical convergence and ideal convergence are two distinct modes of convergence used in mathematical analysis and related fields to study sequences of real or complex numbers. These convergence concepts differ in their underlying principles and criteria for determining the convergence of a sequence. Both statistical convergence and ideal convergence offer alternative ways to study the behavior of sequences beyond the classical notion of pointwise convergence, and they have applications in areas such as functional analysis, number theory, and summability theory.

Zadeh [34], is credited with pioneering the concept of fuzzy sets. Over the last half-century, there has been a robust exploration of fuzzy set theory, which has found wide application in fields such as cybernetics, artificial intelligence, expert systems, and fuzzy control. Additionally, this concept has been applied in diverse domains, including projectiles, image analysis, probability theory, pattern recognition, operational research, decision making, agriculture, and weather forecasting. Furthermore, fuzzy set theory has been extensively applied in various engineering applications, such as the bifurcation analysis of nonlinear dynamical systems, the control of chaotic systems, nonlinear operators, and the modeling of population dynamics. Research on the applicability of fuzziness has extended across the entirety of mathematical sciences. Moreover, the introduction of various types of sequence spaces and the investigation of their distinctive properties have attracted significant attention from researchers in sequence space and summability theory. For more details, the reader refer to [7, 19, 30].

Phu's pioneering work [27], marked the inception of rough convergence exploration within finite-dimensional normed spaces. In his comprehensive study, he established key properties such as the closedness, convexity, and boundedness of the set LIM_x^r. Furthermore, Phu introduced the novel concept of a rough Cauchy sequence. His research also delved into the intricate interplay between rough convergence and various other forms of convergence, shedding light on how the set LIM_x^r is contingent upon the degree of roughness denoted as r. Additionally, Phu extended these insights to infinite-dimensional normed spaces in his subsequent work [28], see also [18, 26, 31].

In a related investigation by Aytar [1], the focus was shifted towards rough statistical convergence. Aytar identified the set of rough statistical limit points for a given sequence and established two statistical convergence criteria related to this set. Notably, he proved that this set is both closed and convex. Aytar's subsequent research [2] further illuminated the relationship between the r-limit set of a sequence and the intersection of these sets, as well as the r-core of the sequence being equivalent to the union of these sets.

Recently, Dündar and Çakan [10, 11] introduced the concepts of rough $\mathcal{I}$-convergence and the set of rough $\mathcal{I}$-limit points for sequences. In a parallel study, Dündar [12] conducted an in-depth exploration of rough convergence, I_2-convergence, and the sets of rough limit points and rough I_2-limit points for double sequences. Notably, within the context of 2-normed spaces, Arslan and Dündar [3, 4, 5, 6] and Dündar and Ulusu [13] developed various novel concepts related to rough convergence (see also [18, 20]).

The paper is organized as follows: Section 2 presents a comprehensive overview of the fundamental principles and definitions concerning 2-normed spaces, emphasizing the significance of fuzzy numbers. It delves into the concepts of fuzzy normed and fuzzy 2-normed spaces, also discussing the topic of ideal convergence. Moving to Section 3, the focus is on the elucidation of rough $\mathcal{I}_2^{\mathfrak{E}}$-convergence and rough $\mathcal{I}_2^{*,\mathfrak{E}}$-convergence for a double sequence within the context of a fuzzy 2-normed space denoted as $(\mathfrak{E}, \|\cdot,\cdot\|)$, providing essential insights into this form of convergence and introducing the concepts of rough $(\mathfrak{E}, \|\cdot,\cdot\|)$-limit point and rough $(\mathfrak{E}, \|\cdot,\cdot\|)$-cluster point for a double sequence. In Section 4, we present the notion of rough $\mathcal{I}_2$-limit points and rough $\mathcal{I}_2$-cluster points for double sequences within the framework of fuzzy 2-normed linear spaces. Additionally, we introduce the concept of a rough $\mathcal{I}_2$-Cauchy sequence in a fuzzy 2-normed space $(\mathfrak{E}, \|\cdot,\cdot\|)$ and explore significant outcomes within this specific framework. Finally, Section 5 concludes by summarizing the key findings related to rough $\mathcal{I}_2$-convergence in fuzzy 2-normed spaces, discussing potential future research directions, and identifying areas for further investigation.

2. Preliminaries

In this section, we provide an overview of key principles and definitions related to 2-normed spaces, fuzzy numbers, as well as fuzzy normed and fuzzy 2-normed spaces. We also delve into the topic of ideal convergence.

Definition 2.1. ([16]) Let $\mathfrak{X}$ be a real vector space of dimension ν, $2 \leq \nu < \infty$. A *2-norm on* $\mathfrak{X}$ is a function $\|\cdot,\cdot\| : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}$ which satisfies:

- (i) $\|x, y\| = 0$ if and only if x and y are linearly dependent;
- (ii) $\|x, y\| = \|y, x\|$ for all $x, y \in \mathfrak{X}$;
- (iii) $\|cx, y\| = |c| \, \|x, y\|$ for all $x, y \in \mathfrak{X}$ and $c \in \mathbb{R}$;
- (iv) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for all $x, y, z \in \mathfrak{X}$.

The pair $(\mathfrak{X}, \|\cdot,\cdot\|)$ is called a *2-normed space.*

Example 2.1. *Let* $\mathfrak{X} = \mathbb{R}^2$. *Define* $\|\cdot,\cdot\|$ *on* $\mathbb{R}^2$ *by* $\|x, y\| = |x_1y_2 - x_2y_1|$, *where* $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. *Then* $(\mathfrak{X}, \|\cdot,\cdot\|)$ *is a 2-normed space.*

Definition 2.2. ([9], [15], [22]) A *fuzzy real number*, or simply *fuzzy number*, is a fuzzy set $\mathfrak{X} : \mathbb{R} \to [0, 1]$ having the following properties:

- (a) $\mathfrak{X}$ is normal (i.e., there exists a $t_0 \in \mathbb{R}$ such that $\mathfrak{X}(t_0) = 1$);
- (b) $\mathfrak{X}$ is fuzzy convex (i.e., for $r, s \in \mathbb{R}$ and $\lambda \in J = [0, 1]$, $\mathfrak{X}(\lambda r + (1 - \lambda)s) \geq \min\{\mathfrak{X}(r), \mathfrak{X}(s)\}$);
- (c) $\mathfrak{X}$ is upper semi-continuous (i.e., $\mathfrak{X}^{\leftarrow}([0, t + \varepsilon))$ is open in $\mathbb{R}$ for each $t \in J$ and each $\varepsilon > 0$);
- (d) The closure of the set $[\mathfrak{X}]_0 := \{t \in \mathbb{R} : \mathfrak{X}(t) > 0\}$ is compact.

Let $\mathfrak{F}(\mathbb{R})$ be the set of all fuzzy real numbers. For $\mathfrak{X} \in \mathfrak{F}(\mathbb{R})$, the *$\alpha$-level set* of $\mathfrak{X}$ [15] is defined as:

$$[\mathfrak{X}]_\alpha = \begin{cases} \{t \in \mathbb{R} : \mathfrak{X}(t) \geq \alpha\}, & \text{if } 0 < \alpha \leq 1; \\ \mathrm{Cl}(\{t \in \mathbb{R} : \mathfrak{X}(t) > 0\}), & \text{if } \alpha = 0. \end{cases}$$

i.e., $\mathbb{R}$ can be embedded in $\mathfrak{F}(\mathbb{R})$.
It is easy to show that $\mathfrak{X}$ is a fuzzy number if and only if $[\mathfrak{X}]_\alpha$ is a nonempty bounded and closed interval for each $\alpha \in [0, 1]$. We denote this interval $[\mathfrak{X}]_\alpha = [\mathfrak{X}^-_\alpha, \mathfrak{X}^+_\alpha]$ (see [17]).

Remark 2.1. *The definition of fuzzy numbers presented earlier exhibits a slight variation compared to the one found in* [15]. *In the latter, it allows for the inclusion of* $\mathfrak{X}^-_\alpha = -\infty$ *and* $\mathfrak{X}^+_\alpha = +\infty$ *as admissible values, and it does not take into account the zero-level set.*

A fuzzy number $\mathfrak{X}$ is referred to as a "non-negative fuzzy number" when $\mathfrak{X}(t) = 0$ for $t < 0$. Let's denote the set of all non-negative fuzzy numbers as $\mathfrak{F}^*(\mathbb{R})$. It's evident that $\mathfrak{X} \in \mathcal{F}^*(\mathbb{R})$ if and only if, $\mathfrak{X}^-_\alpha \geq 0$ for each $\alpha \in J$, and $\overline{0} \in \mathfrak{F}^*(\mathbb{R})$.

We define a partial order $\preceq$ on $\mathfrak{F}(\mathbb{R})$ such that $\mathfrak{X} \preceq \mathfrak{Y}$ if and only if $\mathfrak{X}^-_\alpha \leq \mathfrak{Y}^-_\alpha$ and $\mathfrak{X}^+_\alpha \leq \mathfrak{Y}^+_\alpha$ for all $\alpha \in J$. Additionally, we establish a strict inequality $\prec$ on $\mathfrak{F}(\mathbb{R})$, defined

as $\mathfrak{X} \prec \mathfrak{Y}$ if and only if $\mathfrak{X}^-_\alpha < \mathfrak{Y}^-_\alpha$ and $\mathfrak{X}^+_\alpha < \mathfrak{Y}^+_\alpha$ for all $\alpha \in J$.
Let $\mathfrak{X}, \mathfrak{Y} \in \mathfrak{F}(\mathbb{R})$, define

$$\overline{\varrho}(\mathfrak{X}, \mathfrak{Y}) = \sup_{\alpha \in [0,1]} \max\{|\mathfrak{X}^-_\alpha - \mathfrak{Y}^-_\alpha|, |\mathfrak{X}^+_\alpha - \mathfrak{Y}^+_\alpha|\}.$$

Then $\overline{\varrho}$ is called the *supremum metric* on $\mathfrak{F}(\mathbb{R})$. It is known that $(\mathfrak{F}(\mathbb{R}), \overline{\varrho})$ is a complete metric space (for details see [22]).

Let $(\mathfrak{X}_k)$ be a sequence in $\mathfrak{F}(\mathbb{R})$ and $\mathfrak{X}_0 \in \mathfrak{F}(\mathbb{R})$. We say that $(\mathfrak{X}_k)$ *converges to* $\mathfrak{X}_0$ *with respect to the metric* $\overline{\varrho}$ if $\lim_{k\to\infty} \overline{\varrho}(\mathfrak{X}_k, \mathfrak{X}_0) = 0$. In this case, we write $\mathfrak{X}_k \xrightarrow{\overline{\varrho}} \mathfrak{X}_0$ or $\overline{\varrho} - \lim_{k\to\infty} \mathfrak{X}_k = \mathfrak{X}_0$.
Now, we define the notion of fuzzy 2-normed space.

Let $\mathfrak{E}$ be a real vector space with the zero element θ, let $\|\cdot,\cdot\| : \mathfrak{E} \times \mathfrak{E} \to \mathfrak{F}(\mathbb{R})$, and let the mappings $\mathfrak{L}, \mathfrak{R} : [0,1] \times [0,1] \to [0,1]$ be symmetric, non-decreasing in both arguments and satisfy $\mathfrak{L}(0,0) = 0$ and $\mathfrak{R}(1,1) = 1$.

Definition 2.3. The quadruple $(\mathfrak{E}, \|\cdot,\cdot\|, \mathfrak{L}, \mathfrak{R})$ is called a *fuzzy 2-normed space* and $\|\cdot,\cdot\|$ a *fuzzy 2-norm*, if the following axioms are satisfied:
(2FN1) $\|\mathfrak{X}, \mathfrak{Y}\| = \overline{0}$ if and only if $\mathfrak{X}$ and $\mathfrak{Y}$ are linearly dependent;
(2FN2) $\|\lambda\mathfrak{X}, \mathfrak{Y}\| = |\lambda| \|\mathfrak{X}, \mathfrak{Y}\|$, $\lambda \in \mathbb{R}$;
(2FN3) For all $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z} \in \mathfrak{E}$,
(i) $\|\mathfrak{X}+\mathfrak{Y}, \mathfrak{Z}\|(r+s) \geq \mathfrak{L}(\|\mathfrak{X}, \mathfrak{Z}\|(r), \|\mathfrak{Y}, \mathfrak{Z}\|(s))$, whenever $r \leq \|\mathfrak{X}, \mathfrak{Z}\|^-_1$, $s \leq \|\mathfrak{Y}, \mathfrak{Z}\|^-_1$ and $r+s \leq \|\mathfrak{X}+\mathfrak{Y}, \mathfrak{Z}\|^-_1$,
(ii) $\|\mathfrak{X}+\mathfrak{Y}, \mathfrak{Z}\|(r+s) \geq \mathfrak{R}(\|\mathfrak{X}, \mathfrak{Z}\|(r), \|\mathfrak{Y}, \mathfrak{Z}\|(s))$, whenever $r \geq \|\mathfrak{X}, \mathfrak{Z}\|^-_1$, $s \geq \|\mathfrak{Y}, \mathfrak{Z}\|^-_1$ and $r+s \geq \|\mathfrak{X}+\mathfrak{Y}, \mathfrak{Z}\|^-_1$.

In the sequel we take $\mathfrak{L}(p,q) = \min\{p,q\}$ and $\mathfrak{R}(p,q) = \max\{p,q\}$, for all $p, q \in [0,1]$ and write $(\mathfrak{E}, \|\cdot,\cdot\|)$ or simply $\mathfrak{E}$, for such $\mathfrak{L}$ and $\mathfrak{R}$.

For $\mathfrak{X} \in \mathfrak{E}$, $\varepsilon > 0$ and $\alpha \in [0,1]$, the (ε,α)*-neighborhood of* $\mathfrak{X}$ is the set

$$\mathfrak{U}_{\mathfrak{X}}(\varepsilon,\alpha) = \{\mathfrak{Y} \in \mathfrak{E} : \|\mathfrak{X} - \mathfrak{Y}, \mathfrak{Z}\|^+_\alpha < \varepsilon, \text{ for all } \mathfrak{Z} \in \mathfrak{E}\}.$$

The (ε,α)*-neighborhood system at* $\mathfrak{X}$ is the collection

$$\mathfrak{U}_{\mathfrak{X}} = \{\mathfrak{U}_{\mathfrak{X}}(\varepsilon,\alpha) : \varepsilon > 0, \alpha \in [0,1]\}$$

and the (ε,α)*-neighborhood system for* $\mathfrak{E}$ is the union $\mathfrak{U} = \bigcup_{\mathfrak{X}\in\mathfrak{E}} \mathfrak{U}_{\mathfrak{X}}$. It is easy to see that $\mathfrak{U}$ generates a *first countable Hausdorff topology* on $\mathfrak{E}$.

Definition 2.4. Let $(\mathfrak{E}, \|\cdot,\cdot\|)$ be a fuzzy 2-norm space. A sequence $\{\mathfrak{X}_k\}$ in $\mathfrak{E}$ is said to be *convergent to* $\mathfrak{X}_0 \in \mathfrak{E}$ *with respect to the norm on* $\mathfrak{E}$, and we denote this by $\mathfrak{X}_k \to \mathfrak{X}_0$, provided $\overline{\varrho} - \lim_{k\to\infty} \|\mathfrak{X}_k - \mathfrak{X}_0, \mathfrak{Z}\| = \overline{0}$ for all $\mathfrak{Z} \in \mathfrak{E}$, i.e., for every $\varepsilon > 0$ there exists an integer $k_0 = k_0(\varepsilon)$ in $\mathbb{N}$ such that $\overline{\varrho}(\|\mathfrak{X}_k - \mathfrak{X}_0, \mathfrak{Z}\|, \overline{0}) < \varepsilon$, for $k \geq k_0$.

This can be restated as follows: For any given $\varepsilon > 0$, there exists an integer $k_0(\varepsilon)$ in the set of natural numbers $\mathbb{N}$ such that, for all $k \geq k_0$, it holds that

$$\sup_{\alpha\in[0,1]} \|\mathfrak{X}_k - \mathfrak{X}_0, \mathfrak{Z}\| \alpha^+ = \|\mathfrak{X}_k - \mathfrak{X}_0, \mathfrak{Z}\| 0^+ < \varepsilon.$$

Regarding neighborhoods, the convergence $\mathfrak{X}_k \to \mathfrak{X}_0$ is assured if, for any given $\varepsilon > 0$, there exists an integer $k_0(\varepsilon)$ in the set of natural numbers $\mathbb{N}$ such that $\mathfrak{X}_k$ belongs to the neighborhood $\mathfrak{U}_{\mathfrak{X}_0}(\varepsilon, 0)$ for all $k \geq k_0$ and for all $\mathfrak{Z} \in \mathfrak{E}$.

Now, we provide some fundamental information concerning classical concepts of ideals and filters.

Definition 2.5. [32] *A non-empty set $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is termed an "ideal" when it satisfies two properties: additivity (i.e., if $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$) and heredity (i.e., if $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$). An ideal $\mathcal{I}$ is considered "non-trivial" if it is not equal to the entire power set $2^{\mathbb{N}}$. If a non-trivial ideal $\mathcal{I}$ includes every finite subset of $\mathbb{N}$, it is designated as an "admissible" ideal. For any given non trivial ideal, there exists a corresponding filter denoted as $\mathcal{F}(\mathcal{I})$, which can be defined as follows:*

$$\mathcal{F}(\mathcal{I}) = \{K \subseteq \mathbb{N} : \mathbb{N} \setminus K \in \mathcal{I}\}.$$

In what follows the symbol $\mathcal{I}_2$ denotes an ideal on $\mathbb{N} \times \mathbb{N}$, and $(\mathfrak{E}, \|\cdot, \cdot\|)$ is a fuzzy 2-normed space.

3. Rough ideal convergence in fuzzy 2-normed linear space

In this section, we present the notions of rough $\mathcal{I}_2^{\mathfrak{E}}$-convergence and rough $\mathcal{I}_2^{*,\mathfrak{E}}$-convergence for a double sequence within the framework of a fuzzy 2-normed space denoted as $(\mathfrak{E}, \|\cdot, \cdot\|)$. We also offer some foundational insights into this type of convergence. Additionally, we introduce the concepts of rough $(\mathfrak{E}, \|\cdot, \cdot\|)$-limit point and rough $(\mathfrak{E}, \|\cdot, \cdot\|)$-cluster point for a double sequence with respect to an ideal $\mathcal{I}_2 \subseteq 2^{\mathbb{N}\times\mathbb{N}}$.

We commence with the following definition.

Definition 3.1. Let r be a nonnegative real number. A double sequence denoted as $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$, residing within a fuzzy 2-normed space $(\mathfrak{E}, \|\cdot, \cdot\|)$, is considered to exhibit "rough $\mathfrak{E}$-convergence to $\mathfrak{X}_0$" if, for any $\varepsilon > 0$ and for each $\mathfrak{Z} \in \mathfrak{E}$, there exists a positive integer $n_0 = n_0(\varepsilon)$ such that:

$$\mathfrak{X}_{jk}, \mathfrak{Z} \in \mathfrak{U}_{\mathfrak{X}_0}(r + \varepsilon, 0) \text{ for each } j, k \geq n_0.$$

In this scenario, it is denoted as $r\mathfrak{E}\text{-}\lim \|\mathfrak{X}_{jk} - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ = 0$.

This can equivalently be expressed as follows: For any given $\varepsilon > 0$, there exists an integer $N_0(\varepsilon)$ in the set of natural numbers $\mathbb{N}$ such that:

$$\|\mathfrak{X}_{jk} - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ < r + \varepsilon \text{ for all } j, k \geq N_0.$$

Definition 3.2. Let $(\mathfrak{E}, \|\cdot, \cdot\|)$ be a fuzzy 2-normed space, r be a nonnegative real number and $\mathcal{I}_2$ an ideal on $\mathbb{N} \times \mathbb{N}$. A double sequence $\{\mathfrak{X}_{jk}\}$ in $\mathfrak{E}$ is said to be rough $\mathcal{I}_2^{\mathfrak{E}}$*-convergent to $\mathfrak{X}_0 \in \mathfrak{E}$ with respect to the fuzzy 2-norm on* $\mathfrak{E}$ if for each $\varepsilon > 0$ and each $\mathfrak{Z} \in \mathfrak{E}$, the set $A(r, \varepsilon) := \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|\mathfrak{X}_{jk} - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ \geq r + \varepsilon\}$ belongs to $\mathcal{I}_2$.

In this case, we write $\mathfrak{X}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{X}_0$. In general, it is important to note that the rough $\mathcal{I}_2$-limit of a sequence $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$ may not have a unique value when considering a roughness degree $r > 0$. To address this, we introduce the notation:

$$\mathcal{I}_2 - \mathrm{LIM}_{\mathfrak{X}}^r = \left\{\mathfrak{X}_0 \in \mathfrak{E} : \mathfrak{X} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{X}_0\right\}.$$

We say that the sequence $\mathfrak{X}$ is "$r\mathcal{I}_2$-convergent" if and only if $\mathcal{I}_2 - \mathrm{LIM}_{\mathfrak{X}}^r$ is not an empty set. The element $\mathfrak{X}_0$ is called the *rough $\mathcal{I}_2^{\mathfrak{E}}$-limit* of $\{\mathfrak{X}_{jk}\}$ in $\mathfrak{E}$.

Remark 3.1. *The $\mathcal{I}_2$-convergence of a sequence $\{\mathfrak{X}_{jk}\}$ in $\mathfrak{E}$ doesn't necessarily guarantee the existence of a sequence $\{\mathfrak{Y}_{jk}\}$ in $\mathfrak{E}$ that is $\mathcal{I}_2$-convergent and fulfills the condition that*

$\left\{(j,k)\in\mathbb{N}^2 : \|\mathfrak{X}_{jk}-\mathfrak{Y}_{jk},\mathfrak{Z}\|_0^+ \geq r\right\}\in\mathcal{I}_2$ *for every non-zero* $\mathfrak{Z}\in\mathfrak{E}$. *For any* $\varepsilon>0$ *and a non-zero* $\mathfrak{Z}\in\mathfrak{E}$, *the following relationship holds:*

$$\begin{aligned}\left\{(j,k)\in\mathbb{N}^2 : \|\mathfrak{X}_{jk}-\mathfrak{X}_0,\mathfrak{Z}\|_0^+ \geq r+\varepsilon\right\} &\subset \left\{(j,k)\in\mathbb{N}^2; \|\mathfrak{X}_{jk}-\mathfrak{Y}_{jk},\mathfrak{Z}\|_0^+ \geq r\right\}\\ &\cup \left\{(j,k)\in\mathbb{N}^2 : \|\mathfrak{Y}_{jk}-\mathfrak{X}_0,\mathfrak{Z}\|_0^+ \geq \varepsilon\right\}.\end{aligned}$$

Theorem 3.1. *Consider a fuzzy 2-normed space denoted as* $(\mathfrak{E},\|\cdot,\cdot\|)$, *and let* r *be a nonnegative real number. For a given sequence* $\mathfrak{X}=\{\mathfrak{X}_{jk}\}$ *within* $\mathfrak{E}$, *it can be observed that the diameter of* $\mathcal{I}_2-\mathrm{LIM}_{\mathfrak{X}}^r$ *is bounded by* $2r$, *i.e., diam*$(\mathcal{I}_2-\mathrm{LIM}_{\mathfrak{X}}^r)\leq 2r$. *However, it should be noted that in general, there is no smaller bound for the diameter of* $\mathcal{I}_2-\mathrm{LIM}_{\mathfrak{X}}^r$.

Proof. Assume that *diam*$(\mathcal{I}_2-\mathrm{LIM}_{\mathfrak{X}}^r)>2r$. Then there exists $\mathfrak{X}_1,\mathfrak{X}_2\in\mathcal{I}_2-\mathrm{LIM}_{\mathfrak{X}}^r$ such that $\|\mathfrak{X}_1-\mathfrak{X}_2,\mathfrak{Z}\|_0^+>2r$ for each $\mathfrak{Z}\in\mathfrak{E}$. Choose $0<\varepsilon<\dfrac{\|\mathfrak{X}_1-\mathfrak{X}_2,\mathfrak{Z}\|_0^+}{2}-r$.

$$\begin{aligned}A_1(r,\varepsilon) &= \left\{(j,k)\in\mathbb{N}\times\mathbb{N} : \|\mathfrak{X}_{jk}-\mathfrak{X}_1,\mathfrak{Z}\|_0^+\geq r+\varepsilon\right\} \text{ and}\\ A_2(r,\varepsilon) &= \left\{(j,k)\in\mathbb{N}\times\mathbb{N} : \|\mathfrak{X}_{jk}-\mathfrak{X}_2,\mathfrak{Z}\|_0^+\geq r+\varepsilon\right\}.\end{aligned}$$

Then $A_1(r,\varepsilon),A_2(r,\varepsilon)\in\mathcal{I}_2$ and hence $B=\mathbb{N}\times\mathbb{N}\setminus(A_1(r,\varepsilon)\cup A_2(r,\varepsilon))\in F(\mathcal{I}_2)$ and so $B\neq\emptyset$. Let $(j,k)\in B$. Then $(j,k)\notin A_1(r,\varepsilon)$ and $(j,k)\notin A_2(r,\varepsilon)$ and so we have $\|\mathfrak{X}_{jk}-\mathfrak{X}_1,\mathfrak{Z}\|_0^+<r+\varepsilon$ and $\|\mathfrak{X}_{jk}-\mathfrak{X}_2,\mathfrak{Z}\|_0^+<r+\varepsilon$. Consequently,

$$\|\mathfrak{X}_1-\mathfrak{X}_2,\mathfrak{Z}\|_0^+\leq\|\mathfrak{X}_{jk}-\mathfrak{X}_1,\mathfrak{Z}\|_0^++\|\mathfrak{X}_{jk}-\mathfrak{X}_2,\mathfrak{Z}\|_0^+<2(r+\varepsilon)<\|\mathfrak{X}_1-\mathfrak{X}_2,\mathfrak{Z}\|_0^+,$$

which is a contradiction. Thus *diam*$(\mathcal{I}_2-\mathrm{LIM}_{\mathfrak{X}}^r)\leq 2r$.

To prove the converse part, consider a sequence $\mathfrak{X}=\{\mathfrak{X}_{jk}\}$ such that $\mathcal{I}_2$-lim $\mathfrak{X}=\mathfrak{X}_0$. Let $\varepsilon>0$ be given. Then for each $\mathfrak{Z}\in\mathfrak{E}$, the set

$$A=\left\{(j,k)\in\mathbb{N}\times\mathbb{N} : \|\mathfrak{X}_{jk}-\mathfrak{X}_0,\mathfrak{Z}\|_0^+\geq\varepsilon\right\}\in\mathcal{I}_2.$$

Now for each $\mathfrak{Q}\in\overline{B}^r=\{\mathfrak{Q}\in\mathfrak{E} : \|\mathfrak{Q}-\mathfrak{X}_0,\mathfrak{Z}\|_0^+\leq r\}$, we have

$$\|\mathfrak{X}_{jk}-\mathfrak{Q},\mathfrak{Z}\|_0^+\leq\|\mathfrak{X}_{jk}-\mathfrak{X}_0,\mathfrak{Z}\|_0^++\|\mathfrak{X}_0-\mathfrak{Q},\mathfrak{Z}\|_0^+<r+\varepsilon$$

whenever $(j,k)\notin A$. This shows that $\mathfrak{Q}\in\mathcal{I}_2-\mathrm{LIM}_{\mathfrak{X}}^r$ and hence we can write $\mathcal{I}_2-\mathrm{LIM}_{\mathfrak{X}}^r=\overline{B}^r(\mathfrak{X}_0)$. This shows that in general upper bound $2r$ of the diameter of the set $\mathcal{I}_2-\mathrm{LIM}_{\mathfrak{X}}^r$ can not be decreased anymore. ■

Remark 3.2. *Let* $(\mathfrak{E},\|\cdot,\cdot\|)$ *be a fuzzy 2-normed linear space and* r *be a nonnegative real number. Then*

(i) *In terms of neighborhoods, we have* $\mathfrak{X}_{jk}\xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}}\mathfrak{X}_0$, *provided that for each* $\varepsilon>0$ *and* $\mathfrak{Z}\in\mathfrak{E}$,

$$\{(j,k)\in\mathbb{N}\times\mathbb{N} : \mathfrak{X}_{jk},\mathfrak{Z}\notin\mathfrak{U}_{\mathfrak{X}_0}(r+\varepsilon,0)\}\in\mathcal{I}_2.$$

The above definition can be expressed also in the following way:

$$\mathfrak{X}_{jk}\xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}}\mathfrak{X}_0\Longleftrightarrow r\mathcal{I}_2^{\mathfrak{E}}-\lim_{j,k\to\infty}\|\mathfrak{X}_{jk}-\mathfrak{X}_0,\mathfrak{Z}\|_0^+=0,\ \textit{for all } \mathfrak{Z}\in\mathfrak{E}.$$

(ii) *Note that* $r\mathcal{I}_2^{\mathfrak{E}}-\lim\limits_{j,k\to\infty}\|\mathfrak{X}_{jk}-\mathfrak{X}_0,\mathfrak{Z}\|_0^+=0$, *for all* $\mathfrak{Z}\in\mathfrak{E}$ *implies*

$$r\mathcal{I}_2^{\mathfrak{E}}-\lim\|\mathfrak{X}_{jk}-\mathfrak{X}_0,\mathfrak{Z}\|_\alpha^-=r\mathcal{I}_2^{\mathfrak{E}}-\lim\|\mathfrak{X}_{jk}-\mathfrak{X}_0,\mathfrak{Z}\|_\alpha^+$$

for each $\alpha\in[0,1]$ *and each* $\mathfrak{Z}\in\mathfrak{E}$. *(It is because* $0\leq\|\mathfrak{X}_{jk}-\mathfrak{X}_0,\mathfrak{Z}\|_\alpha^-\leq\lim\|\mathfrak{X}_{jk}-\mathfrak{X}_0,\mathfrak{Z}\|_\alpha^+\leq\|\mathfrak{X}_{jk}-\mathfrak{X}_0,\mathfrak{Z}\|_0^+$, *holds for each* $(j,k)\in\mathbb{N}\times\mathbb{N}$, $\alpha\in[0,1]$ *and each* $\mathfrak{Z}\in\mathfrak{E}$.*)*

Proposition 3.1. *Let $(\mathfrak{E}, \|\cdot,\cdot\|)$ be a fuzzy 2-normed linear space and r be a nonnegative real number. Then we have*

(i) *If $r\mathfrak{E}\text{-}\lim \|\mathfrak{X}_{jk} - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ = 0$, $r\mathcal{I}_2^{\mathfrak{E}}\text{-}\lim \|\mathfrak{X}_{jk} - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ = 0$;*

(ii) *If $\mathfrak{X}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{X}_0$ and $\mathfrak{Y}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{Y}_0$, $\mathfrak{X}_{jk} + \mathfrak{Y}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{X}_0 + \mathfrak{Y}_0$;*

(iii) *If $\mathfrak{X}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{X}_0$ and $c \in \mathbb{R}$, $c\mathfrak{X}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} c\mathfrak{X}_0$;*

(iv) *If $\mathfrak{X}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{X}_0$ and $\mathfrak{Y}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{Y}_0$, $\mathfrak{X}_{jk}\mathfrak{Y}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{X}_0\mathfrak{Y}_0$;*

(v) *If $\mathfrak{X}_{jk} \preceq \mathfrak{Y}_{jk} \preceq \mathfrak{Z}_{jk}$ for all $(j,k) \in \mathbb{N}\times\mathbb{N}$ belonging to the set $B \in \mathcal{F}(\mathcal{I}_2)$, and $\mathfrak{X}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{X}_0$ and $\mathfrak{Z}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{X}_0$, $\mathfrak{Y}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{X}_0$.*

Proof. (i) Assume that $r\mathfrak{E}\text{-}\lim \|\mathfrak{X}_{jk} - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ = 0$. Take any $\varepsilon > 0$, and let $\mathfrak{Z}$ be any nonzero element in $\mathfrak{E}$. We can then find a positive integer $N_0 \in \mathbb{N}$ such that $\|\mathfrak{X}jk - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ < r + \varepsilon$ for all pairs of indices j and k greater than or equal to N_0. This holds true because

$$A = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \|\mathfrak{X}_{jk} - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ \geq \varepsilon + r\} \subseteq \{1, 2, \cdots, N_0 - 1\} \times \{1, 2, \cdots, N_0 - 1\}$$

and the ideal $\mathcal{I}_2$ is admissible, we have $A \in \mathcal{I}_2$. This shows that $r\mathcal{I}_2^{\mathfrak{E}}\text{-}\lim \|\mathfrak{X}_{jk} - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ = 0$.

(ii) Suppose that $\mathfrak{X}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{X}_0$ and $\mathfrak{Y}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{Y}_0$. Since $\|\cdot,\cdot\|_0^+$ is a 2-norm in the usual sense, we get

$$\|(\mathfrak{X}_{jk} + \mathfrak{Y}_{jk}) - (\mathfrak{X}_0 + \mathfrak{Y}_0), \mathfrak{Z}\|_0^+ \leq \|\mathfrak{X}_{jk} - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ + \|\mathfrak{Y}_{jk} - \mathfrak{Y}_0, \mathfrak{Z}\|_0^+ \tag{1}$$

for all $(j,k) \in \mathbb{N}\times\mathbb{N}$. Put

$B(r,\varepsilon) = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \|(\mathfrak{X}_{jk} + \mathfrak{Y}_{jk}) - (\mathfrak{X}_0 + \mathfrak{Y}_0), \mathfrak{Z}\|_0^+ \geq 2r + 2\varepsilon\}$,
$A_1(r,\varepsilon) = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \|\mathfrak{X}_{jk} - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ \geq r + \varepsilon\}$,
$A_2(r,\varepsilon) = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \|\mathfrak{Y}_{jk} - \mathfrak{Y}_0, \mathfrak{Z}\|_0^+ \geq r + \varepsilon\}$.

By assumption, we have that $A_1(r,\varepsilon)$ and $A_2(r,\varepsilon)$ belong to $\mathcal{I}_2$. So, $A_1(r,\varepsilon) \cup A_2(r,\varepsilon) \in \mathcal{I}_2$. From (1) it follows that $B(r,\varepsilon) \subseteq A_1(r,\varepsilon) \cup A_2(r,\varepsilon)$. This implies that $B(r,\varepsilon) \in \mathcal{I}_2$. Consequently, $\mathfrak{X}_{jk} + \mathfrak{Y}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{X}_0 + \mathfrak{Y}_0$.

(iii) Let $c \in \mathbb{R}$. If $c = 0$, we have nothing to prove, so we assume that $c \neq 0$. Let $\varepsilon > 0$ be given. Since $\|\cdot,\cdot\|_0^+$ is a 2-norm in the usual sense, $\|c\mathfrak{X}_{jk}, \mathfrak{Z}\|_0^+ = |c| \, \|\mathfrak{X}_{jk}, \mathfrak{Z}\|_0^+$.

Since $\mathfrak{X}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{X}_0$, we have

$$A(r,\varepsilon) = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \|\mathfrak{X}_{jk} - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ \geq r + \varepsilon\} \in \mathcal{I}_2.$$

Let $A_1(r,\varepsilon) = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \|c\mathfrak{X}_{jk} - c\mathfrak{X}_0, \mathfrak{Z}\|_0^+ \geq r+\varepsilon\}$. We need to show that $A_1(r,\varepsilon)$ is contained in $A(r_1, \varepsilon_1)$. Let $(p,q) \in A_1(r,\varepsilon)$, then $r+\varepsilon \leq \|c\mathfrak{X}_{pq} - c\mathfrak{X}_0\|_0^+ = |c| \, \|\mathfrak{X}_{pq} - \mathfrak{X}_0\|_0^+$. This implies that $\|\mathfrak{X}_{pq} - \mathfrak{X}_0\|_0^+ \geq \dfrac{\varepsilon + r}{|c|} = r_1 + \varepsilon_1$. Therefore $(p,q) \in A(r_1, \varepsilon_1)$. Then we have $A_1(r,\varepsilon) \subset A(r_1,\varepsilon_1)$. By the definition of the ideal, we get $A_1(r,\varepsilon) \in \mathcal{I}_2$ and hence $c\mathfrak{X}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} c\mathfrak{X}_0$.

(iv) Since $\mathfrak{X}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{X}_0$, we have

$$A(1) = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \|\mathfrak{X}_{jk} - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ < 1\} \in \mathcal{F}(\mathcal{I}_2).$$

Now $\|\cdot,\cdot\|_0^+$ is a 2-norm in the usual sense, so that

$$\|\mathfrak{X}_{jk}\mathfrak{Y}_{jk} - \mathfrak{X}_0\mathfrak{Y}_0, \mathfrak{Z}\|_0^+ \leq \|\mathfrak{X}_{jk}, \mathfrak{Z}\|_0^+ \, \|\mathfrak{Y}_{jk} - \mathfrak{Y}_0, \mathfrak{Z}\|_0^+ + \|\mathfrak{Y}_0, \mathfrak{Z}\|_0^+ \, \|\mathfrak{X}_{jk} - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ .$$

For $(j,k) \in A(1)$, we have $\|\mathfrak{X}_{jk}, \mathfrak{Z}\|_0^+ \leq \|\mathfrak{X}_0, \mathfrak{Z}\|_0^+ + 1$ and it follows that

$$\|\mathfrak{X}_{jk}\mathfrak{Y}_{jk} - \mathfrak{X}_0\mathfrak{Y}_0, \mathfrak{Z}\|_0^+ \leq \left(\|\mathfrak{X}_0, \mathfrak{Z}\|_0^+ + 1\right) \|\mathfrak{Y}_{jk} - \mathfrak{Y}_0, \mathfrak{Z}\|_0^+ + \|\mathfrak{Y}_0, \mathfrak{Z}\|_0^+ \|\mathfrak{X}_{jk} - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ . \quad (2)$$

Let $\varepsilon > 0$ be given. Choose $\gamma > 0$ such that

$$0 < \gamma < \frac{\varepsilon - r\left(\|\mathfrak{Y}_0, \mathfrak{Z}\|_0^+ + \|\mathfrak{X}_0, \mathfrak{Z}\|_0^+\right)}{\|\mathfrak{Y}_0, \mathfrak{Z}\|_0^+ + \|\mathfrak{X}_0, \mathfrak{Z}\|_0^+ + 1} \quad (3)$$

Since $\mathfrak{X}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{X}_0$ and $\mathfrak{Y}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{Y}_0$, the sets

$$A_1(r,\gamma) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|\mathfrak{X}_{jk} - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ < r + \gamma\}$$

and

$$A_2(r,\gamma) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|\mathfrak{Y}_{jk} - \mathfrak{Y}_0, \mathfrak{Z}\|_0^+ < r + \gamma\}$$

belong to $\mathcal{F}(\mathcal{I}_2)$.

Obviously, $A(1) \cap A_1(r,\gamma) \cap A_2(r,\gamma) \in \mathcal{F}(\mathcal{I}_2)$ and for each $(j,k) \in A(1) \cap A_1(r,\gamma) \cap A_2(r,\gamma)$, we have from (2) and (3),

$$\|\mathfrak{X}_{jk}\mathfrak{Y}_{jk} - \mathfrak{X}_0\mathfrak{Y}_0\|_0^+ < r + \varepsilon.$$

This implies that $\{(j,k) \in \mathbb{N}\times\mathbb{N} : \|\mathfrak{X}_{jk} \cdot \mathfrak{Y}_{jk} - \mathfrak{X}_0 \cdot \mathfrak{Y}_0\|_0^+ \geq r+\varepsilon\} \in \mathcal{I}_2$, i.e., $\mathfrak{X}_{jk} \cdot \mathfrak{Y}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{X}_0 \cdot \mathfrak{Y}_0$.

(v) Let $\varepsilon > 0$ and $\mathfrak{W} \in \mathfrak{E}$ be given. From $\mathfrak{X}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{X}_0$ it follows

$$A_1(r,\varepsilon) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|\mathfrak{X}_{jk} - \mathfrak{X}_0, \mathfrak{W}\|_0^+ \geq r + \varepsilon\} \in \mathcal{I}_2,$$

and from $\mathfrak{Z}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{X}_0$ it follows

$$A_2(r,\varepsilon) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|\mathfrak{Z}_{jk} - \mathfrak{X}_0, \mathfrak{W}\|_0^+ \geq r + \varepsilon\} \in \mathcal{I}_2.$$

We shall prove

$$C := \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|\mathfrak{Y}_{jk} - \mathfrak{X}_0, \mathfrak{W}\|_0^+ \geq r + \varepsilon\} \subset A_1(r,\varepsilon) \cup A_2(r,\varepsilon) \cup (\mathbb{N} \times \mathbb{N} \setminus B).$$

Let $(m,n) \in C$. If $(m,n) \in \mathbb{N} \times \mathbb{N} \setminus B$, then $(m,n) \in A_1(r,\varepsilon) \cup A_2(r,\varepsilon) \cup (\mathbb{N} \times \mathbb{N} \setminus B)$. Assume now $(m,n) \in B$. Then $\|\mathfrak{Y}_{pq} - \mathfrak{X}_0, \mathfrak{W}\|_0^+ \geq r + \varepsilon$. Since $\mathfrak{Z}_{mn} \succeq \mathfrak{Y}_{mn}$ we have $\|\mathfrak{Z}_{mn} - \mathfrak{X}_0, \mathfrak{W}\|_0^+ \geq r+\varepsilon$, hence $(m,n) \in A_2(r,\varepsilon)$. Therefore, $(m,n) \in A_1(r,\varepsilon) \cup A_2(r,\varepsilon) \cup (\mathbb{N} \times \mathbb{N} \setminus B)$. Since the last set is in $\mathcal{I}_2$, we get $C \in \mathcal{I}_2$, i.e., $\mathfrak{Y}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{X}_0$. ■

Lemma 3.1. [21] *Let $\mathcal{I}_2$ be an admissible ideal with the property* (AP). *If $\{P_j\}_{j=1}^{\infty}$ is a countable collection of subsets of $\mathbb{N} \times \mathbb{N}$ such that $P_j \in \mathcal{F}(\mathcal{I}_2)$ for each j, then there exists a set $P \subset \mathbb{N} \times \mathbb{N}$ such that $P \in \mathcal{F}(\mathcal{I}_2)$ and the set $P \setminus P_j$ is finite for all j.*

Theorem 3.2. *Let $\mathcal{I}_2$ be an admissible ideal with the property* (AP) *and r be a nonnegative real number. Let $(\mathfrak{E}, \|\cdot,\cdot\|)$ be a fuzzy 2-normed space and $\{\mathfrak{X}_{jk}\}$ be a double sequence in $\mathfrak{E}$. Then $\{\mathfrak{X}_{jk}\}$ is an $r\mathcal{I}_2^{\mathfrak{E}}$-convergent sequence in $\mathfrak{E}$ if and only if there is an $r\mathfrak{E}$-convergent double sequence $\{\mathfrak{Y}_{jk}\}$ such that $\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mathfrak{X}_{jk} \neq \mathfrak{Y}_{jk}\} \in \mathcal{I}_2$.*

Proof. Suppose $\mathfrak{X}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{X}_0$. For each $n \in \mathbb{N}$ and a non-zero $\mathfrak{Z} \in \mathfrak{E}$, let

$$A_n = \left\{(j,k) \in \mathbb{N} \times \mathbb{N} : \|\mathfrak{X}_{jk} - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ < r + \frac{1}{n}\right\}.$$

Then $A_n \in \mathcal{F}(\mathcal{I}_2)$ for each $n \in \mathbb{N}$.

Since $\mathcal{I}_2$ is admissible ideal with the property (AP), by Lemma 3.1 there exists $A \subset \mathbb{N}\times\mathbb{N}$ such that $A \in \mathcal{F}(\mathcal{I})$ and the set $A \setminus A_n$ is finite for each n. Observe that $\mathfrak{X}_{jk} \xrightarrow{r}{}_{(A)}\mathfrak{X}_0$, i.e., for each $\varepsilon > 0$, there exists an integer $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $j,k \geq n_0$ and $(j,k) \in A$ implies $\|\mathfrak{X}_{jk} - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ < r + \varepsilon$.

Define a sequence $\{\mathfrak{Y}_{jk}\}$ in $\mathfrak{E}$ as

$$\mathfrak{Y}_{jk} = \begin{cases} \mathfrak{X}_{jk}, & \text{for } (j,k) \in A; \\ \mathfrak{X}_0, & \text{for } (j,k) \in (\mathbb{N}\times\mathbb{N}) \setminus A. \end{cases}$$

The sequence $\{\mathfrak{Y}_{jk}\}$ is $r\mathfrak{E}$-convergent to $\mathfrak{X}_0$ with respect to the fuzzy norm on $\mathfrak{E}$. Thus we have $\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mathfrak{X}_{jk} \neq \mathfrak{Y}_{jk}\} \in \mathcal{I}_2$.

Next suppose that $\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mathfrak{X}_{jk} \neq \mathfrak{Y}_{jk}\} \in \mathcal{I}_2$ and $\mathfrak{Y}_{jk} \xrightarrow{r} \mathfrak{X}_0$. Let $\varepsilon > 0$ be given. Then for each n and a non-zero $\mathfrak{Z} \in \mathfrak{E}$, we can write

$$\begin{aligned} &\{j,k \leq n : \|\mathfrak{Y}_{jk} - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ \geq r + \varepsilon\} \subseteq \\ &\{j,k \leq n : \mathfrak{X}_{jk} \neq \mathfrak{Y}_{jk}\} \cup \{j,k \leq n : \|\mathfrak{X}_{jk} - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ > r + \varepsilon\}. \end{aligned} \tag{4}$$

Since first set on the right side of (4) belongs to $\mathcal{I}_2$, and the second set is contained in a fixed number of integers and thus belongs to $\mathcal{I}_2$, we conclude that $\{(j,k) : j,k \leq n, \|\mathfrak{X}_{jk} - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ \geq r + \varepsilon\}$ belongs to $\mathcal{I}_2$. This achieves the proof. ■

Now we prove a *decomposition theorem for rough* $\mathcal{I}_2^{\mathfrak{E}}$-convergent sequences.

Theorem 3.3. *Let $\{\mathfrak{X}_{jk}\}$ be a double sequence in a fuzzy 2-normed space $(\mathfrak{E}, \|\cdot,\cdot\|)$, r be a nonnegative real number and $\mathcal{I}_2$ be an admissible ideal. If there exist two sequences $\{\mathfrak{Y}_{jk}\}$ and $\{\mathfrak{Z}_{jk}\}$ in $\mathfrak{E}$ such that $\mathfrak{X}_{jk} = \mathfrak{Y}_{jk} + \mathfrak{Z}_{jk}$; $\mathfrak{Y}_{jk}$ $r\mathfrak{E}$-converges to $\mathfrak{X}_0$ and* $\operatorname{supp}(\mathfrak{Z}_{jk}) = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \mathfrak{Z}_{jk} \neq \theta\} \in \mathcal{I}_2$, *then $\mathfrak{X}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{X}_0$.*

Proof. Let $\{\mathfrak{Y}_{jk}\}$ and $\{\mathfrak{Z}_{jk}\}$ be double sequences in $\mathfrak{E}$ as in the statement of the theorem and $H = \operatorname{supp}(\mathfrak{Z}_{jk})$. Let $\varepsilon > 0$ and $\mathfrak{W} \in \mathfrak{E}$ be given. Since $A_1 = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \|\mathfrak{Z}_{jk} - \overline{0}, \mathfrak{W}\|_0^+ \geq r + \varepsilon/2\} \subset \operatorname{supp}(\mathfrak{Z}_{jk}) = H$, we have $A_1 \in \mathcal{I}_2$. Further,

$$\|\mathfrak{X}_{jk} - \mathfrak{X}_0, \mathfrak{W}\|_0^+ = \|\mathfrak{Y}_{jk} + \mathfrak{Z}_{jk} - \overline{0} - \mathfrak{X}_0, \mathfrak{W}\|_0^+ \leq \|\mathfrak{Y}_{jk} - \mathfrak{X}_0, \mathfrak{W}\|_0^+ + \|\mathfrak{Z}_{jk} - \overline{0}, \mathfrak{W}\|_0^+$$

implies

$$\begin{aligned} \{(j,k) \in \mathbb{N}\times\mathbb{N} : \|\mathfrak{X}_{jk} - \mathfrak{X}_0, \mathfrak{W}\|_0^+ < 2r + \varepsilon\} \supset \{(j,k) \in \mathbb{N}\times\mathbb{N} : \|\mathfrak{Y}_{jk} - \mathfrak{X}_0, \mathfrak{W}\|_0^+ < r + \varepsilon/2\} \\ \cap \{(j,k) \in \mathbb{N}^2 : \|\mathfrak{Z}_{jk} - \overline{0}\|_0^+ < r + \varepsilon/2\}. \end{aligned}$$

The sets on the right side are both in $\mathcal{F}(\mathcal{I}_2)$, so that the set on the left side is also in $\mathcal{F}(\mathcal{I}_2)$. Therefore, $\{(j,k) \in \mathbb{N}^2 : \|\mathfrak{X}_{jk} - \mathfrak{X}_0, \mathfrak{W}\|_0^+ \geq 2r + \varepsilon\} \in \mathcal{I}_2$, i.e., $\mathfrak{X}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{X}_0$. ■

Definition 3.3. *Let $(\mathfrak{E}, \|\cdot,\cdot\|$ be a fuzzy 2-normed space. We say that a double sequence $\{\mathfrak{X}_{jk}\}$ in $\mathfrak{E}$ is rough $\mathcal{I}_2^{*,\mathfrak{E}}$*-convergent *to $\mathfrak{X}_0 \in \mathfrak{E}$ with respect to the 2-norm on $\mathfrak{E}$ if there exists a subset*

$$K = \{(j_m, k_m) : j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots\} \subset \mathbb{N}\times\mathbb{N}$$

such that $K \in \mathcal{F}(\mathcal{I}_2)$ and $r\mathfrak{E}\text{-}\lim_{m\to\infty} \|\mathfrak{X}_{j_m k_m} - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ = 0$ for each non-zero $\mathfrak{Z} \in \mathfrak{E}$.

In this case, we write $\mathfrak{X}_{jk} \xrightarrow{r\mathcal{I}_2^{*,\mathfrak{E}}} \mathfrak{X}_0$.

Remark 3.3. *Based on the Definition 3.3, we can construct the following set*

$$\mathcal{I}_2^* - \mathrm{LIM}_{\mathfrak{X}}^r = \left\{ \mathfrak{X}_0 : \mathfrak{X}_{jk} \xrightarrow{r\mathcal{I}_2^{*,\mathfrak{E}}} \mathfrak{X}_0 \right\}.$$

Hence, we have

$$\mathcal{I}_2^* - \mathrm{LIM}_{\mathfrak{X}}^r \subseteq \mathcal{I}_2 - \mathrm{LIM}_{\mathfrak{X}}^r.$$

Theorem 3.4. *Let $(\mathfrak{E}, \|\cdot,\cdot\|)$ be a fuzzy 2-normed space, r be a nonnegative real number and $\mathcal{I}_2$ be an admissible ideal. If $\mathfrak{X}_{jk} \xrightarrow{r\mathcal{I}_2^{*,\mathfrak{E}}} \mathfrak{X}_0$, then $\mathfrak{X}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{X}_0$.*

Proof. Suppose that $\mathfrak{X}_{jk} \xrightarrow{r\mathcal{I}_2^{*,\mathfrak{E}}} \mathfrak{X}_0$. Then by definition, there exists

$$K = \{(j_m, k_m) \in \mathbb{N} \times \mathbb{N} : j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots\} \in \mathcal{F}(\mathcal{I}_2)$$

such that $r\mathfrak{E}\text{-}\lim_{m\to\infty} \|\mathfrak{X}_{j_m k_m} - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ = 0$. Let $\varepsilon > 0$ and non-zero $\mathfrak{Z} \in \mathfrak{E}$ be given. Since $r\mathfrak{E}\text{-}\lim_{m\to\infty} \|\mathfrak{X}_{j_m k_m} - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ = 0$, there exists $n_0 \in \mathbb{N}$ such that $\|\mathfrak{X}_{j_m k_m} - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ < r + \varepsilon$ for every $m \geq n_0$. Since

$$A = \{(j_m, k_m) \in K : \|\mathfrak{X}_{j_m k_m} - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ \geq r + \varepsilon\}$$

is contained in

$$B = \{j_1, j_2, \cdots, j_{n_0 - 1}; k_1, k_2, \cdots, k_{n_0 - 1}\}$$

and the ideal $\mathcal{I}_2$ is admissible, we have $A \in \mathcal{I}_2$. Hence

$$\{(j,k) \in \mathbb{N} \times \mathbb{N} : \|\mathfrak{X}_{j_m k_m} - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ \geq r + \varepsilon\} \subseteq K \cup B \in \mathcal{I}_2$$

for $\varepsilon > 0$ and nonzero $\mathfrak{Z} \in \mathfrak{E}$. Therefore, we conclude that

$$\mathfrak{X}_{jk} \xrightarrow{r\mathcal{I}_2^{\mathfrak{E}}} \mathfrak{X}_0.$$

■

Remark 3.4. *In general, Theorem 3.4 does not hold in the converse. However, if the admissible ideal possesses the (AP) property, the converse becomes true, establishing the equivalence of the two concepts.*

Now, we study the concepts of rough $I_2^{\mathfrak{E}}$-Cauchy and rough $I_2^{*,\mathfrak{E}}$-Cauchy double sequences in $(E, \|\cdot,\cdot\|)$. Moreover, we will study the relations between them. The investigation of I_2-Cauchy and I_2^*-Cauchy double sequences was done in [21].

Definition 3.4. Let $(\mathfrak{E}, \|\cdot,\cdot\|)$ be a fuzzy 2-normed space, r be a non-negative real number and $\mathcal{I}_2$ be an admissible ideal of $\mathbb{N} \times \mathbb{N}$. A double sequence $\{\mathfrak{X}_{jk}\}$ of elements in $\mathfrak{E}$ is said to be

(i) a rough $\mathcal{I}_2^{\mathfrak{E}}$-*Cauchy sequence* in $\mathfrak{E}$ if for every $\varepsilon > 0$ and a nonzero $\mathfrak{Z} \in \mathfrak{E}$, there exist $s = s(\varepsilon), t = t(\varepsilon)$ such that

$$\{(j,k) \in \mathbb{N} \times \mathbb{N} : \|\mathfrak{X}_{jk} - \mathfrak{X}_{st}, \mathfrak{Z}\|_0^+ \geq r + \varepsilon\} \in \mathcal{I}_2.$$

(ii) a rough $\mathcal{I}_2^{*,\mathfrak{E}}$-*Cauchy sequence* in $\mathfrak{E}$ if for every $\varepsilon > 0$ and a nonzero $\mathfrak{Z} \in \mathfrak{E}$, there exists

$$K = \{(j_m, k_m) : j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots\} \subset \mathbb{N} \times \mathbb{N}$$

such that $K \in \mathcal{F}(\mathcal{I}_2)$ and $\{\mathfrak{X}_{j_m k_m}\}$ is an ordinary $\mathfrak{E}$-Cauchy sequence in $\mathfrak{E}$.

The next theorem gives a relation between $\mathcal{I}_2^{\mathfrak{E}}$- and $\mathcal{I}_2^{*\mathfrak{E}}$-double Cauchy sequences.

Theorem 3.5. *Let $(\mathfrak{E}, \|\cdot,\cdot\|)$ be a fuzzy 2-normed space, r be a non-negative real number and $\mathcal{I}_2$ be an admissible ideal of $\mathbb{N}\times\mathbb{N}$. If $\{\mathfrak{X}_{jk}\}$ is a rough $\mathcal{I}_2^{*,\mathfrak{E}}$-double Cauchy sequence, then $\{\mathfrak{X}_{jk}\}$ is a rough $\mathcal{I}_2^{\mathfrak{E}}$-double Cauchy sequence.*

Proof. Since $\{\mathfrak{X}_{jk}\}$ is a rough $\mathcal{I}_2^{*,\mathfrak{E}}$-double Cauchy sequence, for any $\varepsilon > 0$ and any non-zero $\mathfrak{Z} \in \mathfrak{E}$, there exist

$$K = \{(j_m, k_m) : j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots\} \in \mathcal{F}(\mathcal{I}_2)$$

and a number $n_0 \in \mathbb{N}$ such that

$$\left\|\mathfrak{X}_{j_m k_m} - \mathfrak{X}_{j_p kp}, \mathfrak{Z}\right\|_0^+ < r + \varepsilon$$

for every $m, p \geq n_0$. Now, fix $p = j_{n_0+1}, s = k_{n_0+1}$. Then for every $\varepsilon > 0$ and a non-zero $\mathfrak{Z} \in \mathfrak{E}$, we have

$$\|\mathfrak{X}_{j_m k_m} - \mathfrak{X}_{ps}, \mathfrak{Z}\|_0^+ < r + \varepsilon$$

for every $m \geq n_0$. Let $H = \mathbb{N}\times\mathbb{N} \setminus K$. It is obvious that $H \in \mathcal{F}(\mathcal{I}_2)$ and

$$\begin{aligned} A(\varepsilon) &= \{(j,k) \in \mathbb{N}\times\mathbb{N} : \|\mathfrak{X}_{j_m k_m} - \mathfrak{X}_{ps}, \mathfrak{Z}\|_0^+ \geq r + \varepsilon\} \\ &\subset H \cup \{j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots\} \in \mathcal{I}_2. \end{aligned}$$

Therefore, for every $\varepsilon > 0$ and non-zero $\mathfrak{Z} \in \mathfrak{E}$, we can find $(p, s) \in \mathbb{N}\times\mathbb{N}$ such that $A(\varepsilon) \in \mathcal{I}_2$, i.e., $\{\mathfrak{X}_{jk}\}$ is a rough $\mathcal{I}_2^{\mathfrak{E}}$-double Cauchy sequence. ■

4. Rough $\mathcal{I}$-limit points and rough $\mathcal{I}$-cluster points

In this section, our aim is to present the concept of rough $\mathcal{I}$-limit points and rough $\mathcal{I}$-cluster points for sequences of real numbers within the setting of fuzzy 2-normed linear spaces. Additionally, we introduce the idea of a rough $\mathcal{I}_2$-Cauchy sequence in the context of a fuzzy 2-normed space $(\mathfrak{E}, \|\cdot,\cdot\|)$, and then delve into the exploration of important findings derived from this context.

Definition 4.1. *Let $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$ be a double sequence in a fuzzy 2-normed space $(\mathfrak{E}, \|\cdot,\cdot\|)$, r be a non-negative real number and $\mathcal{I}_2$ be an ideal on $\mathbb{N}\times\mathbb{N}$. Then an element $\mathfrak{Y} \in \mathfrak{E}$ is said to be a rough $\mathcal{I}_2$-*cluster point *of $\mathfrak{X}_{jk}$ if for each $\varepsilon > 0$ and a non-zero $\mathfrak{Z} \in \mathfrak{E}$, the set $\{(j,k) \in \mathbb{N}\times\mathbb{N} : \|\mathfrak{X}_{jk} - \mathfrak{Y}, \mathfrak{Z}\|_0^+ < r + \varepsilon\} \notin \mathcal{I}_2$.*

Example 4.1. *Let $\mathfrak{E} = \mathbb{R}^2$ with the 2-norm $\|\cdot,\cdot\|$ as defined in Example 2.1. Consider $\mathcal{I}_2$ as an ideal of $\mathbb{N}\times\mathbb{N}$ containing all subsets of $\mathbb{N}\times\mathbb{N}$ with a natural density of zero. Define $\{\mathfrak{X}_{jk}\}$ in $\mathfrak{E}$ as follows:*

$$\mathfrak{X}_{jk} = \begin{cases} \left((-1)^{jk}, 0\right), & \text{if } j \text{ and } k \text{ are not a perfect square;} \\ (jk, jk), & \text{otherwise.} \end{cases}$$

For $r \geq 1$, we have $\mathcal{I}_2 - \mathrm{LIM}_{\mathfrak{X}}^r = \overline{B_r(-1,0)} \cap \overline{B(1,0)}$, where

$$\overline{B_r(\mathfrak{X}_0)} = \left\{\mathfrak{Y} \in \mathfrak{E} : \|\mathfrak{Y} - \mathfrak{X}_0, \mathfrak{Z}\|_0^+ \leq r\right\} \quad \text{for every non-zero } \mathfrak{Z} \in \mathfrak{E}.$$

Consider $F \in \overline{B_r(-1,0)} \cap \overline{B(1,0)}$. For this F, we find that

$$\left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \|\mathfrak{X}_{jk} - F, \mathfrak{Z}\|_0^+ \geq r + \varepsilon\right\} \subset \{(1,1), (1,4), \cdots, (4,1), \cdots\}.$$

Since the later set has a natural density of zero, it follows that

$$\left\{(j,k) \in \mathbb{N}^2 : \|\mathfrak{X}jk - F, \mathfrak{Z}\|_0^+ \geq r + \varepsilon\right\} \in \mathcal{I}_2.$$

Also, if $r < 1$, then $\mathcal{I}_2 - \mathrm{LIM}_{\mathfrak{X}}^r = \emptyset$. Moreover, $\mathcal{I}_2 - \mathrm{LIM}_{\mathfrak{X}}^r = \emptyset$ for any r.

We denote $\mathcal{I}_2 - \mathrm{LIM}^r_{\mathfrak{X}}$ and $\mathcal{I}_2^{\mathfrak{E}} - C_{\mathfrak{X}}$ the set of all rough $\mathcal{I}_2$-limit points and rough $\mathcal{I}_2$-cluster points of a sequence $\{\mathfrak{X}_{jk}\}$ in $(\mathfrak{E}, \|\cdot,\cdot\|)$, respectively.

Definition 4.2. *We define a sequence $\{\mathfrak{X}_{jk}\}$ in $\mathfrak{E}$ as $\mathcal{I}_2$-bounded with respect to the fuzzy 2-norm if there exists a constant $M > 0$ such that for every non-zero $\mathfrak{Z} \in \mathfrak{E}$, the set*

$$\left\{(j,k) \in \mathbb{N}^2 : \|\mathfrak{X}_{jk}, \mathfrak{Z}\|_0^+ \geq M\right\} \in \mathcal{I}_2.$$

It is a known fact that if $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$ in $\mathfrak{E}$ is bounded, then $\mathrm{LIM}^r_{\mathfrak{X}} \neq \emptyset$, consequently implying that $\mathcal{I}_2 - \mathrm{LIM}^r_{\mathfrak{X}} \neq \emptyset$. We can establish a connection between the $\mathcal{I}_2$-boundedness of a sequence and its rough $\mathcal{I}_2$-limit set.

Theorem 4.1. *Let $(\mathfrak{E}, \|\cdot,\cdot\|)$ be a fuzzy 2-normed space and let $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$ be a sequence in $\mathfrak{E}$. Then $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$ is $\mathcal{I}_2$-bounded if and only there exists some $r > 0$ such that $\mathcal{I}_2 - \mathrm{LIM}^r_{\mathfrak{X}} \neq \emptyset$.*

Proof. First, let's consider an $\mathcal{I}_2$-bounded sequence $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$. This implies the existence of a positive number M such that for every non-zero $\mathfrak{Z} \in \mathfrak{E}$, $\left\{(j,k) \in \mathbb{N}^2 : \|\mathfrak{X}_{jk}, \mathfrak{Z}\|_0^+ \geq M\right\} \in \mathcal{I}_2$. Let $A = \left\{(j,k) \in \mathbb{N}^2 : \|\mathfrak{X}jk, \mathfrak{Z}\|_0^+ < M\right\}$ and define $r = \sup\left\{\|\mathfrak{X}jk, \mathfrak{Z}\|_0^+ : (j,k) \in A\right\}$. Consequently, for $(j,k) \in A$ and every non-zero $\mathfrak{Z} \in \mathfrak{E}$, if $\|\mathfrak{X}jk, \mathfrak{Z}\|_0^+ \leq r$, then $\|\mathfrak{X}jk - \Theta, \mathfrak{Z}\|_0^+ < r + \varepsilon$ for any $\varepsilon > 0$, where Θ denotes the zero vector of $\mathfrak{E}$. This implies

$$\left\{(j,k) \in \mathbb{N}^2 : \|\mathfrak{X}jk - \Theta, \mathfrak{Z}\|_0^+ \geq r + \varepsilon\right\} \in \mathcal{I}_2.$$

Thus, $\Theta \in \mathcal{I}_2 - \mathrm{LIM}^r_{\mathfrak{X}}$, leading to the conclusion that $\mathcal{I}_2 - \mathrm{LIM}^r_{\mathfrak{X}} \neq \emptyset$.

Conversely, suppose $\mathcal{I}_2 - \mathrm{LIM}^r_{\mathfrak{X}} \neq \emptyset$ for some $r > 0$, and $\Gamma \in \mathcal{I}_2 - \mathrm{LIM}^r_{\mathfrak{X}}$. Let $\varepsilon > 0$ be given. Then, for each non-zero $\mathfrak{Z} \in \mathfrak{E}$, $\left\{(j,k) \in \mathbb{N}^2 : \|\mathfrak{X}_{jk} - \Gamma, \mathfrak{Z}\|_0^+ \geq r + \varepsilon\right\} \in \mathcal{I}_2$. Define $K = \sup\left\{\|\Gamma, \mathfrak{Z}\|_0^+ : \mathfrak{Z} \in \mathfrak{E}\right\}$. Since $\|\mathfrak{X}_{jk}, \mathfrak{Z}\|_0^+ \leq \|\mathfrak{X}jk - \Gamma, \mathfrak{Z}\|_0^+ + \|\Gamma, \mathfrak{Z}\|_0^+ \leq \|\mathfrak{X}_{jk} - \Gamma, \mathfrak{Z}\|_0^+ + K$, it follows that

$$\left\{(j,k) \in \mathbb{N}^2 : \|\mathfrak{X}jk, \mathfrak{Z}\|_0^+ \geq r + \varepsilon + K\right\} \subseteq \left\{(j,k) \in \mathbb{N}^2 : \|\mathfrak{X}jk - \Gamma, \mathfrak{Z}\|_0^+ \geq r + \varepsilon\right\}.$$

Suppose $M = r + \varepsilon + K$. Consequently, $\left\{(j,k) \in \mathbb{N}^2 : \|\mathfrak{X}jk, \mathfrak{Z}\|_0^+ \geq M\right\} \in \mathcal{I}_2$. This proves that $\mathfrak{X} = \{\mathfrak{X}jk\}$ is $\mathcal{I}_2$-bounded. ■

Next we present some topological and geometrical properties of the r-$\mathcal{I}$-limit set of a sequence.

Theorem 4.2. *Let $(\mathfrak{E}, \|\cdot,\cdot\|)$ be a fuzzy 2-normed space and r be a nonnegative real number. The r-$\mathcal{I}_2$-limit set $\mathcal{I}_2 - \mathrm{LIM}^r_{\mathfrak{X}}$ of a sequence $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$ is closed set.*

Proof. If $\mathcal{I}_2 - \mathrm{LIM}^r_{\mathfrak{X}} = \emptyset$, then there is nothing to prove. So assume that $\mathcal{I}_2 - \mathrm{LIM}^r_{\mathfrak{X}} \neq \emptyset$. Suppose that $\{\mathfrak{Y}_{jk}\} \subset \mathcal{I}_2 - \mathrm{LIM}^r_{\mathfrak{X}}$ and $\mathfrak{Y}_{jk} \to \mathfrak{Y}$ as $j,k \to \infty$. Let $\varepsilon > 0$ be given. Then for each $\mathfrak{Z} \in \mathfrak{E}$ there exists $j(\varepsilon), k(\varepsilon) \in \mathbb{N}$ such that $\|\mathfrak{Y}_{jk} - \mathfrak{Y}, \mathfrak{Z}\|_0^+ < \varepsilon$ for all $j > j(\varepsilon)$ and $k > k(\varepsilon)$ Let $j_0, k_0 \in \mathbb{N}$ such that $\mathfrak{Y}_{j_0k_0} \in \{\mathfrak{Y}_{jk}\} \subset \mathcal{I}_2 - \mathrm{LIM}^r_{\mathfrak{X}}$. Consequently we have

$$A = \left\{(j,k) \in \mathbb{N} \times \mathbb{N}; \|\mathfrak{Y}_{jk} - \mathfrak{Y}_{j_0k_0}, \mathfrak{Z}\|_0^+ \geq r + \varepsilon\right\} \in \mathcal{I}_2.$$

Clearly $M = \mathbb{N} \times \mathbb{N} \setminus A \in \mathcal{F}(\mathcal{I}_2)$ and so $M \neq \emptyset$. Choose $(s,t) \in M$. Choose an $j_0 > j(\varepsilon), k_0 > k(\varepsilon)$. We have

$$\|\mathfrak{Y}_{st} - \mathfrak{Y}, \mathfrak{Z}\|_0^+ \leq \|\mathfrak{Y}_{st} - \mathfrak{Y}_{j_0k_0}, \mathfrak{Z}\|_0^+ + \|\mathfrak{Y}_{j_0k_0} - \mathfrak{Y}, \mathfrak{Z}\|_0^+ < r + 2\varepsilon.$$

Hence

$$\begin{aligned} M &= \left\{(j,k) \in \mathbb{N} \times \mathbb{N} : \|\mathfrak{Y}_{jk} - \mathfrak{Y}_{j_0k_0}, \mathfrak{Z}\|_0^+ < r + \varepsilon\right\} \\ &\subseteq \left\{(j,k) \in \mathbb{N} \times \mathbb{N} : \|\mathfrak{Y}_{jk} - \mathfrak{Y}, \mathfrak{Z}\|_0^+ < r + 2\varepsilon\right\}, \end{aligned}$$

where $\{(j,k) \in \mathbb{N}\times\mathbb{N} : \|\mathfrak{Y}_{jk} - \mathfrak{Y}_{j_0k_0}, \mathfrak{Z}\|_0^+ < r+\varepsilon\} \in \mathcal{F}(\mathcal{I}_2)$. Consequently $\{(j,k) \in \mathbb{N}\times\mathbb{N} : \|\mathfrak{Y}_{jk} - \mathfrak{Y}, \mathfrak{Z}\|_0^+ < r + 2\varepsilon\} \in \mathcal{F}(\mathcal{I}_2)$ and so $\{(j,k) \in \mathbb{N}\times\mathbb{N} : \|\mathfrak{Y}_{jk} - \mathfrak{Y}, \mathfrak{Z}\|_0^+ \geq r + 2\varepsilon\} \in \mathcal{I}$. This completes the proof. ■

Theorem 4.3. *Let $(\mathfrak{E}, \|\cdot,\cdot\|)$ be a fuzzy 2-normed space and r be a nonnegative real number. Then $\mathcal{I}_2 - \mathrm{LIM}^r_{\mathfrak{X}}$ is convex.*

Proof. Let $\Gamma_1, \Gamma_2 \in \mathcal{I}_2 - \mathrm{LIM}^r_{\mathfrak{X}}$. Then for every $\varepsilon > 0$ and each nonzero $\mathfrak{Z} \in \mathfrak{E}$, the sets

$$\begin{aligned} A &= \left\{(j,k) \in \mathbb{N}^2 : \|\mathfrak{X}_{jk} - \Gamma_1, \mathfrak{Z}\|_0^+ \geq r + \varepsilon\right\} \quad \text{and} \\ B &= \left\{(j,k) \in \mathbb{N}^2 : \|\mathfrak{X}_{jk} - \Gamma_2, \mathfrak{Z}\|_0^+ \geq r + \varepsilon\right\} \end{aligned}$$

belong to $\mathcal{I}_2$. Now, for $(j,k) \in A^c \cap B^c$ and for each $\lambda \in [0,1]$,

$$\begin{aligned} &\|\mathfrak{X}_{jk} - (\lambda\Gamma_1 + (1-\lambda)\Gamma_2), \mathfrak{Z}\|_0^+ = \|\lambda(\mathfrak{X}_{jk} - \Gamma_1) + (1-\lambda)(\mathfrak{X}_{jk} - \Gamma_2), \mathfrak{Z}\|_0^+ \\ &\leq \lambda \|\mathfrak{X}_{jk} - \Gamma_1, \mathfrak{Z}\|_0^+ + (1-\lambda)\|\mathfrak{X}_{jk} - \Gamma_2, \mathfrak{Z}\|_0^+ \\ &< \lambda(r+\varepsilon) + (1-\lambda)(r+\varepsilon) = r + \varepsilon. \end{aligned}$$

Consequently,

$$\left\{(j,k) \in \mathbb{N}^2 : \|\mathfrak{X}_{jk} - (\lambda\Gamma_1 + (1-\lambda)\Gamma_2), \mathfrak{Z}\|_0^+ \geq r + \varepsilon\right\} \subset A \cup B \in \mathcal{I}_2.$$

This gives $\lambda\Gamma_1 + (1-\lambda)\Gamma_2 \in \mathcal{I}_2 - \mathrm{LIM}^r_{\mathfrak{X}}$, i.e., $\mathcal{I}_2 - \mathrm{LIM}^r_{\mathfrak{X}}$ is a convex set. ■

Definition 4.3. *($\mathcal{I}_2$-cluster point). Let $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$ be a double sequence in a fuzzy 2-normed space $(\mathfrak{E}, \|\cdot\|)$and $\mathcal{I}_2$ be an ideal on $\mathbb{N}\times\mathbb{N}$. Then an element $\mathfrak{Y} \in \mathfrak{E}$ is said to be an $\mathcal{I}_2$-cluster point of $\{\mathfrak{X}_{jk}\}$ if for each ε and a non-zero $\mathfrak{Z} \in \mathfrak{E}$, the set*

$$\left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \|\mathfrak{X}_{jk} - \mathfrak{Y}, \mathfrak{Z}\|_0^+ < \varepsilon\right\} \notin \mathcal{I}_2. \tag{5}$$

Theorem 4.4. *Let $(\mathfrak{E}, \|\cdot,\cdot\|)$ be a fuzzy 2-normed space. For an arbitrary $\Gamma \in \mathcal{I}_2^{\mathfrak{E}} - C_{\mathfrak{X}}$ and each nonzero $\mathfrak{Z} \in \mathfrak{E}$ we have $\|\Phi - \Gamma, \mathfrak{Z}\|_0^+ \leq r$ for all $\Phi \in \mathcal{I}_2 - \mathrm{LIM}^r_{\mathfrak{X}}$.*

Proof. Suppose there exist $\Gamma \in \mathcal{I}_2^{\mathfrak{E}} - C_{\mathfrak{X}}$ and $\Phi \in \mathcal{I}_2 - \mathrm{LIM}^r_{\mathfrak{X}}$ such that for each non-zero $\mathfrak{Z} \in \mathfrak{E}$, $\|\Phi - \Gamma, \mathfrak{Z}\|_0^+ > r$. Choose $\varepsilon = \dfrac{\|\Phi - \Gamma, \mathfrak{Z}\|_0^+ - r}{2}$. Consequently, for every non-zero $\mathfrak{Z} \in \mathfrak{E}$, the sets

$$\begin{aligned} A_1 &= \left\{(j,k) \in \mathbb{N}^2 : \|\mathfrak{X}_{jk} - \Gamma, \mathfrak{Z}\|_0^+ < \varepsilon\right\} \quad \text{and} \\ A_2 &= \left\{(j,k) \in \mathbb{N}^2 : \|\mathfrak{X}_{jk} - \Phi, \mathfrak{Z}\|_0^+ < r + \varepsilon\right\} \end{aligned}$$

do not belong to $\mathcal{I}_2$. For every $(j,k) \in \mathbb{N}^2$, we have

$$\|\mathfrak{X}_{jk} - \Phi, \mathfrak{Z}\|_0^+ > \|\Gamma - \Phi, \mathfrak{Z}\|_0^+ - \|\mathfrak{X}_{jk} - \Gamma, \mathfrak{Z}\|_0^+ > 2\varepsilon + r - \varepsilon = r + \varepsilon.$$

This indicates that $(j,k) \in A_2$, and thus, $A_1 \subset A_2$. Since $A_1 \in \mathcal{I}_2$, this would imply that $A_1 \in \mathcal{I}_2$, which leads to a contradiction. This completes the proof. ■

Theorem 4.5. *Let $(\mathfrak{E}, \|\cdot,\cdot\|)$ be a fuzzy 2-normed space. A sequence $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$ is $\mathcal{I}_2$-convergent if and only if $\mathcal{I}_2 - \mathrm{LIM}^r_{\mathfrak{X}} = \overline{B}_r(\Phi)$.*

Proof. The necessity component has already been established in Theorem 3.1. Regarding the sufficiency, assuming that $\mathcal{I}_2 - \mathrm{LIM}^r_{\mathfrak{X}} = \overline{B}r(\Phi) \neq \emptyset$, according to Theorem 4.1, the set $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$ is bounded under $\mathcal{I}_2$. Let's suppose the sequence $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$ possesses another

$\mathcal{I}_2$-cluster point Γ distinct from Φ. Let $\Lambda = \Phi + \frac{r}{\|\Phi - \Gamma, \mathfrak{Z}\|_0^+}(\Phi - \Gamma)$. The point Λ satisfies the condition

$$\|\Lambda - \Gamma, \mathfrak{Z}\|_0^+ = \left(\frac{r}{\|\Phi - \Gamma, \mathfrak{Z}\|_0^+} + 1\right) \|\Phi - \Gamma, \mathfrak{Z}\|_0^+ = r + \|\Phi - \Gamma, \mathfrak{Z}\|_0^+ > r.$$

Given that $\Gamma \in \mathcal{I}_2^{\mathfrak{E}} - C_{\mathfrak{X}}$, it follows from Theorem 4.4 that $\Lambda \notin \mathcal{I}_2 - \mathrm{LIM}_{\mathfrak{X}}^r$. However, this is contradictory since $\|\Lambda - \Gamma, \mathfrak{Z}\|_0^+ = r$ and $\mathcal{I}_2^{\mathfrak{E}} - \mathrm{LIM}\mathfrak{X}^r = \overline{B}r(\Phi)$. Thus, Φ stands as the exclusive $\mathcal{I}_2$-cluster point of $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$. Consequently, $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$ converges under $\mathcal{I}_2$ to Φ. This concludes the proof. ■

Theorem 4.6. *Let $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$ be a sequence in a fuzzy 2-normed space $(\mathfrak{E}, \|\cdot,\cdot\|)$. Then the following assertions hold:*

(a) *If $\Gamma \in \mathcal{I}_2^{\mathfrak{E}} - C_{\mathfrak{X}}$, then $\mathcal{I}_2 - \mathrm{LIM}_{\mathfrak{X}}^r \subseteq \overline{B}_r(\Gamma)$,*

(b) $\mathcal{I}_2 - \mathrm{LIM}_{\mathfrak{X}}^r = \bigcap\limits_{\Gamma \in \mathcal{I}_2^{\mathfrak{E}} - C_{\mathfrak{X}}} \overline{B}_r(\Gamma) = \left\{\mathfrak{X}_0 \in \mathfrak{E} : \mathcal{I}_2^{\mathfrak{E}} - C_{\mathfrak{X}} \subseteq \overline{B}_r(\mathfrak{X}_0)\right\}$.

Proof. (a) Consider $\Phi \in \mathcal{I}_2 - \mathrm{LIM}_{\mathfrak{X}}^r$ and $\Gamma \in \mathcal{I}_2^{\mathfrak{E}} - C_{\mathfrak{X}}$. For any nonzero $\mathfrak{Z} \in \mathfrak{E}$ and in accordance with Theorem 4.4, it follows that $\|\Phi - \Gamma, \mathfrak{Z}\|_0^+ \leq r$. This leads to the inference that Φ resides in $\overline{B}_r(\Gamma)$, thus establishing that $\mathcal{I}_2 - \mathrm{LIM}_{\mathfrak{X}}^r$ is a subset of $\overline{B}_r(\Gamma)$.

(b) Utilizing part(a), we establish that $\mathcal{I}_2 - \mathrm{LIM}_{\mathfrak{X}}^r = \bigcap\limits_{\Gamma \in \mathcal{I}_2^{\mathfrak{E}} - C_{\mathfrak{X}}} \overline{B}_r(\Gamma)$. Let $\Lambda \in \bigcap\limits_{\Gamma \in \mathcal{I}_2^{\mathfrak{E}} - C_{\mathfrak{X}}} \overline{B}r(\Gamma)$. Consequently, for any nonzero $\mathfrak{Z} \in \mathfrak{E}$, we find $\|\Lambda - \Gamma, \mathfrak{Z}\|_0^+ \leq r$ for all $\Gamma \in \mathcal{I}_2^{\mathfrak{E}} - C_{\mathfrak{X}} \subseteq \overline{B}_r(\mathfrak{X}_0)$. Now, assume $\Lambda \notin \mathcal{I}_2 - \mathrm{LIM}_{\mathfrak{X}}^r$. This assumption implies the existence of an $\varepsilon > 0$ such that for each nonzero $\mathfrak{Z} \in \mathfrak{E}$, the set $\left\{(j,k) \in \mathbb{N}^2 : \|\mathfrak{X} - \Lambda, \mathfrak{Z}\|_0^+ \geq r + \varepsilon\right\} \notin \mathcal{I}_2$, implying the existence of an $\mathcal{I}_2$-cluster point Γ of the sequence $\{\mathfrak{X}jk\}$ satisfying $\|\Lambda - \Gamma, \mathfrak{Z}\|_0^+ \geq r + \varepsilon$. Hence, $\mathcal{I}_2^{\mathfrak{E}} - C_{\mathfrak{X}} \nsubseteq \overline{B}_r(\Lambda)$ and $\Lambda \notin \left\{\mathfrak{X}_0 \in \mathfrak{E} : \mathcal{I}_2^{\mathfrak{E}} - C_{\mathfrak{X}} \subseteq \overline{B}_r(\mathfrak{X}_0)\right\}$. This leads to the deduction that

$$\left\{\mathfrak{X}_0 \in \mathfrak{E} : \mathcal{I}_2^{\mathfrak{E}} - C_{\mathfrak{X}} \subseteq \overline{B}r(\mathfrak{X}_0)\right\} \subseteq \mathcal{I}_2 - \mathrm{LIM}_{\mathfrak{X}}^r.$$

Thus, we conclude that $\mathcal{I}_2 - \mathrm{LIM}_{\mathfrak{X}}^r = \bigcap\limits_{\Gamma \in \mathcal{I}_2^{\mathfrak{E}} - C_{\mathfrak{X}}} \overline{B}_r(\Gamma) = \left\{\mathfrak{X}_0 \in \mathfrak{E} : \mathcal{I}_2^{\mathfrak{E}} - C_{\mathfrak{X}} \subseteq \overline{B}_r(\mathfrak{X}_0)\right\}$, thereby completing the proof. ■

Theorem 4.7. *Let $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$ be a sequence in a fuzzy 2-normed space $(\mathfrak{E}, \|\cdot,\cdot\|)$. Then $\mathcal{I}_2^{\mathfrak{E}} - C_{\mathfrak{X}} \subseteq \mathcal{I}_2 - \mathrm{LIM}_{\mathfrak{X}}^r$, where $r = diam\left(\mathcal{I}_2^{\mathfrak{E}} - C_{\mathfrak{X}}\right)$.*

Proof. Suppose $\mathfrak{Y} \notin \mathcal{I}_2 - \mathrm{LIM}_{\mathfrak{X}}^r$. Consequently, there exists an $\varepsilon > 0$ such that for every nonzero $\mathfrak{Z} \in \mathfrak{E}$, the set $\left\{(j,k) \in \mathbb{N}^2 : \|\mathfrak{X} - \mathfrak{Y}, \mathfrak{Z}\|_0^+ \geq r + \varepsilon\right\} \notin \mathcal{I}_2$. Furthermore, given that the sequence $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$ is $\mathcal{I}_2$-bounded, there exists another $\mathcal{I}_2$-cluster point $\mathfrak{Y}_1 \in \mathfrak{E}$ such that $\|\mathfrak{Y}_1 - \mathfrak{Y}, \mathfrak{Z}\|_0^+ > r + \frac{\varepsilon}{2}$ for each nonzero $\mathfrak{Z} \in \mathfrak{E}$. As a result, it follows that $\mathfrak{Y} \notin \mathcal{I}_2^{\mathfrak{E}} - C_{\mathfrak{X}}$, thereby establishing the conclusion. ■

Presently, we introduce the concept of a rough $\mathcal{I}_2$-Cauchy sequence within a fuzzy 2-normed space $(\mathfrak{E}, \|\cdot,\cdot\|)$, subsequently exploring significant results within this framework.

Definition 4.4. *Let $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$ be a sequence in a fuzzy 2-normed space $(\mathfrak{E}, \|\cdot,\cdot\|)$. Then $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$ is said to be a rough $\mathcal{I}_2$-Cauchy sequence of roughness degree $\vartheta > 0$ if for every $\varepsilon > 0$ there exists $(j,k) \in \mathbb{N}^2$ such that the set $\left\{(j,k) \in \mathbb{N}^2 : \|\mathfrak{X}_{jk} - \mathfrak{X}_{nm}, \mathfrak{Z}\|_0^+ \geq \varepsilon + \vartheta\right\} \in$*

$\mathcal{I}_2$ for each nonzero $\mathfrak{Z} \in \mathfrak{E}$. Also, we call ϑ as a $\mathcal{I}_2$-Cauchy degree of $\mathfrak{X}$ and the sequence $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$ is called a ϑ-$\mathcal{I}_2$-Cauchy sequence in $\mathfrak{E}$.

Lemma 4.1. *Let $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$ be a ϑ-$\mathcal{I}_2$-Cauchy sequence in $\mathfrak{E}$ and $\vartheta_0 > \vartheta$. Then ϑ_0 is also a $\mathcal{I}_2$-Cauchy degree of $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$.*

Lemma 4.2. *A sequence $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$ in a fuzzy 2-normed space $(\mathfrak{E}, \|\cdot,\cdot\|)$ is $\mathcal{I}_2$-bounded if and only if there exists a $\vartheta > 0$ such that $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$ is a ϑ-$\mathcal{I}_2$-Cauchy sequence in $\mathfrak{E}$.*

Theorem 4.8. *Let $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$ be a sequence in a fuzzy 2-normed space $(\mathfrak{E}, \|\cdot,\cdot\|)$. Then $\mathcal{I}_2 - \mathrm{LIM}^r_{\mathfrak{X}} \neq \emptyset$ if and only if for every $\vartheta \geq 2r$, $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$ is a ϑ-$\mathcal{I}_2$-Cauchy sequence.*

Proof. Suppose $\Phi \in \mathcal{I}_2 - \mathrm{LIM}^r_{\mathfrak{X}}$. For any $\varepsilon > 0$ and any nonzero $\mathfrak{Z} \in \mathfrak{E}$, the set $A = \left\{(j,k) \in \mathbb{N}^2 : \|\mathfrak{X} - \Phi, \mathfrak{Z}\|_0^+ \geq r + \frac{\varepsilon}{2}\right\} \in \mathcal{I}_2$. Hence,

$$A^c = \left\{(j,k) \in \mathbb{N}^2 : \|\mathfrak{X} - \Phi, \mathfrak{Z}\|_0^+ < r + \frac{\varepsilon}{2}\right\} \in \mathcal{F}(\mathcal{I}_2).$$

Consequently, there exists $(n,m) \in A^c$ such that $\|\mathfrak{X}_{nm} - \Phi, \mathfrak{Z}\|_0^+ < r + \frac{\varepsilon}{2}$ for every nonzero $\mathfrak{Z} \in \mathfrak{E}$. Moreover, for $(j,k) \in A^c$, we have

$$\begin{aligned} \|\mathfrak{X}_{nm} - \mathfrak{X}_{jk}, \mathfrak{Z}\|_0^+ &\leq \|\mathfrak{X}_{nm} - \Phi, \mathfrak{Z}\|_0^+ + \|\mathfrak{X}_{jk} - \Phi, \mathfrak{Z}\|_0^+ \\ &\leq r + \frac{\varepsilon}{2} + r + \frac{\varepsilon}{2} = 2r + \varepsilon. \end{aligned}$$

Hence, $\left\{(j,k) \in \mathbb{N}^2 : \|\mathfrak{X}_{nm} - \mathfrak{X}_{jk}, \mathfrak{Z}\|_0^+ \geq 2r + \varepsilon\right\} \in \mathcal{I}_2$ holds. Consequently, according to Lemma 4.1, for every $\vartheta \geq 2r$, $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$ constitutes a ϑ-$\mathcal{I}_2$-Cauchy sequence.

Conversely, assume that ϑ represents a $\mathcal{I}_2$-Cauchy degree of $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$ for every $\vartheta \geq 2r > 0$. It then follows from Lemma 4.2 that the sequence $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$ is $\mathcal{I}_2$-bounded. Consequently, in line with Theorem 4.1, $\mathfrak{X} = \{\mathfrak{X}_{jk}\}$ demonstrates rough $\mathcal{I}_2$-convergence with a roughness degree $\vartheta > 0$. This concludes the proof. ■

5. Conclusions

In this study, we have advanced our understanding of rough convergence and its applications in fuzzy 2-normed linear spaces. By introducing and exploring rough $\mathcal{I}_2$-convergence, rough $\mathcal{I}_2^*$-convergence, rough $\mathcal{I}_2$-limit points, and rough $\mathcal{I}_2$-cluster points, we've established a foundation for further exploration in diverse mathematical domains. The proof of a decomposition theorem specific to rough $\mathcal{I}_2$-convergence for double sequences is a fundamental result, providing insight into their structure within fuzzy 2-normed linear spaces. Definitions and properties of rough $\mathcal{I}_2$-double Cauchy sequences and $\mathcal{I}_2^*$-double Cauchy sequences have expanded our understanding, enabling deeper investigations into their convergence properties.

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Nifeen Hussein Altaweel Currently serving as an assistant professor at Tabuk University in Saudi Arabia, renowned for dedication and industriousness in challenging environments. Embracing diverse ideas, I value fresh perspectives. My academic journey includes a BSc and MSc in Mathematics from King Abdulaziz University, showcasing adaptability and ambition. Specializing in analysis and Pure mathematics, I earned my Ph.D. in Functional analysis from Lancaster University, UK, two years ago, focusing on "Embedded eigenvalues of operators and Fredholm properties of Pencils."

Mohammad Hussein Mohammad Rashid has held the position of Professor in the Department of Mathematics and Statistics within the Faculty of Science at the University of Mut'ah, Jordan since 2018. His primary research focus lies in Functional Analysis/Operator Theory, Inequalities, Linear Algebra, Fuzzy Mathematics, Fuzzy metric space, Fixed point theory and Ideal of convergence. He has authored over 150 papers published in esteemed international peer-reviewed journals indexed in Scopus and Clarivate ISI.

Nidal Hassan Elbadowi Eljaneid is a dedicated and ambitious individual who excels in diverse environments, adapting and collaborating effectively. With a master's degree from Al-Nilein University, Sudan, and another in educational technology, along with a doctorate from Sudan University of Science and Technology, his research focuses on pure mathematics. Currently, he serves as an assistant professor at the University of Tabuk, KSA.

Razan Abdul Karim Al-Balawi currently enrolled as a graduate student at the University of Tabuk, is diligently pursuing a master's degree. Demonstrating exceptional academic prowess, she actively engages in research initiatives. Her scholarly endeavors are guided and supervised by Dr. Nevin.