TWMS J. App. and Eng. Math. V.15, N.5, 2025, pp.1217-1229

m-ETERNAL TOTAL BONDAGE NUMBER IN CIRCULANT GRAPHS

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Abstract. An Eternal dominating set of a graph is defined as a set of guards located at vertices, required to protect the vertices of the graph against infinitely long sequences of attacks, such that the configuration of guards induces a dominating set at all times. The eternal m-security number is defined as the minimum number of guards to handle an arbitrary sequence of single attacks using multiple-guard shifts. Klostermeyer and Mynhardt defined the m-eternal total domination number of a graph G denoted by $\gamma^{\infty}_{mt}(G)$ as the minimum number of guards to handle an arbitrary sequence of single attacks using multiple guard shifts and the configuration of guards always induces a total dominating set. We define the m-Eternal Total bondage number of a graph G denoted by $b_{mt}(G)$ as the minimum cardinality of set of edges $E^{'} \subseteq E(G)$ for which $\gamma^{\infty}_{mt}(G - E^{'}) > \gamma^{\infty}_{mt}(G)$ and $G - E^{'}$ does not contain isolated vertices. In this paper we find the exact values of $b_{mt}(G)$ for Circulant graphs $C_n(1, 2)$ and $C_n(1, 3)$.

Keywords: Eternal total domination, total domination, Bondage Number, m- Eternal total Bondage Number

AMS Subject Classification: 83-02, 99A00

1. Introduction

Let $G = (V, E)$ be a simple and connected graph of order $|V| = n$. For graph theoretic terminology we refer to Harary [3]. For any vertex $v \in V$, the open neighbourhood of v is the set $N(v) = \{v \in V/uv \in E\}$ and the closed neighbouhood is the set $N[v] = N(v) \cup v$. A set S is a dominating set if $N[S] = V(G)$ or equivalently, every vertex in $V - S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G, and a dominating set S of minimum cardinality is called a γ-set of G.

A total dominating set(TDS) of G is a set $D \subseteq V$ with the property that for each $u \in V$, there exists $x \in D$ adjacent to u. The minimum cardinality amongst all total dominating sets is the total domination number $\gamma_t(G)$. A TDS S of minimum cardinality

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§ Manuscript received: December 26, 2023; accepted: September 30, 2024.
TWMS Journal of Applied and Engineering Mathematics, Vol.15, No.5;

is called γ_t-set of G. Note that this parameter is only defined for graphs without isolated vertices.

Klostermeyer et al. [4] defined an eternal dominating set(EDS) of G to be a set D such that for each sequence of attacks $R = r_1, r_2, \ldots$ with $r_i \in V$ there exists a sequence $D = D_1, D_2, \ldots$ of dominating sets and a sequence of vertices $s_1, s_2, \ldots$ where $s_i \in D_i \cap N[r_i]$, such that $D_{i+1} = (D_i - \{s_i\}) \cup \{r_i\}$. Note that $s_i = r_i$ is possible. The set D_{i+1} is the set of locations of guards after the attack at r_i is defended. If $s_i \neq r_i$, we say that the guard at s_i has moved to r_i. The minimum cardinality amongst all eternal dominating sets is the eternal domination number $\gamma^{\infty}(G)$.For the m-eternal dominating set problem, each D_i, $i \geq 1$, is required to be a dominating set, $r_i \in V$ (assuming without loss of generality $r_i \notin D_i$), and D_{i+1} is obtained from D_i by moving the guards to neighbouring vertices. That is each guard in D_i may move to an adjacent vertex, as long as one guard moves to r_i. Thus it is required that $r_i \in D_{i+1}$. The size of a smallest m-eternal dominating set (defined similar to an eternal dominating set) of G is the m-eternal domination number $\gamma_m^{\infty}(G)$. This "multiple guards move" version of the problem was introduced by Goddard et al. [2]. It is also called the "all guards move" model. It is clear that $\gamma^{\infty}(G) \geq \gamma_m^{\infty}(G) \geq \gamma(G)$ for all graphs G. They defined an m-eternal total dominating set (m-ETDS) of G to be an EDS except that all the sets D_i are total dominating sets. The minimum cardinality amongst all m-ETDSs is the m-eternal total domination number$\gamma_{mt}^{\infty}(G)$. A m-ETDS of minimum cardinality is called a γ_{mt}^{∞}-set of G.

Fink et al.[1] initiated the study of bondage number of a graph G, where the bondage number b(G) was defined to be cardinality of the smallest number of edges $F \subset E(G)$ such that $\gamma(G - F) > \gamma(G)$ and sharp bounds were obtained for $b(G)$ and exact values were determined for several classes of graphs.

The super computers are developed rapidly and the growth of faster processors are in demand. The interconnection network can effieciently hold and connect such massive system. Usage of circulant graph topology is ideal as it satisfy small network diameter, large bisection with topological simplicity,symmetry and maximum connectivity. It is also used for studying the reliability of certain communication networks. The circulant graphs are decomposed and used for various research.

Applications of circulant graphs are in digital signal processing,Image compression, Physics/Engineering simulation, number theory and cryptography. They are an important class of interconnection networks in parallel and distributed computing. The circulant topologies are used as a promising deadlock-free topology for networks on chip. Deadlock free routing algorithms are presented for the high level modeling and comparison of the peak throughout the network in chips. An extensive study on this graph topology is done in [7] and [5]. Hence the research on m-eternal total bondage number was initiated in [9] for a graph.

We define m-eternal total bondage number $b_{mt}(G)$ to be the minimum cardinality of set of edges $E^{'} \subseteq E(G)$ for which $\gamma_{mt}^{\infty}(G - E^{'}) > \gamma_{mt}^{\infty}(G)$ and $G - E^{'}$ does not contain isolated vertices. In this paper, we find the exact values of $b_{mt}(G)$ for circulant graphs $C_n(1,2)$ and $C_n(1,3)$.

2. Circulant graphs

The circulant graph $C_n(S_c)$ is the graph with the vertex set $V(C_n(S_n)) = \{v_i : 0 \leq i \leq n-1\}$ and the edge set $E(C_n(S_c)) = \{0 \leq i, j \leq n-1 (mod\ n) \in S_n\}$, $S_n \subseteq \{1, 2, 3, \cdots, \lceil \frac{n}{2} \rceil\}$ where subscripts are taken modulo n. In this section we find the value of γ_{mt}^{∞} for the circulant graphs $C_n(1,2)$ and $C_n(1,3)$.

Theorem 2.1. [8]*For the circulant graph* $G = C_n(1, 2)$,

$$\gamma_t(G) = \begin{cases} \dfrac{2n}{7}, & n \equiv 0 \ (mod\ 7) \\ \dfrac{2(n-i)}{7} + 1, & n \equiv i \ (mod\ 7), i = 1, 2 \\ \dfrac{2(n-i)}{7} + 2, & n \equiv i \ (mod\ 7), i = 3, 4, 5, 6 \end{cases}$$

Theorem 2.2. [8]*For the circulant graph* $G = C_n(1, 2)$, $\gamma^{\infty}_{mt}(G) = \gamma_t(G)$.

Theorem 2.3. [6] *For any integer* $n \geq 4$,

$$\gamma_t(C_n(1,3)) = \begin{cases} \lceil \frac{n}{4} \rceil + 1, & n \equiv 2, 4 (mod\ 8) \\ \lceil \frac{n}{4} \rceil, & otherwise. \end{cases}$$

Theorem 2.4. [8]*For the circulant graph* $G = C_n(1, 3)$, $n \geq 4$,

$$\gamma^{\infty}_{mt}(G) = \gamma_t(G)$$

Theorem 2.5. *For circulant graphs* $G = C_n(1, 2)$,

$$b_{mt}(G) = \begin{cases} 1 & , n \equiv 0, 2 \ \ (mod\ 7) \\ 2 & , n \equiv 1, 5, 6 \quad (mod\ 7) \end{cases} \tag{1}$$

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. When n is even $V(G)$ is partitioned in three cycles and when n is odd $V(G)$ is partitioned in two cycles. For any γ_t-set S of G, $d(u, v) \leq 5$, for every $u, v \in S$.

Case(i): $n \equiv 0(mod\ 7)$
Consider the graph $G_1 = G - e$. If e is an edge in the inner cycle, without loss of generality let $e = v_4v_6$. The possible γ_t-sets of G_1 which are also the γ_t-sets of G are listed below.
$S_1 = \{v_{7k-6}, v_{7k-4} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\}$
$S_2 = \{v_{7k-4}, v_{7k-2} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\}$
$S_3 = \{v_{7k-2}, v_{7k} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\}$

When the guards are placed in S_1, to defend an attack at v_4, the guards move one step along the edges of the outer cycle in the clockwise direction, leaving v_6 undefended. When the guards are placed in S_2, to defend an attack at v_4, $v_3 \to v_4$, leaving v_1 undefended. When the guards are placed in S_3, to defend an attack at v_4, the guards move one step along the edges of the outer cycle in the clockwise direction and the resulting configuration of guards is not a TDS.

In all the above cases, we see that $\gamma^{\infty}_{mt}(G_1) > \gamma_t(G_1) = \gamma_t(G)$. By Theorem 2.2, $\gamma^{\infty}_{mt}(G) = \gamma_t(G)$. Hence $\gamma^{\infty}_{mt}(G_1) > \gamma^{\infty}_{mt}(G)$. Therefore $b_{mt}(G) = 1$.

Case(ii):$n \equiv 1(mod\ 7)$
Consider the graph $G_1 = G - e$. If e is an edge in the outer cycle, without loss of generality let $e = v_1v_2$. Let $S = \{v_{7k-6}, v_{7k-4} : k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\} \cup \{v_n\}$ be a γ_t-set of G_1. We observe that S is also a γ_t-set of G. Now, we place the guards in S. We partition $V \setminus S$ as follows:
$A = \{v_{7k-3} \quad k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\}$
$B = \{v_{7k-2} \quad k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\}$
$C = \{v_{7k-1} \quad k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\}$
$D = \{v_{7k} \quad k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\}$

$E = \{v_{7k-5} \quad k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\}$

To defend an attack at a vertex $r \in A \cup B$, the guards in S move in the clockwise direction leaving no vertex undefended. To defend an attack at a vertex $r \in C \cup D \cup E$, the guards in S move in the anticlockwise direction leaving no vertex undefended.

If e is not an edge in the outer cycle without loss of generality $e = v_1v_3$. Let $S = \{v_{7k-5}, v_{7k-3} : k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\} \cup \{v_n\}$ be a γ_t-set of G_1. We observe that S is also a γ_t-set of G. Now, we place the guards in S. We partition $V \setminus S$ as follows:

$A = \{v_{7k-2} \quad k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\}$
$B = \{v_{7k-1} \quad k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\}$
$C = \{v_{7k} \quad k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\}$
$D = \{v_{7k+1} \quad k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\}$
$E = \{v_{7k-4} \quad k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\}$

To defend an attack at a vertex $r \in A \cup B$, the guards in S move in the clockwise direction leaving no vertex undefended. To defend an attack at a vertex $r \in C \cup D \cup E$, the guards in S move in the anticlockwise direction leaving no vertex undefended.

In both the above cases, we see that $\gamma^{\infty}_{mt}(G_1) = \gamma_t(G_1) = \gamma_t(G)$. By Theorem 2.2, $\gamma^{\infty}_{mt}(G) = \gamma_t(G)$. Hence $\gamma^{\infty}_{mt}(G_1) = \gamma^{\infty}_{mt}(G)$. Therefore $b_{mt}(G) \geq 2$.

Now, consider the graph $G_2 = G - \{e_1, e_2\}$. If e_1, e_2 are the edges in the inner cycle, without loss of generality let $e_1 = v_5v_6$ and $e_2 = v_6v_7$. The possible γ_t-sets of G_2 which are also γ_t-sets of G are listed below.

$S_1 = \{v_{7k-6}, v_{7k-4} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\} \cup \{v_n\}$
$S_2 = \{v_{7k-5}, v_{7k-3} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\} \cup \{v_n\}$
$S_3 = \{v_{7k-1}, v_{7k+1} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\} \cup \{v_4\}$

When the guards are placed in S_1, to defend an attack at v_5, the guards placed in $S_1 \setminus v_n$ moves one step in the clockwise direction along the edges of the inner cycle and to maintain totality $v_n \to v_1$, leaving v_6 undefended. When the guards are placed in S_2, to defend an attack at v_5, the guards move one step in the clockwise direction along the edges of the outer cycle, leaving v_6 undefended. When the guards are placed in S_3, to defend an attack at v_5, the guard at v_4 move to v_5. Now the resulting configuration of guards do not form a TDS as v_5 and v_6 are isolated.

In all the above cases, we see that $\gamma^{\infty}_{mt}(G_2) > \gamma_t(G_2) = \gamma_t(G)$. By Theorem 2.2, $\gamma^{\infty}_{mt}(G) = \gamma_t(G)$. Hence $\gamma^{\infty}_{mt}(G_2) > \gamma^{\infty}_{mt}(G)$. Therefore $b_{mt}(G) = 2$.

Case(iii): $n \equiv 2 (mod\ 7)$

Consider the graph $G_1 = G - e$. If e is an edge in the inner cycle, without loss of generality let $e = v_5v_7$. The possible γ_t-sets of G_1 which are also γ_t-sets of G are listed below.

$S_1 = \{v_{7k-6}, v_{7k-4} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\} \cup \{v_{n-1}\}$
$S_2 = \{v_{7k+1}, v_{7k+3} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\} \cup \{v_3\}$
$S_3 = \{v_{7k+3}, v_{7k+5} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\} \cup \{v_5\}$

When the guards are placed in S_i, $i = 1, 2, 3$, to defend an attack at v_5, the guards move one step clockwise direction along the edges of the inner cycle leaving v_7 undefended. In all the above cases, we see that $\gamma^{\infty}_{mt}(G_1) > \gamma_t(G_1) = \gamma_t(G)$. By Theorem 2.2, $\gamma^{\infty}_{mt}(G) = \gamma_t(G)$. Hence $\gamma^{\infty}_{mt}(G_1) > \gamma^{\infty}_{mt}(G)$. Therefore $b_{mt}(G) = 1$.

Case(iv):$n \equiv 5 (mod\ 7)$

Consider the graph $G_1 = G - e$. If e is an edge in the outer cycle, without loss of generality let $e = v_1v_2$. Let $S = \{v_{7k-6}, v_{7k-4} : k = 1, 2, \ldots, \lceil \frac{n}{7} \rceil\}$ be a γ_t-set of G_1. We observe that S is also a γ_t-set G. Now, we place the guards in S. We partition $V \setminus S$ as follows:

$A = \{v_{7k-2} \quad k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\}$

$B = \{v_{7k-1} \quad k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\}$
$C = \{v_{7k} \quad k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\}$
$D = \{v_{7k+1} \quad k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\}$
$E = \{v_{7k-4} \quad k = 1, 2, \ldots, \lceil \frac{n}{7} \rceil\}$

To defend an attack at a vertex $r \in A \cup B$, the guards in S move in the clockwise direction leaving no vertex undefended. To defend an attack at a vertex $r \in C \cup D \cup E$, the guards in S move in the anticlockwise direction leaving no vertex undefended.

Without Loss of generality consider the graph $G_1 = G - e$, where $e = v_1v_3$. Let $S = \{v_{7k-5}, v_{7k-3} : k = 1, 2, \ldots, \lceil \frac{n}{7} \rceil\}$ be the γ_t-set of G_1. We observe that S is also the γ_t-set of G. Now, we place the guards in S. We partition $V \setminus S$ as follows:
$A = \{v_{7k-2} \quad k = 1, 2, \ldots, \lceil \frac{n}{7} \rceil\}$
$B = \{v_{7k-1} \quad k = 1, 2, \ldots, \lceil \frac{n}{7} \rceil\}$
$C = \{v_{7k} \quad k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\}$
$D = \{v_{7k+1} \quad k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\}$
$E = \{v_{7k-4} \quad k = 1, 2, \ldots, \lceil \frac{n}{7} \rceil\}$

To defend an attack at a vertex $r \in A \cup B$, the guards in S move in the clockwise direction leaving no vertex undefended. To defend an attack at a vertex $r \in C \cup D \cup E$, the guards in S move in the anticlockwise direction leaving no vertex undefended.

In both the above cases, we see that $\gamma_{mt}^{\infty}(G_1) = \gamma_t(G_1) = \gamma_t(G)$. By Theorem 2.2, $\gamma_{mt}^{\infty}(G) = \gamma_t(G)$. Hence $\gamma_{mt}^{\infty}(G_1) = \gamma_{mt}^{\infty}(G)$. Therefore $b_{mt}(G) \geq 2$.
Similar argument follows for $G - e$, where $e = \{v_5v_7\}$. Let $S = \{v_{7k-5}, v_{7k-3} : k = 1, 2, \ldots, \lceil \frac{n}{7} \rceil\}$. In both the above cases, we see that $\gamma_{mt}^{\infty}(G_1) = \gamma_t(G_1) = \gamma_t(G)$. By Theorem 2.2, $\gamma_{mt}^{\infty}(G) = \gamma_t(G)$. Hence $\gamma_{mt}^{\infty}(G_1) = \gamma_{mt}^{\infty}(G)$. Therefore $b_{mt}(G) \geq 2$.

Now, consider the graph $G_2 = G - \{e_1, e_2\}$. If e_1, e_2 are the edges in the inner cycle, without loss of generality let $e_1 = v_5v_6$ and $e_2 = v_6v_7$. The possible γ_t-sets of G_2 which are also a γ_t-set of G are listed below.
$S_1 = \{v_{7k-5}, v_{7k-3} \ : \ k = 1, 2, \ldots, \lceil \frac{n}{7} \rceil\}$
$S_2 = \{v_{7k+2}, v_{7k+4} \ : \ k = 1, 2, \ldots, \lceil \frac{n}{7} \rceil\}$
$S_3 = \{v_{7k+1}, v_{7k+3} \ : \ k = 1, 2, \ldots, \lceil \frac{n}{7} \rceil\}$
$S_4 = \{v_{7k+3}, v_{7k+5} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\} \cup \{v_4, v_5\}$

When the guards are placed in S_1, to defend an attack at v_5, the guards move one step in the clockwise direction along the edges of the outer cycle, leaving v_6 undefended. When the guards are placed in S_2, to defend an attack at v_n, the guards placed in $S_2 \setminus \{v, v_6\}$ moves one step in the clockwise direction along the edges of the outer cycle, leaving v_7 undefended. When the guards are placed in S_3, to defend an attack at v_6, the guards move one step in the anti-clockwise direction along the edges of the inner cycle, leaving no vertex undefended. If there is a subsequent attack at v_7, $v_8 \to v_7$, $v_6 \to v_8$ leaving no vertex undefended. If there is a attack at v_5, $v_7 \to v_5$ the resulting configuration do not form a TDS. When the guards are placed in S_4, to defend an attack at v_7, the guards move as follows: $v_5 \to v_7$,$v_4 \to v_5$, leaving v_6 undefended.
In both the above cases, we see that $\gamma_{mt}^{\infty}(G_2) > \gamma_t(G_2) = \gamma_t(G)$. By Theorem 2.2, $\gamma_{mt}^{\infty}(G) = \gamma_t(G)$. Hence $\gamma_{mt}^{\infty}(G_2) > \gamma_{mt}^{\infty}(G)$. Therefore $b_{mt}(G) = 2$.

Case(v): $n \equiv 6(mod\ 7)$
Consider the graph $G_1 = G - e$. If e is an edge in the outer cycle, without loss of generality let $e = v_1v_2$. Let $S = \{v_{7k-3}, v_{7k-1} : k = 1, 2, \ldots, \lceil \frac{n}{7} \rceil\}$ be a γ_t-set of G_1. We observe that S is also a γ_t-set G. Now, we place the guards in S. We partition $V \setminus S$ as follows:
$A = \{v_{7k} \quad k = 1, 2, \ldots, \lceil \frac{n}{7} \rceil\}$
$B = \{v_{7k+1} \quad k = 1, 2, \ldots, \lceil \frac{n}{7} \rceil\}$

$C = \{v_{7k+2} \quad k = 1, 2, \ldots, \lceil \frac{n}{7} \rceil\}$
$D = \{v_{7k+3} \quad k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\}$
$E = \{v_{7k-2} \quad k = 1, 2, \ldots, \lceil \frac{n}{7} \rceil\}$

To defend an attack at a vertex $r \in A \cup E$, the guards in S move in the clockwise direction along the edges of the outer cycle leaving no vertex undefended. To defend an attack at a vertex $r \in B$, the guards in S move one step in the clockwise direction along the edges of the inner cycle leaving no vertex undefended. To defend an attack at a vertex $r \in C$, the guards in S move one step in the anticlockwise direction along the edges of the inner cycle leaving no vertex undefended. To defend an attack at a vertex $r \in D$, the guards in S move one step in the anticlockwise direction along the edges of the outer cycle leaving no vertex undefended.

If e is not an edge in the outer cycle without loss of generality let $e = v_5v_7$. Let $S = \{v_{7k-6}, v_{7k-4} : k = 1, 2, \ldots, \lceil \frac{n}{7} \rceil\}$ be a γ_t-set of G_1. We observe that S is also a γ_t-set of G. Now, we place the guards in S. We partition $V \setminus S$ as follows:

$A = \{v_{7k-3} \quad k = 1, 2, \ldots, \lceil \frac{n}{7} \rceil\}$
$B = \{v_{7k-2} \quad k = 1, 2, \ldots, \lceil \frac{n}{7} \rceil\}$
$C = \{v_{7k-1} \quad k = 1, 2, \ldots, \lceil \frac{n}{7} \rceil\}$
$D = \{v_{7k} \quad k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\}$
$E = \{v_{7k-5} \quad k = 1, 2, \ldots, \lceil \frac{n}{7} \rceil\}$

To defend an attack at a vertex $r \in A \cup E$, the guards in S move in the clockwise direction along the edge of the outer cycle leaving no vertex undefended. To defend an attack at a vertex $r \in B$, the guards in S move in the clockwise direction along the edge of the inner cycle leaving no vertex undefended. To defend an attack at a vertex $r \in C$, the guards in S move in the anticlockwise direction along the edge of the inner cycle leaving no vertex undefended. To defend an attack at a vertex $r \in D$, the guards in S move in the anticlockwise direction along the edge of the outer cycle leaving no vertex undefended.

In both the above cases, we see that $\gamma_{mt}^{\infty}(G_1) = \gamma_t(G_1) = \gamma_t(G)$. By Theorem 2.2, $\gamma_{mt}^{\infty}(G) = \gamma_t(G)$. Hence $\gamma_{mt}^{\infty}(G_1) = \gamma_{mt}^{\infty}(G)$. Therefore $b_{mt}(G) \geq 2$.
Now, consider the graph $G_2 = G - \{e_1, e_2\}$. If e_1, e_2 are the edges in the inner cycle, without loss of generality let $e_1 = v_5v_6$ and $e_2 = v_6v_7$. The possible γ_t-sets of G_2 which are also γ_t-sets of G are listed below.

$S_1 = \{v_{7k-6}, v_{7k-4} \ : \ k = 1, 2, \ldots, \lceil \frac{n}{7} \rceil\}$
$S_2 = \{v_{7k-5}, v_{7k-3} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\} \cup \{v_n\}$
$S_3 = \{v_{7k-1}, v_{7k+1} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\} \cup \{v_4\}$

When the guards are placed in S_1, to defend an attack at v_5, the guards placed in $S_1 \setminus v_1, v_3$ moves one step along the edges of the outer cycle in the clockwise direction and $v_3 \to v_5$, $v_1 \to v_3$, leaving v_6 undefended. When the guards are placed in S_2, to defend an attack at v_5, the guards move as follows: $v_4 \to v_5$, $v_2 \to v_3$, leaving v_6 undefended.

In all the above cases, we see that $\gamma_{mt}^{\infty}(G_2) > \gamma_t(G_2) = \gamma_t(G)$. By Theorem 2.2, $\gamma_{mt}^{\infty}(G) = \gamma_t(G)$. Hence $\gamma_{mt}^{\infty}(G_2) > \gamma_{mt}^{\infty}(G)$. Therefore $b_{mt}(G) = 2$. □

Theorem 2.6. *For circulant graphs $G = C_n(1, 2)$, $b_{mt}(G) \leq 3, if \ \ n \equiv 3, 4 \ (mod\ 7)$*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. The pattern in which the guards are placed is such that $d(u, v) \leq 5$, where $u, v \in V(G)$.

Case(i): $n \equiv 3 (mod\ 7)$
Consider the graph $G_1 = G - \{e_1, e_2, e_3\}$, where $e_1 = v_4v_5$, $e_2 = v_5v_6$, $e_3 = v_6v_7$. The possible γ-sets of G_1 which are also a γ-set of G are listed below.

$S_1 = \{v_{7k-6}, v_{7k-4} \ : \ k = 1, 2, \ldots, \lceil \frac{n}{7} \rceil\} \cup \{v_{n-1}\}$
$S_2 = \{v_{7k+5}, v_{7k+7} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\} \cup \{v_7, v_8\}$
$S_3 = \{v_{7k}, v_{7k+2} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\} \cup \{v_4, v_6\}$
$S_4 = \{v_{7k}, v_{7k+2} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\} \cup \{v_3, v_4\}$

When the guards are placed in S_1, to defend an attack at v_4, the guards move as follows: $v_3 \to v_4$, $v_1 \to v_2$, $v_8 \to v_7$, $v_{10} \to v_9$, $v_{15} \to v_{14}$, $v_{17} \to v_{16}$ leaving v_6 undefended. When the guards are placed in S_2, to defend a 1^{st} attack at v_5, the guards move as follows: $v_7 \to v_5$, $v_8 \to v_7$, leaving no vertex undefended. To defend a 2^{nd} attack at v_6, then $v_4 \to v_6$, $v_2 \to v_4$, leaving v_{17} and v_1 undefended. When the guards are placed in S_3, to defend a 1^{st} attack at v_5, the guards move as follows. $v_7 \to v_5$, $v_4 \to v_3$, $v_6 \to v_8$, $v_9 \to v_{10}$,leaving no vertex undefended. To defend a 2^{nd} attack at v_4, then $v_3 \to v_4$, $v_5 \to v_3$, $v_8 \to v_7$, $v_{10} \to v_9$,leaving no vertex undefended. To defend a 3^{rd} attack at v_{11}, then $v_9 \to v_{11}$, $v_7 \to v_9$ leaving v_5 undefended. When the guards are placed in S_4, to defend a 1^{st} attack at v_5, the guards move as follows: $v_3 \to v_5$, $v_4 \to v_3$, $v_9 \to v_8$, $v_{11} \to v_{10}$,leaving no vertex undefended. To defend a 2^{nd} attack at v_6, then $v_{10} \to v_8$, $v_{15} \to v_{13}$, $v_{17} \to v_{15}$ leaving no vertex undefended. To defend a 3^{rd} attack at v_4, then $v_3 \to v_4$ leaving v_1 undefended. Hence S_4 is not a γ_{mt}^{∞}-set of G_1.

In all the above cases, we see that $\gamma_{mt}^{\infty}(G_1) > \gamma_t(G_1) = \gamma_t(G)$. By Theorem 2.2, $\gamma_{mt}^{\infty}(G) = \gamma_t(G)$. Hence $\gamma_{mt}^{\infty}(G_1) > \gamma_{mt}^{\infty}(G)$. Therefore $b_{mt}(G) \leq 3$.

Case(ii): $n \equiv 4(mod\ 7)$

Consider the graph $G_1 = G - \{e_1, e_2, e_3\}$, where $e_1 = v_5v_6$, $e_2 = v_6v_7$ and $e_3 = v_7v_8$. The possible γ-sets of G_1 which are also a γ-sets of G are listed below.

$S_1 = \{v_{7k-5}, v_{7k-3} \ : \ k = 1, 2, \ldots, \lceil \frac{n}{7} \rceil\}$
$S_2 = \{v_{7k+2}, v_{7k+4} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\} \cup \{v_4, v_6\}$
$S_3 = \{v_{7k+3}, v_{7k+5} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\} \cup \{v_4, v_5\}$
$S_4 = \{v_{7k+5}, v_{7k+7} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{7} \rfloor\} \cup \{v_8, v_9\}$

When the guards are placed in S_1, to defend an 1^{st} attack at v_5, the guards move as follows. $v_4 \to v_5$, $v_2 \to v_4$ leaving no vertex undefended. To defend a 2^{nd} attack at v_7, then $v_5 \to v_7$, $v_4 \to v_5$ leaving v_6 undefended. When the guards are placed in S_2, to defend an 1^{st} attack at v_7, the guards move as follows: $v_9 \to v_7$, $v_{16} \to v_{14}$, $v_{11} \to v_9$, $v_{18} \to v_{16}$, $v_2 \to v_4$, $v_6 \to v_4$ leaving no vertex undefended.

To defend an 2^{nd} attack at v_8, then $v_9 \to v_8$, $v_7 \to v_9$, $v_4 \to v_5$, $v_{14} \to v_{15}$, $v_4 \to v_5$, $v_2 \to v_4$, $v_{16} \to v_{17}$ leaving v_{12} undefended.

When the guards are placed in S_3, to defend an attack at v_7, then the guards move as follows: $v_5 \to v_7$, $v_4 \to v_5$, $v_{10} \to v_9$,$v_{12} \to v_{11}$, $v_{17} \to v_{16}$, $v_1 \to v_{18}$ leaving v_6 undefended. When the guards are placed in S_4, to defend an attack at v_6, the guards move as follows: $v_8 \to v_6$, $v_9 \to v_8$ leaving v_7 undefended.

In all the above cases, we see that $\gamma_{mt}^{\infty}(G_1) > \gamma_t(G_1) = \gamma_t(G)$. By Theorem 2.2, $\gamma_{mt}^{\infty}(G) = \gamma_t(G)$. Hence $\gamma_{mt}^{\infty}(G_1) > \gamma_{mt}^{\infty}(G)$. Therefore $b_{mt}(G) \leq 3$.

□

Theorem 2.7. *For circulant graphs $G = C_n(1, 3)$,*

$$b_{mt}(G) = \begin{cases} 1 & , n \equiv 0, 1, 6, 7 \ \ (mod\ 8) \\ 2 & , n \equiv 2, 3, 4, 5 \ \ (mod\ 8) \end{cases} \tag{2}$$

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. The pattern in which the guards are placed is such that $d(u, v) \leq 7$, where $u, v \in V(G)$.

We claim that $b_{mt}(G) = 1$ when $n \equiv 0, 1, 6, 7 \pmod 8$.

Case(i): $n \equiv 0 (mod\, 8)$
Without loss of generality consider the graph $G_1 = G - e$, where $e = v_4v_6$. The possible γ_t-sets of G_1 which are also the γ_t-sets of G are listed below.
$S_1 = \{v_{8k-7}, v_{8k-6} \ : \ k = 1, 2, \ldots, \frac{n}{8}$
$S_2 = \{v_{8k-5}, v_{8k-2} \ : \ k = 1, 2, \ldots, \frac{n}{8}$
$S_3 = \{v_{8k-1}, v_{8k} \ : \ k = 1, 2, \ldots, \frac{n}{8}$
$S_4 = \{v_{8k}, v_{8k+1} \ : \ k = 1, 2, \ldots, \frac{n}{8}$
$S_5 = \{v_{8k+1}, v_{8k+2} \ : k = 1, 2, \ldots, \frac{n}{8}$

When the guards are placed in S_1, to defend an attack at v_4, the guards move as follows: $v_1 \to v_4$, $v_2 \to v_3$, $v_9 \to v_{12}$, $v_{10} \to v_{11}$, leaving v_5 undefended. When the guards are placed in S_2, to defend an attack at v_5, the guards move as follows: $v_6 \to v_5$, $v_3 \to v_2$, $v_{11} \to v_{10}$, $v_{14} \to v_{13}$, leaving v_4 undefended. When the guards are placed in S_3, to defend an attack at v_4, the guards move as follows: $v_7 \to v_4$, $v_8 \to v_7$, $v_{15} \to v_{12}$, $v_{16} \to v_{15}$, leaving v_5 undefended. When the guards are placed in S_4, to defend an attack at v_5, the guards move as follows: $v_8 \to v_5$, $v_9 \to v_6$, $v_{16} \to v_{13}$, $v_1 \to v_{14}$, leaving v_4 undefended.

When the guards are placed in S_5, to defend an attack at v_4, the guards move as follows: $v_1 \to v_4$, $v_2 \to v_3$, $v_9 \to v_{12}$, $v_{10} \to v_{11}$, leaving v_5 undefended.
In all the above cases, we see that $\gamma_{mt}^{\infty}(G_1) > \gamma_t(G_1) = \gamma_t(G)$. By Theorem 2.4, $\gamma_{mt}^{\infty}(G) = \gamma_t(G)$. Hence $\gamma_{mt}^{\infty}(G_1) > \gamma_{mt}^{\infty}(G)$. Therefore $b_{mt}(G) = 1$.

Case(ii): $n \equiv 1 (mod\, 8)$
Without loss of generality consider the graph $G_1 = G - e$, where $e = v_5v_6$. The possible γ_t-sets of G_1 which are also the γ_t-sets of G are listed below.
$S_1 = \{v_{8k-7}, v_{8k-6} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{8} \rfloor \cup \{v_{n-1}\}$
$S_2 = \{v_{8k-6}, v_{8k-5} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{8} \rfloor \cup \{v_{n-1}\}$
$S_3 = \{v_{8k-4}, v_{8k-1} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{8} \rfloor \cup \{v_1\}$
$S_4 = \{v_{8k-1}, v_{8k} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{8} \rfloor \cup \{v_4\}$
$S_5 = \{v_{8k}, v_{8k+1} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{8} \rfloor \cup \{v_3\}$
$S_6 = \{v_{8k+1}, v_{8k+2} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{8} \rfloor \cup \{v_4\}$
$S_7 = \{v_{8k+2}, v_{8k+3} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{8} \rfloor \cup \{v_7\}$
$S_8 = \{v_{8k+3}, v_{8k+4} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{8} \rfloor \cup \{v_8\}$

When the guards are placed in S_1, to defend an attack at v_5, the guards move as follows: $v_2 \to v_5$, $v_1 \to v_4$, $v_9 \to v_{12}$, $v_{10} \to v_{13}$, leaving v_6 undefended. When the guards are placed in S_2, to defend an attack at v_5, the guards move as follows: $v_2 \to v_5$, $v_3 \to v_2$, leaving v_6 undefended. When the guards are placed in S_3, to defend an attack at v_6, the guards move as follows: $v_7 \to v_6$, $v_4 \to v_3$, $v_{12} \to v_{11}$, $v_{15} \to v_{14}$, $v_1 \to v_n$, leaving v_5 undefended. When the guards are placed in S_4, to defend an attack at v_5, the guards move as follows: $v_4 \to v_5$, $v_8 \to v_9$, $v_7 \to v_8$, $v_{16} \to v_{17}$, $v_{15} \to v_{16}$, leaving v_6 undefended. When the guards are placed in S_5, to defend an attack at v_6, the guards move as follows: $v_3 \to v_6$, leaving v_4 and v_5 undefended. When the guards are placed in S_6, to defend an attack at v_5, the guards move as follows: $v_4 \to v_5$, $v_1 \to v_2$, $v_n \to v_1$, $v_{10} \to v_{11}$, $v_9 \to v_{10}$, leaving v_6 undefended. When the guards are placed in S_7, to defend an attack at v_6, the guards move as follows: $v_7 \to v_6$, $v_{10} \to v_9$, $v_{11} \to v_{10}$, $v_1 \to v_n$, $v_2 \to v_1$, leaving v_5 undefended. When the guards are placed in S_8, to defend an attack at v_5, the guards move as follows: $v_2 \to v_5$, $v_3 \to v_4$, $v_{11} \to v_{14}$, $v_{12} \to v_{13}$, $v_8 \to v_{11}$, leaving v_6 undefended.

In all the above cases, we see that $\gamma_{mt}^{\infty}(G_1) > \gamma_t(G_1) = \gamma_t(G)$. By Theorem 2.4,

$\gamma_{mt}^{\infty}(G) = \gamma_t(G)$. Hence $\gamma_{mt}^{\infty}(G_1) > \gamma_{mt}^{\infty}(G)$. Therefore $b_{mt}(G) = 1$.

Case(iii): $n \equiv 6(mod\, 8)$
Let $S = \{v_{8k-6}, v_{8k-5} : k = 1, 2, \ldots, \lceil \frac{n}{8} \rceil\}$ be a γ_t-set of G. Consider a graph $G_1 = G - e$ where $e = v_1v_2$. Now the γ_t-set of G_1 is given by $S' = \{v_{8k-6}, v_{8k-5} : k = 1, 2, \ldots, \lceil \frac{n}{8} \rceil\} \cup \{v_n\}$. Clearly $|S| < |S'|$. Therefore $\gamma_t(G) < \gamma_t(G_1)$. By Theorem 2.4, $\gamma_{mt}^{\infty}(G) = \gamma_t(G)$. Hence $\gamma_{mt}^{\infty}(G_1) > \gamma_{mt}^{\infty}(G)$. Therefore $b_{mt}(G) = 1$.

Case(iv): $n \equiv 7(mod\, 8)$
Let $S = \{v_{8k-6}, v_{8k-5} : k = 1, 2, \ldots, \lceil \frac{n}{8} \rceil\}$ be a γ_t-set of G. Consider a graph $G_1 = G - e$ where $e = v_1v_2$. Now the γ_t-set of G_1 is given by $S' = \{v_{8k-6}, v_{8k-5} : k = 1, 2, \ldots, \lceil \frac{n}{8} \rceil\} \cup \{v_{n-2}\}$. Clearly $|S| < |S'|$. Therefore $\gamma_t(G) < \gamma_t(G_1)$. By Theorem 4.4, $\gamma_{mt}^{\infty}(G) = \gamma_t(G)$. Hence $\gamma_{mt}^{\infty}(G_1) > \gamma_{mt}^{\infty}(G)$. Therefore $b_{mt}(G) = 1$.
Now we claim that $b_{mt}(G) = 2$, when $n \equiv 2, 3, 4, 5 \ (mod\, 8)$.

Case(i): $n \equiv 2(mod\, 8)$
Consider the graph $G_1 = G - e$, where $e = \{v_1v_2\}$. Let $S = \{v_{8k-6}, v_{8k-5} : k = 1, 2, \ldots, \lfloor \frac{n}{8} \rfloor\} \cup \{v_n, v_{n-1}\}$ be the γ_t-sets of G_1. We observe that S is also the γ_t-set G. Now, we place the guards in S. We partition the vertices $v \notin S$ as follows:
$A = \{v_{8k-4} \quad k = 1, 2, \ldots \lfloor \frac{n}{8} \rfloor\}$
$B = \{v_{8k-3} \quad k = 1, 2, \ldots \lfloor \frac{n}{8} \rfloor\}$
$C = \{v_{8k-2} \quad k = 1, 2, \ldots \lfloor \frac{n}{8} \rfloor\}$
$D = \{v_{8k-1} \quad k = 1, 2, \ldots \lfloor \frac{n}{8} \rfloor\}$
$E = \{v_{8k} \quad k = 1, 2, \ldots \lfloor \frac{n}{8} \rfloor\}$
$F = \{v_{8k+1} \quad k = 1, 2, \ldots \lfloor \frac{n}{8} \rfloor\}$

To defend an attack at a vertex $r \in A$ or $r \in B$ or $r \in C$, the guards in S move in the clockwise direction leaving no vertex undefended. To defend an attack at a vertex $r \in D$ or $r \in E$ or $r \in F$, the guards in S move in the anticlockwise direction leaving no vertex undefended.
Without Loss of generality consider the graph $G_1 = G - e$, where $e = v_1v_4$. Let $S = \{v_{8k-6}, v_{8k-5} : k = 1, 2, \ldots, \lceil \frac{n}{8} \rceil\}$ be the γ_t-sets of G_1. We observe that S is also the γ_t-set G. Now, we place the guards in S. We partition the vertices $v \notin S$ as discussed above. Similar argument follows as discussed above when there is an attack.

In both the above cases, we see that $\gamma_{mt}^{\infty}(G_1) = \gamma_t(G_1) = \gamma_t(G)$. By Theorem 2.4, $\gamma_{mt}^{\infty}(G) = \gamma_t(G)$. Hence $\gamma_{mt}^{\infty}(G_1) = \gamma_{mt}^{\infty}(G)$. Therefore $b_{mt}(G) \geq 2$.
Now, consider the graph $G_1 = G - \{e_1, e_2\}$, where $e_1 = v_1v_2$ and $e_2 = v_2v_4$. The possible γ_t-set of G_1 which are also a γ_t-set of G are listed below.
$S_1 = \{v_{8k-5}, v_{8k-4} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{8} \rfloor\} \cup \{v_{n-1}, v_{n-2}\}$
$S_2 = \{v_{8k-4}, v_{8k-3} \ : \ k = 1, 2, \ldots, \lceil \frac{n}{8} \rceil\}$

When the guards are placed in S_1, to defend an 1^{st} attack at v_2, the guards move as follows: $v_{n-1} \to v_2$, $v_{n-2} \to v_{n-1}$. To defend an 2^{nd} attack at v_1, the guards move as follows: $v_4 \to v_1$, $v_3 \to v_4$, $v_{n-1} \to v_{n-2}$, $v_2 \to v_{n-1}$ leaving v_2 undefended. When the guards are placed in S_2, to defend an 1^{st} attack at v_1, the guards move as follows: $v_4 \to v_1$, $v_3 \to v_n$, $v_2 \to v_3$, $v_5 \to v_4$, $v_{12} \to v_{11}$, $v_{13} \to v_{12}$. To defend an 2^{nd} attack at v_5, the guards move as follows: $v_4 \to v_5$, $v_3 \to v_4$, $v_{18} \to v_{17}$, $v_1 \to v_{18}$. To defend an 3^{rd} attack at v_8, the guards move as follows: $v_5 \to v_8$, $v_4 \to v_7$. To defend an 4^{th} attack at v_{14}, the guards move as follows: $v_{17} \to v_{14}$, $v_{18} \to v_{15}$, $v_7 \to v_4$, $v_8 \to v_5$ leaving v_2 undefended.

In all the above cases, we see that $\gamma_{mt}^{\infty}(G_1) > \gamma_t(G_1) = \gamma_t(G)$. By Theorem 2.4, $\gamma_{mt}^{\infty}(G) = \gamma_t(G)$. Hence $\gamma_{mt}^{\infty}(G_1) > \gamma_{mt}^{\infty}(G)$. Therefore $b_{mt}(G) = 2$.

Case(ii): $n \equiv 3(mod\, 8)$
Consider the graph $G_1 = G - e$, where $e = \{v_1v_2\}$. Let $S = \{v_{8k-6}, v_{8k-5} : k = 1, 2, \ldots, \lfloor \frac{n}{8} \rfloor\}$ be the γ_t-sets of G_1. We observe that S is also the γ_t-set G. Now, we place the guards in S. We partition the vertices $v \notin S$ as follows:
$A = \{v_{8k-4} \quad k = 1, 2, \ldots \lceil \frac{n}{8} \rceil\}$
$B = \{v_{8k-3} \quad k = 1, 2, \ldots \lfloor \frac{n}{8} \rfloor\}$
$C = \{v_{8k-2} \quad k = 1, 2, \ldots \lfloor \frac{n}{8} \rfloor\}$
$D = \{v_{8k-1} \quad k = 1, 2, \ldots \lfloor \frac{n}{8} \rfloor\}$
$E = \{v_{8k} \quad k = 1, 2, \ldots \lfloor \frac{n}{8} \rfloor\}$
$F = \{v_{8k+1} \quad k = 1, 2, \ldots \lfloor \frac{n}{8} \rfloor\}$

To defend an attack at a vertex $r \in A$ or $r \in B$ or $r \in C$, the guards in S move in the clockwise direction leaving no vertex undefended. To defend an attack at a vertex $r \in D$ or $r \in E$ or $r \in F$, the guards in S move in the anticlockwise direction leaving no vertex undefended.
Without Loss of generality consider the graph $G_1 = G - e$, where $e = v_1v_4$. Let $S = \{v_{8k-6}, v_{8k-5} : k = 1, 2, \ldots, \lceil \frac{n}{8} \rceil\}$ be the γ_t-sets of G_1. We observe that S is also the γ_t-set G. Now, we place the guards in S. We partition the vertices $v \notin S$ as discussed above. Similar argument follows as discussed above when there is an attack.

In both the above cases, we see that $\gamma^{\infty}_{mt}(G_1) = \gamma_t(G_1) = \gamma_t(G)$. By Theorem 2.4, $\gamma^{\infty}_{mt}(G) = \gamma_t(G)$. Hence $\gamma^{\infty}_{mt}(G_1) = \gamma^{\infty}_{mt}(G)$. Therefore $b_{mt}(G) \geq 2$.
Now, consider the graph $G_1 = G - \{e_1, e_2\}$, where $e_1 = v_1v_2$ and $e_2 = v_2v_3$. The possible γ_t-set of G_1 which are also a γ_t-set of G are listed below.
$S_1 = \{v_{8k-5}, v_{8k-4} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{8} \rfloor\} \cup \{v_n, v_{n-1}\}$
$S_2 = \{v_{8k-4}, v_{8k-3} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{8} \rfloor\} \cup \{v_n, v_{n-1}\}$
$S_3 = \{v_{8k-3}, v_{8k-2} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{8} \rfloor\} \cup \{v_n, v_{n-1}\}$
$S_4 = \{v_{8k-2}, v_{8k-1} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{8} \rfloor\} \cup \{v_n, v_{n-1}\}$
$S_5 = \{v_{8k-1}, v_{8k} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{8} \rfloor\} \cup \{v_n, v_{n-1}\}$

When the guards are placed in S_1, to defend an attack at v_2, the guards move as follows: $v_{n-1} \to v_2$, $v_n \to v_{n-1}$, $v_{12} \to v_{13}$, $v_{11} \to v_{12}$, $v_4 \to v_5$, $v_3 \to v_4$,leaving v_1 undefended. When the guards are placed in S_2, to defend an attack at v_2, the guards move as follows: $v_{n-1} \to v_2$, $v_n \to v_{n-1}$ leaving v_1 undefended. When the guards are placed in S_3, to defend an attack at v_2, the guards move as follows: $v_{n-1} \to v_2$, $v_n \to v_{n-1}$ leaving v_1 undefended. When the guards are placed in S_4, to defend an attack at v_2, the guards move as follows: $v_{n-1} \to v_2$, $v_n \to v_{n-1}$ leaving v_1 undefended. When the guards are placed in S_5, to defend an attack at v_2, the guards move as follows: $v_{n-1} \to v_2$, $v_n \to v_{n-1}$ leaving v_1 and v_3 undefended.

In all the above cases, we see that $\gamma^{\infty}_{mt}(G_1) > \gamma_t(G_1) = \gamma_t(G)$. By Theorem 2.4, $\gamma^{\infty}_{mt}(G) = \gamma_t(G)$. Hence $\gamma^{\infty}_{mt}(G_1) > \gamma^{\infty}_{mt}(G)$. Therefore $b_{mt}(G) = 2$.

Case(iii): $n \equiv 4(mod\, 8)$
Consider the graph $G_1 = G - e$, where $e = \{v_1v_2\}$. Let $S = \{v_{8k-6}, v_{8k-5} : k = 1, 2, \ldots, \lceil \frac{n}{8} \rceil\}$ be the γ_t-sets of G_1. We observe that S is also the γ_t-set G. Now, we place the guards in S. We partition the vertices $v \notin S$ as follows:
$A = \{v_{8k-4} \quad k = 1, 2, \ldots \lceil \frac{n}{8} \rceil\}$
$B = \{v_{8k-3} \quad k = 1, 2, \ldots \lceil \frac{n}{8} \rceil\}$
$C = \{v_{8k-2} \quad k = 1, 2, \ldots \lfloor \frac{n}{8} \rfloor\}$
$D = \{v_{8k-1} \quad k = 1, 2, \ldots \lfloor \frac{n}{8} \rfloor\}$
$E = \{v_{8k} \quad k = 1, 2, \ldots \lfloor \frac{n}{8} \rfloor\}$
$F = \{v_{8k+1} \quad k = 1, 2, \ldots \lfloor \frac{n}{8} \rfloor\}$

To defend an attack at a vertex $r \in A$ or $r \in B$ or $r \in C$, the guards in S move in the clockwise direction leaving no vertex undefended. To defend an attack at a vertex $r \in D$ or $r \in E$ or $r \in F$, the guards in S move in the anticlockwise direction leaving no vertex undefended.
Without Loss of generality consider the graph $G_1 = G - e$, where $e = v_1v_4$. Let $S = \{v_{8k-6}, v_{8k-5} : k = 1, 2, \ldots, \lceil \frac{n}{8} \rceil\}$ be the γ_t-sets of G_1. We observe that S is also the γ_t-set G. Now, we place the guards in S. We partition the vertices $v \notin S$ as discussed above. Similar argument follows as discussed above when there is an attack.

In both the above cases, we see that $\gamma_{mt}^{\infty}(G_1) = \gamma_t(G_1) = \gamma_t(G)$. By Theorem 2.4, $\gamma_{mt}^{\infty}(G) = \gamma_t(G)$. Hence $\gamma_{mt}^{\infty}(G_1) = \gamma_{mt}^{\infty}(G)$. Therefore $b_{mt}(G) \geq 2$.
Now, consider the graph $G_1 = G - \{e_1, e_2\}$, where $e_1 = v_1v_2$ and $e_2 = v_2v_3$. The possible γ_t-set of G_1 which are also a γ_t-set of G are listed below.
$S_1 = \{v_{8k-4}, v_{8k-3} \ : \ k = 1, 2, \ldots, \lceil \frac{n}{8} \rceil\}$
$S_2 = \{v_{8k-3}, v_{8k-2} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{8} \rfloor\} \cup \{v_n, v_{n-1}\}$
$S_3 = \{v_{8k-2}, v_{8k-1} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{8} \rfloor\} \cup \{v_n, v_{n-1}\}$

When the guards are placed in S_1, to defend an attack at v_2, the guards move as follows: $v_5 \to v_2$, $v_4 \to v_5$ leaving v_7 undefended. When the guards are placed in S_2, to defend an 1^{st} attack at v_2, the guards move as follows: $v_5 \to v_2$, $v_6 \to v_5$, $v_{13} \to v_{12}$, $v_{14} \to v_{13}$. To defend an 2^{nd} attack at v_3, the guards move as follows: $v_n \to v_3$, $v_{n-1} \to v_n$, leaving v_7 and v_{n-2} undefended. When the guards are placed in S_3, to defend an attack at v_2, the guards move as follows: $v_{n-1} \to v_2$, $v_6 \to v_5$, $v_7 \to v_6$, $v_{14} \to v_{13}$, $v_{15} \to v_{14}$, $v_n \to v_{n-1}$, leaving v_1 undefended.

In all the above cases, we see that $\gamma_{mt}^{\infty}(G_1) > \gamma_t(G_1) = \gamma_t(G)$. By Theorem 2.4, $\gamma_{mt}^{\infty}(G) = \gamma_t(G)$. Hence $\gamma_{mt}^{\infty}(G_1) > \gamma_{mt}^{\infty}(G)$. Therefore $b_{mt}(G) = 2$.
Case(iv): $n \equiv 5(mod\ 8)$
Consider the graph $G_1 = G - e$, where $e = \{v_1v_2\}$. Let $S = \{v_{8k-6}, v_{8k-5} : k = 1, 2, \ldots, \lceil \frac{n}{8} \rceil\}$ be the γ_t-sets of G_1. We observe that S is also the γ_t-set G. Now, we place the guards in S. We partition the vertices $v \notin S$ as follows:
$A = \{v_{8k-4} \quad k = 1, 2, \ldots \lceil \frac{n}{8} \rceil\}$
$B = \{v_{8k-3} \quad k = 1, 2, \ldots \lceil \frac{n}{8} \rceil\}$
$C = \{v_{8k-2} \quad k = 1, 2, \ldots \lceil \frac{n}{8} \rceil\}$
$D = \{v_{8k-1} \quad k = 1, 2, \ldots \lfloor \frac{n}{8} \rfloor\}$
$E = \{v_{8k} \quad k = 1, 2, \ldots \lfloor \frac{n}{8} \rfloor\}$
$F = \{v_{8k+1} \quad k = 1, 2, \ldots \lfloor \frac{n}{8} \rfloor\}$

To defend an attack at a vertex $r \in A$ or $r \in B$ or $r \in C$, the guards in S move in the clockwise direction leaving no vertex undefended. To defend an attack at a vertex $r \in D$ or $r \in E$ or $r \in F$, the guards in S move in the anticlockwise direction leaving no vertex undefended.
Without Loss of generality consider the graph $G_1 = G - e$, where $e = v_1v_4$. Let $S = \{v_{8k-6}, v_{8k-5} : k = 1, 2, \ldots, \lceil \frac{n}{8} \rceil\}$ be the γ_t-sets of G_1. We observe that S is also the γ_t-set G. Now, we place the guards in S. We partition the vertices $v \notin S$ as discussed above. Similar argument follows as discussed above when there is an attack.

In both the above cases, we see that $\gamma_{mt}^{\infty}(G_1) = \gamma_t(G_1) = \gamma_t(G)$. By Theorem 2.4, $\gamma_{mt}^{\infty}(G) = \gamma_t(G)$. Hence $\gamma_{mt}^{\infty}(G_1) = \gamma_{mt}^{\infty}(G)$. Therefore $b_{mt}(G) \geq 2$.
Now, consider the graph $G_1 = G - \{e_1, e_2\}$, where $e_1 = v_1v_2$ and $e_2 = v_2v_3$. The possible γ_t-set of G_1 which are also a γ_t-set of G are listed below.
$S_1 = \{v_{8k-4}, v_{8k-3} \ : \ k = 1, 2, \ldots, \lceil \frac{n}{8} \rceil\}$
$S_2 = \{v_{8k-3}, v_{8k-2} \ : \ k = 1, 2, \ldots, \lceil \frac{n}{8} \rceil\}$
$S_3 = \{v_{8k-5}, v_{8k-4} \ : \ k = 1, 2, \ldots, \lfloor \frac{n}{8} \rfloor\}$

When the guards are placed in S_1, to defend an attack at v_2, the guards move as follows: $v_5 \to v_2$, $v_4 \to v_5$ leaving v_7 undefended. When the guards are placed in S_2, to defend an attack at v_2, the guards move as follows: $v_5 \to v_2$, $v_6 \to v_5$, $v_{13} \to v_{12}$, $v_{14} \to v_{13}$, $v_n \to v_{n-1}$, $v_1 \to v_n$, leaving v_7 undefended. When the guards are placed in S_3, to defend an attack at v_2, the guards move as follows: $v_{n-1} \to v_2$, $v_{n-2} \to v_{n-1}$, leaving v_1, v_{16} and v_{18} leaving undefended.

In all the above cases, we see that $\gamma^{\infty}_{mt}(G_1) > \gamma_t(G_1) = \gamma_t(G)$. By Theorem 2.4, $\gamma^{\infty}_{mt}(G) = \gamma_t(G)$. Hence $\gamma^{\infty}_{mt}(G_1) > \gamma^{\infty}_{mt}(G)$. Therefore $b_{mt}(G) = 2$.

□

3. Conclusion

In conclusion, the study demonstrated the m-eternal total bondage number for Circulant graphs $C_n(1,2)$ and $C_n(1,3)$. The future research can be extended to Circulant graphs for $n \geq 3$. One can also find the m eternal total bondage number for other regular graphs.

Acknowledgement. Our sincere thanks to the annoymous reviewers of the paper for their valuable suggesstions and guidance in improving the standard of the paper.

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