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# FOURTH-ORDER DIFFERENTIAL SUBORDINATION RESULTS FOR ANALYTIC FUNCTIONS INVOLVING THE GENERALIZED BESSEL FUNCTIONS

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Abstract. In this current paper, we obtain fourth-order differential subordination results for analytic functions in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ associated with a new operator $\mathcal{S}^{\gamma,k}_{\alpha,\beta} f(z)$ which involves the generalized Bessel function of the 1st kind of order $p$ and the Carlson-Shaffer operator. We also derive some interesting new results for this operator. The results are obtained by considering suitable classes of admissible functions.

Keywords: Admissible functions; Bessel functions; Carlson-Shaffer Operator; Fourth-order differential subordination; sandwich-type results.

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## 1. Introduction and Preliminaries

Let $\mathcal{H}(\Delta)$ be the class of functions which are analytic in open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$, $\mathcal{H}[a, n]$ denote the subclass of functions $f \in \mathcal{H}(\Delta)$ of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (z \in \Delta;\ a \in \mathbb{C};\ n \in \mathbb{N} = \{1, 2, 3, \cdots\})$$

and $\mathcal{H}[1, n] = \mathcal{H}_n$. We denote by $\mathcal{A}$ the class of all normalized analytic functions in $\Delta$ of the form:

$$f(z) = z + \sum_{n=1}^{\infty} a_{n+1} z^{n+1} \quad (z \in \Delta). \tag{1}$$

Also, let the Hadamard product (or convolution) of two functions

$$g_l(z) = z + \sum_{n=1}^{\infty} b_{n+1,l} z^{n+1} \quad (l = 1, 2)$$

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be defined by

$$(g_1 \star g_2)(z) := z + \sum_{n=1}^{\infty} b_{n+1,1} b_{n+1,2} z^{n+1} =: (g_2 \star g_1)(z).$$

For two functions $f,\ G \in \mathcal{A}$, the function $f(z)$ is said to be subordinate to $G(z)$, denoted by $f(z) \prec G(z)$, if there exists a Schwarz function $\vartheta(z)$ analytic in $\Delta$ with

$$\vartheta(0) = 0 \quad \text{and} \quad |\vartheta(z)| < 1 \quad (z \in \Delta)$$

such that

$$f(z) = G(\vartheta(z)) \quad (z \in \Delta).$$

Moreover, if the function $G$ is univalent in $\Delta$, then $f(z) \prec G(z)$ if and only if $f(0) = G(0)$ and $f(\Delta) \subset G(\Delta)$ (see, for details, [22, 24]).

We recall the function $\omega_{\gamma,b,p}(z)$, the generalized Bessel function of the first kind of order $p$ is defined as a particular solution of the second order linear homogenous differential equation (see, for details, [11]):

$$z^2\omega''(z) + bz\omega'(z) + [\gamma z^2 - p^2 + (1-b)p]\omega(z) = 0 \quad (\gamma; b; p \in \mathbb{C}).$$

Moreover, the function $\omega_{\gamma,b,p}(z)$ has the familiar presentation of the form

$$\omega_{\gamma,b,p}(z) = \sum_{n=0}^{\infty} \frac{(-\gamma)^n}{n!\Gamma(p+n+(b+1)/2)} \left(\frac{z}{2}\right)^{2n+p} \quad (z \in \mathbb{C}), \tag{2}$$

where $\Gamma$ is the Euler Gamma function. This series permits one to study in a unified manner of Bessel function, modified Bessel function and spherical Bessel function which we noted below as particular values of $b$ and $\gamma$.

(1) For $b = 1$ and $\gamma = 1$ in (2), the familiar Bessel function defined by (see [11], [34])

$$J_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(p+n+1)} \left(\frac{z}{2}\right)^{2n+p} \quad (z \in \mathbb{C}).$$

(2) For $b = 1$ and $\gamma = -1$ in (2), the modified Bessel function defined by (see [11], [34])

$$I_p(z) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(p+n+1)} \left(\frac{z}{2}\right)^{2n+p} \quad (z \in \mathbb{C}).$$

(3) For $b = 2$ and $\gamma = 1$ in (2), the function reduces to $\sqrt{2}j_p(z)/\sqrt{\pi}$, where $j_p$ is the spherical Bessel function defined by (see [11])

$$j_p(z) = \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(p+n+3/2)} \left(\frac{z}{2}\right)^{2n+p} \quad (z \in \mathbb{C}).$$

Deniz [14] and Deniz *et al.* [15] (see [2, 9, 10, 11, 12, 30, 31]) introduced the function $\varphi_{\gamma,b,p}(z)$ in terms of the generalized Bessel function $\omega_{\gamma,b,p}(z)$ by

$$\varphi_{\gamma,b,p}(z) = 2^p \Gamma\left(p + \frac{b+1}{2}\right) z^{1-p/2} \omega_{\gamma,b,p}(\sqrt{z}), \tag{3}$$

where the function $\omega_{\gamma,b,p}(z)$ is given in (2). The equation (3) can be written as

$$\varphi_{\gamma,b,p}(z) := \varphi_{\gamma,k}(z) = z + \sum_{n=1}^{\infty} \frac{(-\gamma)^n}{4^n (k)_n} \frac{z^{n+1}}{n!} \quad \left(k = p + \frac{b+1}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-\right),$$

where $\mathbb{Z}_0^- = \{0, -1, -2, \cdots\}$ and $(\delta)_m$ is the Pochhammer symbol:

$$(\delta)_m = \frac{\Gamma(\delta+m)}{\Gamma(\delta)} = \begin{cases} 1, & if\ m = 0, \\ \delta(\delta+1)\cdots(\delta+m-1), & if\ m \in \mathbb{N}. \end{cases}$$

The Carlson-Shaffer Operator $\mathcal{L}(\alpha, \beta)f(z)$ [13] is defined by

$$\mathcal{L}(\alpha, \beta)f(z) = \theta(\alpha, \beta; z) * f(z) = z + \sum_{n=1}^{\infty} \frac{(\alpha)_n}{(\beta)_n} a_{n+1} z^{n+1},$$

where $\theta(\alpha, \beta; z)$ is the incomplete beta function

$$\theta(\alpha, \beta; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n} z^{n+1} \quad (\beta \neq 0, -1, -2, \ldots;\ z \in \Delta).$$

We now define the operator $\mathcal{S}_{\alpha,\beta}^{\gamma,k} : \mathcal{A} \longrightarrow \mathcal{A}$ for a function $f$ of the form (1) by (using Hadamard product of the Carlson-Shaffer Operator and the generalized Bessel function)

$$\mathcal{S}_{\alpha,\beta}^{\gamma,k} f(z) = \varphi_{\gamma,k}(z) * \mathcal{L}(\alpha, \beta)f(z) = z + \sum_{n=1}^{\infty} \frac{(-\gamma)^n (\alpha)_n a_{n+1}}{4^n (k)_n (\beta)_n} \frac{z^{n+1}}{n!}. \tag{4}$$

From (4) we have the identity relation

$$z\left(\mathcal{S}_{\alpha,\beta}^{\gamma,k} f(z)\right)' = \alpha \mathcal{S}_{\alpha+1,\beta}^{\gamma,k} f(z) - (\alpha - 1)\mathcal{S}_{\alpha,\beta}^{\gamma,k} f(z). \tag{5}$$

Indeed, the function $\mathcal{S}_{\alpha,\beta}^{\gamma,k} f(z)$ is an elementary transform of the generalized hypergeometric function defined by (see [23, 25, 27, 28, 29]; also [16, 17])

$${}_qF_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z) = \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{z^n}{n!}$$

$$(\alpha_i \in \mathbb{C};\ \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ q \leq s+1;\ q, s \in \mathbb{N} \cup \{0\};\ i = 1, 2, \cdots, q;\ j = 1, 2, \cdots, s).$$

For instance

$$\mathcal{S}_{\alpha,\beta}^{\gamma,k} f(z) = z\ {}_0F_1\left(k; -\frac{\gamma}{4} z\right) * \mathcal{L}(\alpha, \beta)f(z).$$

We notice that, for suitable choices of the parameters $b$ and $\gamma$, we derive some new operators:

(i) Taking $b = \gamma = 1$ in (4), we have the operator $\mathbb{J}_p : \mathcal{A} \longrightarrow \mathcal{A}$ related with Bessel function, defined by

$$\begin{aligned} \mathbb{J}_p f(z) = \varphi_{1,1,p} * \mathcal{L}(\alpha, \beta)f(z) &= [2^p \Gamma(p+1) z^{1-p/2} J_p(\sqrt{z})] * \mathcal{L}(\alpha, \beta)f(z) \\ &= z + \sum_{n=1}^{\infty} \frac{(-1)^n (\alpha)_n a_{n+1}}{4^n (p+1)_n (\beta)_n} \frac{z^{n+1}}{n!} \end{aligned} \tag{6}$$

(ii) Taking $b = 1$ and $\gamma = -1$ in (4), we have the operator $\mathbb{I}_p : \mathcal{A} \longrightarrow \mathcal{A}$ related with modified Bessel function, defined by

$$\begin{aligned} \mathbb{I}_p f(z) = \varphi_{1,-1,p} * \mathcal{L}(\alpha, \beta)f(z) &= [2^p \Gamma(p+1) z^{1-p/2} I_p(\sqrt{z})] * \mathcal{L}(\alpha, \beta)f(z) \\ &= z + \sum_{n=1}^{\infty} \frac{(\alpha)_n a_{n+1}}{4^n (p+1)_n (\beta)_n} \frac{z^{n+1}}{n!} \end{aligned} \tag{7}$$

(iii) Taking $b = 2$ and $\gamma = 1$ in (4), we have the operator $\mathbb{S}_p : \mathcal{A} \longrightarrow \mathcal{A}$ related with modified Bessel function, defined by

$$\begin{aligned}\mathbb{S}_p f(z) &= [\pi^{-1/2} 2^{p+1/2} \Gamma(p + 3/2) z^{1-p/2} j_p(\sqrt{z})] * \mathcal{L}(\alpha, \beta) f(z) \\ &= z + \sum_{n=1}^{\infty} \frac{(-1)^n (\alpha)_n a_{n+1}}{4^n (p + 3/2)_n (\beta)_n} \frac{z^{n+1}}{n!}. \end{aligned} \tag{8}$$

There are many research articles in the literature dealing with the first-order and second-order differential subordination problems for analytic functions in the open unit disk. In 1992, Ponnusamy and Juneja [26] introduced the concepts of third-order differential inequalities in the complex plane and then the third-order differential subordination theory was introduced by Antonino and Miller [3] in 2011. In recent years, by using the concepts of third-order differential subordination, several researchers obtained many interesting results involving various linear and nonlinear operators and study is continuing on it (see, for example, [6, 8, 18, 19, 20, 32, 33]) and the references cited therein. Recently, Atshan *et al.* [7] introduced and studied the concepts of fourth-order differential subordination in 2020 (which is a generalization of third-order differential subordination results obtained by Antonino and Miller [3]), and in this context, there are only a few articles dealing with the fourth-order differential subordination problems (see, for examples, [1, 4, 7, 5, 21]). This idea proved to be a significant application in the field of Geometric Function Theory of Complex Analysis. Their work has motivated and encouraged many further developments in this direction.

Now, we provide the context of well-known notations and definitions used for obtaining the main results.

**Definition 1.1** (see [3]). *Let $\mathcal{Q}$ be the set of all functions $\varrho$ that are analytic and univalent on $\overline{\Delta} \setminus E(\varrho)$ where*

$$E(\varrho) = \left\{ \zeta \in \partial\Delta : \lim_{\omega \longrightarrow \zeta} \varrho(\omega) = \infty \right\},$$

*and are such that* $\min \mid \varrho'(\zeta) \mid = \rho > 0$ *for $\zeta \in \partial\Delta \setminus E(\varrho)$. Further, let $\mathcal{Q}(a)$ denote the subclass of $\mathcal{Q}$ consisting of functions $\varrho$ for which $\varrho(0) = a$ and $\mathcal{Q}(1) = \mathcal{Q}_1 = \{\varrho(z) \in \mathcal{Q} : \varrho(0) = 1\}$.*

**Definition 1.2** (see [7]). *Assume that $h$ is univalent in $\Delta$ and $\psi : \mathbb{C}^5 \times \Delta \longrightarrow \mathbb{C}$. If the analytic function $p$ fulfills the fourth-order differential subordination*

$$\psi\left(g(z), zg'(z), z^2 g''(z), z^3 g'''(z), z^4 g''''(z); z\right) \prec h(z) \quad (z \in \Delta), \tag{9}$$

*then the function $g$ is called a solution of the differential subordination (9). A univalent function $\varrho$ is called a dominant of the solutions of the differential subordination if $g \prec \varrho$ for all $g$ satisfying (9). A dominant $\tilde{\varrho}(z)$ that fulfils $\tilde{\varrho} \prec \varrho$ for all dominants $\tilde{\varrho}$ of (9) is called the best dominant.*

**Definition 1.3** (see [7]). *If $\Omega \subseteq \mathbb{C}, \varrho \in \mathcal{Q}$ and $n \in \mathbb{N} \setminus \{2\}$. Let $\Psi_j[\Omega, \varrho]$ be the family of admissible functions consisting of functions $\psi : \mathbb{C}^5 \times \Delta \longrightarrow \mathbb{C}$, which fulfill the admissibility condition:*

$$\psi(\boldsymbol{r}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{u}, \boldsymbol{v}; z) \notin \Omega$$

*whenever*

$$\boldsymbol{r} = \varrho(\zeta), \quad \boldsymbol{s} = m\zeta\varrho'(\zeta), \quad \Re\left\{\frac{\boldsymbol{t}}{\boldsymbol{s}} + 1\right\} \geq m\Re\left\{1 + \frac{\zeta\varrho''(\zeta)}{\varrho'(\zeta)}\right\},$$

$$\Re\left\{\frac{\boldsymbol{u}}{\boldsymbol{s}}\right\} \geq m^2\Re\left\{\frac{\zeta^2\varrho'''(\zeta)}{\varrho'(\zeta)}\right\} \text{ and } \Re\left\{\frac{\boldsymbol{v}}{\boldsymbol{s}}\right\} \geq m^3\Re\left\{\frac{\zeta^3\varrho''''(\zeta)}{\varrho'(\zeta)}\right\},$$

*where* $z \in \Delta, \zeta \in \partial\Delta \setminus E(\varrho)$ *and* $m \geq n$.

**Lemma 1.1** (see [7]). *Let* $g \in \mathcal{H}[a, n]$ *with* $n \geq 3$. *Furthermore, let* $\varrho \in \mathcal{Q}$ *and fulfill the following conditions:*

$$\Re\left\{\frac{\zeta^2\varrho'''(\zeta)}{\varrho'(\zeta)}\right\} \geq 0 \ \ and \ \ \left|\frac{z^2 g''(z)}{\varrho'(\zeta)}\right| \leq m^2,$$

*where* $z \in \Delta,\ \zeta \in \partial\Delta \setminus E(\varrho)$ *and* $m \geq n$. *If* $\Omega$ *is a set in* $\mathbb{C}, \psi \in \Psi_j[\Omega, \varrho]$ *and*

$$\psi\left(g(z), zg'(z), z^2g''(z), z^3g'''(z), z^4g''''(z); z\right) \in \Omega,$$

*then*

$$g(z) \prec \varrho(z) \quad (z \in \Delta).$$

The main purpose of this paper is to consider certain appropriate classes of admissible functions and investigate some fourth-order differential subordination results of analytic functions associated with new operator defined by (4).

This work is organized into three sections. Section 1 overviews some deep results in the theory of differential subordination and uses them to prove the existence of a general algorithm for solving all fourth-order differential subordination results. In Section 2, we prove our main results of differential subordination by using new operator and some corollaries are also deduced. Section 3 concludes the work.

## 2. Fourth-order subordination results associated with the operator $\mathcal{S}_{\alpha,\beta}^{\gamma,k}$

We give the class of admissible functions, which is required in proving differential subordination theorems using the operator $\mathcal{S}_{\alpha,\beta}^{\gamma,k} f(z)$ given by (4).

**Definition 2.1.** *If* $\Omega \subseteq \mathbb{C}$ *and* $\varrho \in \mathcal{Q}_1 \cap \mathcal{H}_n$. *Let* $\Phi_1[\Omega, \varrho]$ *be the family of admissible functions which consists of functions* $\phi : \mathbb{C}^5 \times \Delta \longrightarrow \mathbb{C}$ *that satisfy the condition of admissibility:*

$$\phi(\mathfrak{a}, \mathfrak{b}, \mathfrak{x}, \mathfrak{y}, \mathfrak{w}; z) \notin \Omega,$$

*whenever*

$$\mathfrak{a} = \varrho(\zeta), \quad \mathfrak{b} = \frac{m\zeta\varrho'(\zeta) + (\alpha-1)\varrho(z)}{\alpha},$$

$$\Re\left\{\frac{\alpha\mathfrak{b} + \alpha(\alpha+1)\mathfrak{x} - (\alpha^2-1)\mathfrak{a}}{\alpha\mathfrak{b} - (\alpha-1)\mathfrak{a}} - 2\alpha\right\} \geq m\Re\left\{1 + \frac{\zeta\varrho''(\zeta)}{\varrho'(\zeta)}\right\},$$

$$\Re\left\{\frac{\alpha(\alpha+1)[(\alpha+2)\mathfrak{y} - 3(\alpha+1)\mathfrak{x} + 2(\alpha-1)\mathfrak{a}]}{\alpha\mathfrak{b} - (\alpha-1)\mathfrak{a}} + 3\alpha(\alpha+1)\right\} \geq m^2\Re\left\{\frac{\zeta^2\varrho'''(\zeta)}{\varrho'(\zeta)}\right\} \ \ and$$

$$\Re\left\{\frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)\mathfrak{w} - 4\alpha(\alpha+1)(\alpha+2)^2\mathfrak{y} + 6\alpha(\alpha+1)^2(\alpha+2)\mathfrak{x}}{\alpha\mathfrak{b} - (\alpha-1)\mathfrak{a}} - \frac{\alpha(\alpha^2-1)(2\alpha+5)\mathfrak{a}}{\alpha\mathfrak{b} - (\alpha-1)\mathfrak{a}} - 4\alpha(\alpha+1)\right\} \geq m^3\Re\left\{\frac{\zeta^3\varrho''''(\zeta)}{\varrho'(\zeta)}\right\},$$

*where* $z \in \Delta, \zeta \in \partial\Delta \setminus E(\varrho)$ *and* $m \geq 3$.

**Theorem 2.1.** *Assume that* $\Omega \subseteq \mathbb{C}$ *and* $\phi \in \Phi_1[\Omega, \varrho]$. *If* $f \in \mathcal{A}$ *and* $\varrho \in \mathcal{Q}_1$ *satisfy the following conditions:*

$$\Re\left(\frac{\zeta^2\varrho'''(\zeta)}{\varrho'(\zeta)}\right) \geq 0, \qquad \left|\left(\frac{\mathcal{S}_{\alpha+2,\beta}^{\gamma,k} f(z)}{\varrho'(\zeta)}\right)\right| \leq m^2 \tag{10}$$

*and*

$$\phi\left(\mathcal{S}^{\gamma,k}_{\alpha,\beta}f(z),\mathcal{S}^{\gamma,k}_{\alpha+1,\beta}f(z),\mathcal{S}^{\gamma,k}_{\alpha+2,\beta}f(z),\mathcal{S}^{\gamma,k}_{\alpha+3,\beta}f(z),\mathcal{S}^{\gamma,k}_{\alpha+4,\beta}f(z);z\right)\subset\Omega, \tag{11}$$

*then*

$$\mathcal{S}^{\gamma,k}_{\alpha,\beta}f(z)\prec\varrho(z)\quad(z\in\Delta).$$

*Proof.* Define the function $g(z)$ in $\Delta$ by

$$\mathcal{S}^{\gamma,k}_{\alpha,\beta}f(z)=g(z)\quad(z\in\Delta). \tag{12}$$

Differentiating (12) with respect to $z$ and by using the recurrence relation (5), we have

$$\mathcal{S}^{\gamma,k}_{\alpha+1,\beta}f(z)=\frac{zg'(z)+(\alpha-1)g(z)}{\alpha}. \tag{13}$$

Again, differentiating (13) with respect to $z$ and by making use of (5), we obtain

$$\mathcal{S}^{\gamma,k}_{\alpha+2,\beta}f(z)=\frac{z^2g''(z)+2\alpha zg'(z)+\alpha(\alpha-1)g(z)}{\alpha(\alpha+1)}. \tag{14}$$

Further computations show that

$$\mathcal{S}^{\gamma,k}_{\alpha+3,\beta}f(z)=\frac{z^3g'''(z)+3(\alpha+1)z^2g''(z)+3\alpha(\alpha+1)zg'(z)+\alpha(\alpha^2-1)g(z)}{\alpha(\alpha+1)(\alpha+2)}. \tag{15}$$

and

$$\mathcal{S}^{\gamma,k}_{\alpha+4,\beta}f(z)=\frac{z^4g''''(z)+(\alpha+2)[4z^3g'''(z)+6(\alpha+1)z^2g''(z)]}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}+\frac{4\alpha(\alpha+1)(\alpha+2)zg'(z)+\alpha(\alpha^2-1)(\alpha+2)g(z)}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}. \tag{16}$$

Let

$$\begin{aligned}\mathtt{a}=\mathtt{r},\ \mathtt{b}&=\frac{\mathtt{s}+(\alpha-1)\mathtt{r}}{\alpha},\ \mathtt{x}=\frac{\mathtt{t}+2\alpha\mathtt{s}+\alpha(\alpha-1)\mathtt{r}}{\alpha(\alpha+1)},\\ \mathtt{y}&=\frac{\mathtt{u}+3(\alpha+1)\mathtt{t}+3\alpha(\alpha+1)\mathtt{s}+\alpha(\alpha^2-1)\mathtt{r}}{\alpha(\alpha+1)(\alpha+2)}\\ \text{and } \mathtt{w}&=\frac{\mathtt{v}+(\alpha+2)[4\mathtt{u}+6(\alpha+1)\mathtt{t}]+4\alpha(\alpha+1)(\alpha+2)\mathtt{s}+\alpha(\alpha^2-1)(\alpha+2)\mathtt{r}}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}.\end{aligned}$$

We now define the transformation $\psi(\mathtt{r},\mathtt{s},\mathtt{t},\mathtt{u},\mathtt{v};z):\mathbb{C}^5\times\Delta\longrightarrow\mathbb{C}$ by

$$\begin{aligned}\psi(\mathtt{r},\mathtt{s},\mathtt{t},\mathtt{u},\mathtt{v};z)&=\phi(\mathtt{a},\mathtt{b},\mathtt{x},\mathtt{y},\mathtt{w};z)=\phi\left(\mathtt{r},\frac{\mathtt{s}+(\alpha-1)\mathtt{r}}{\alpha},\frac{\mathtt{t}+2\alpha\mathtt{s}+\alpha(\alpha-1)\mathtt{r}}{\alpha(\alpha+1)},\right.\\ &\left.\frac{\mathtt{u}+(\alpha+1)[3\mathtt{t}+3\alpha\mathtt{s}+\alpha(\alpha-1)\mathtt{r}]}{\alpha(\alpha+1)(\alpha+2)},\frac{\mathtt{v}+(\alpha+2)[4\mathtt{u}+(\alpha+1)\{6\mathtt{t}+4\alpha\mathtt{s}+\alpha(\alpha-1)\mathtt{r}\}]}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)};z\right).\end{aligned} \tag{17}$$

Making use of the equations (12) to (16), we find from (17) that

$$\begin{aligned}&\psi\left(g(z),zg'(z),z^2g''(z),z^3g'''(z),z^4g''''(z);z\right)\\ &\qquad=\phi\left(\mathcal{S}^{\gamma,k}_{\alpha,\beta}f(z),\mathcal{S}^{\gamma,k}_{\alpha+1,\beta}f(z),\mathcal{S}^{\gamma,k}_{\alpha+2,\beta}f(z),\mathcal{S}^{\gamma,k}_{\alpha+3,\beta}f(z),\mathcal{S}^{\gamma,k}_{\alpha+4,\beta}f(z);z\right).\end{aligned}$$

Therefore, (11) transforms into

$$\psi\left(g(z),zg'(z),z^2g''(z),z^3g'''(z),z^4g''''(z);z\right)\in\Omega.$$

We observe that

$$\frac{\mathtt{t}}{\mathtt{s}}+1=\frac{\alpha(\alpha+1)\mathtt{x}+\alpha\mathtt{b}-(\alpha^2-1)\mathtt{a}}{\alpha\mathtt{b}-(\alpha-1)\mathtt{a}}-2\alpha,$$

$$\frac{\mathtt{u}}{\mathtt{s}}=\frac{\alpha(\alpha+1)[(\alpha+2)\mathtt{y}-3(\alpha+1)\mathtt{x}+2(\alpha-1)\mathtt{a}]}{\alpha\mathtt{b}-(\alpha-1)\mathtt{a}}+3\alpha(\alpha+1)$$

and

$$\frac{\mathtt{v}}{\mathtt{s}}=\frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)\mathtt{w}-4\alpha(\alpha+1)(\alpha+2)^2\mathtt{y}}{\alpha\mathtt{b}-(\alpha-1)\mathtt{a}}+\frac{6\alpha(\alpha+1)^2(\alpha+2)\mathtt{x}-\alpha(\alpha^2-1)(2\alpha+5)\mathtt{a}}{\alpha\mathtt{b}-(\alpha-1)\mathtt{a}}-4\alpha(\alpha+1).$$

Hence, the admissibility condition for $\phi\in\Phi_1[\Omega,\varrho]$ of Definition 2.1 is equivalent to the admissibility condition for the function $\psi\in\Psi_j[\Omega,\varrho]$. Thus, by Lemma 1.1, we have $g(z)\prec\varrho(z)$ or

$$\mathcal{S}^{\gamma,k}_{\alpha,\beta}f(z)\prec\varrho(z)\quad(z\in\Delta).$$

□

The following result will be an extension of Theorem 2.1 when the behavior of the function $\varrho(\omega)$ on $\partial\Delta$ is unknown.

**Corollary 2.1.** *If $\Omega\subseteq\mathbb{C}$ and $\varrho$ is univalent in $\Delta$ with $\varrho\in\mathcal{Q}_1$. Let $\phi\in\Phi_1[\Omega,\varrho_\rho]$ for some $\rho\in(0,1)$,where $\varrho_\rho(z)=\varrho(\rho z)$. If $f\in\mathcal{A}$ and $\varrho_\rho(z)$ satisfy the following conditions:*

$$\Re\left\{\frac{\zeta^2\varrho_\rho'''(\zeta)}{\varrho_\rho'(\zeta)}\right\}\geq 0\quad\text{and}\quad\left|\left(\frac{\mathcal{S}^{\gamma,k}_{\alpha+2,\beta}f(z)}{\varrho_\rho'(\zeta)}\right)\right|\leq m^2\quad(z\in\Delta)\tag{18}$$

*and*

$$\phi\left(\mathcal{S}^{\gamma,k}_{\alpha,\beta}f(z),\mathcal{S}^{\gamma,k}_{\alpha+1,\beta}f(z),\mathcal{S}^{\gamma,k}_{\alpha+2,\beta}f(z),\mathcal{S}^{\gamma,k}_{\alpha+3,\beta}f(z),\mathcal{S}^{\gamma,k}_{\alpha+4,\beta}f(z);z\right)\subset\Omega,$$

*then*

$$\mathcal{S}^{\gamma,k}_{\alpha,\beta}f(z)\prec\varrho(z)\quad(z\in\Delta).$$

*Proof.* We observe from Theorem 2.1 that $\mathcal{S}^{\gamma,k}_{\alpha,\ beta}f(z)\prec\varrho_\rho(z)\quad(z\in\Delta)$. The result claimed by Corollary 2.1 is now deduced from the following subordination relationship:

$$\varrho_\rho(z)\prec\varrho(z)\quad(z\in\Delta).$$

□

If $\Omega\neq\mathbb{C}$ is a simply connected domain, then $\Omega=h(\Delta)$ for some conformal mapping $h$ from $\Delta$ into the domain $\Omega$. We denote the class $\Phi[h(\Delta),\varrho]$ by $\Phi[h,\varrho]$. Next two results are an immediate consequences of Theorem 2.1 and Corollary 2.1.

**Theorem 2.2.** *Let $\phi\in\Phi[h,\varrho]$. If $f\in\mathcal{A}$ and $\varrho\in\mathcal{Q}_1$ satisfies (10) and*

$$\phi\left(\mathcal{S}^{\gamma,k}_{\alpha,\beta}f(z),\mathcal{S}^{\gamma,k}_{\alpha+1,\beta}f(z),\mathcal{S}^{\gamma,k}_{\alpha+2,\beta}f(z),\mathcal{S}^{\gamma,k}_{\alpha+3,\beta}f(z),\mathcal{S}^{\gamma,k}_{\alpha+4,\beta}f(z);z\right)\prec h(z),\tag{19}$$

*then*

$$\mathcal{S}^{\gamma,k}_{\alpha,\beta}f(z)\prec\varrho(z)\quad(z\in\Delta).$$

**Corollary 2.2.** *If $\varrho$ is univalent function in $\Delta$ with $\varrho \in \mathcal{Q}_1$ and $\phi \in \Phi[h, \varrho_\rho]$ for some $\varrho \in (0,1)$, where $\varrho_\rho(\omega) = \varrho(\rho\omega)$. If $f \in \mathcal{A}$ and $\varrho_\rho$ satisfies (18) and*

$$\phi\left(\mathcal{S}^{\gamma,k}_{\alpha,\beta}f(z), \mathcal{S}^{\gamma,k}_{\alpha+1,\beta}f(z), \mathcal{S}^{\gamma,k}_{\alpha+2,\beta}f(z), \mathcal{S}^{\gamma,k}_{\alpha+3,\beta}f(z), \mathcal{S}^{\gamma,k}_{\alpha+4,\beta}f(z); z\right) \prec h(z),$$

*then*

$$\mathcal{S}^{\gamma,k}_{\alpha,\beta}f(z) \prec \varrho(z) \quad (z \in \Delta).$$

Now, the next theorem gives the best dominant of the differential subordination (19).

**Theorem 2.3.** *Suppose that the function $h$ is univalent in $\Delta$. Also suppose that $\phi : \mathbb{C}^5 \times \Delta \longrightarrow \mathbb{C}$ and the differential equation*

$$\begin{aligned}\phi\Bigg(\varrho(z), &\frac{z\varrho'(z) + (\alpha-1)\varrho(z)}{\alpha}, \frac{z^2\varrho''(z) + 2\alpha z\varrho'(z) + \alpha(\alpha-1)\varrho(z)}{\alpha(\alpha+1)}, \frac{z^3\varrho'''(z) + 3(\alpha+1)z^2\varrho''(z)}{\alpha(\alpha+1)(\alpha+2)} \\ &+ \frac{3\alpha(\alpha+1)z\varrho'(z) + \alpha(\alpha^2-1)\varrho(z)}{\alpha(\alpha+1)(\alpha+2)}, \frac{z^4\varrho''''(z) + 4(\alpha+2)z^3\varrho'''(z) + 6(\alpha+1)(\alpha+2)z^2\varrho''(z)}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} \\ &+ \frac{4\alpha(\alpha+1)(\alpha+2)z\varrho'(z) + \alpha(\alpha^2-1)(\alpha+2)\varrho(z)}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}; z\Bigg) = h(z) \quad (20)\end{aligned}$$

*has a solution $\varrho(z)$ with $\varrho(0) = 1$ and $\varrho(z)$ verifies the condition (10). If $f \in \mathcal{A}$, $\phi \in \Phi[h, \varrho_\rho]$ and*

$$\phi\left(\mathcal{S}^{\gamma,k}_{\alpha,\beta}f(z), \mathcal{S}^{\gamma,k}_{\alpha+1,\beta}f(z), \mathcal{S}^{\gamma,k}_{\alpha+2,\beta}f(z), \mathcal{S}^{\gamma,k}_{\alpha+3,\beta}f(z), \mathcal{S}^{\gamma,k}_{\alpha+4,\beta}f(z); z\right)$$

*is analytic in $\Delta$, then (19) implies that*

$$\mathcal{S}^{\gamma,k}_{\alpha,\beta}f(z) \prec \varrho(z) \quad (z \in \Delta)$$

*and $\varrho(z)$ is the best dominant.*

*Proof.* Applying Theorem 2.2, it can be shown that $\varrho(z)$ is a dominant of equation (19), because $\varrho(z)$ satisfies (20), so that $\varrho(z)$ is a solution of (19) and hence $\varrho(z)$ will be dominated by all dominants. Therefore $\varrho(z)$ is the best dominant. □

Now, we put $\varrho(z) = \mathtt{M}z, \mathtt{M} > 0$, and apropos of definition (2.1), the class of admissible function $\phi[\Omega, \varrho]$, denoted by $\phi[\Omega, \mathtt{M}]$, is depicted below.

**Definition 2.2.** *Let $\Omega \subseteq \mathbb{C}$ and $M > 0$. The family of admissible functions $\Phi[\Omega, M]$ consists of the functions $\phi : \mathbb{C}^5 \times \Delta \longrightarrow \mathbb{C}$, which satisfy the following admissibility condition*

$$\begin{aligned}\phi\Bigg(&Me^{i\theta}, \frac{n + (\alpha-1)}{\alpha}Me^{i\theta}, \frac{L + [2\alpha n + \alpha(\alpha-1)]Me^{i\theta}}{\alpha(\alpha+1)}, \frac{N + 3(\alpha+1)L + [3\alpha(\alpha+1)n + \alpha(\alpha^2-1)]Me^{i\theta}}{\alpha(\alpha+1)(\alpha+2)}, \\ &\frac{X + 4(\alpha+2)N + 6(\alpha+1)(\alpha+2)L + [4\alpha(\alpha+1)(\alpha+2)n + \alpha(\alpha^2-1)(\alpha+2)]Me^{i\theta}}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}; z\Bigg) \notin \Omega\end{aligned}$$

*whenever $\omega \in \Delta, \Re\left\{Le^{-i\theta}\right\} \geq (n-1)nM, \Re\left\{Ne^{-i\theta}\right\} \geq 0$ and $\Re\left\{Xe^{-i\theta}\right\} \geq 0$ for every $\theta \in \mathbb{R}$ and $n \geq 3$.*

Using the definition of the family of admissible functions, from the result in Theorem 2.1 we have the following result.

**Theorem 2.4.** *Assume that $\phi \in \Phi[\Omega, M]$. If $f \in \mathcal{A}$ fulfill the conditions:*

$$\left|\mathcal{S}^{\gamma,k}_{\alpha+2,\beta}f(z)\right| \leq n^2 M \quad (n \geq 3,\ M > 0)$$

*and*

$$\phi\left(\mathcal{S}^{\gamma,k}_{\alpha,\beta}f(z), \mathcal{S}^{\gamma,k}_{\alpha+1,\beta}f(z), \mathcal{S}^{\gamma,k}_{\alpha+2,\beta}f(z), \mathcal{S}^{\gamma,k}_{\alpha+3,\beta}f(z), \mathcal{S}^{\gamma,k}_{\alpha+4,\beta}f(z); z\right) \in \Omega,$$

*then*

$$\left|\mathcal{S}_{\alpha,\beta}^{\gamma,k}f(z)\right| < \mathtt{M} \quad (z \in \Delta).$$

Now, taking $\Omega = \varrho(\Delta) = \{w : |w| < \mathtt{M}\}$, the class $\Phi[\Omega, \mathtt{M}]$ is simply denoted by $\Phi[\mathtt{M}]$.

**Corollary 2.3.** *Let $\phi \in \Phi[\mathtt{M}]$. If $f \in \mathcal{A}$ satisfies $\left|\mathcal{S}_{\alpha+2,\beta}^{\gamma,k}f(z)\right| \leq n^2M$ $(n \geq 3;\ \mathtt{M} > 0)$ and*

$$\left|\phi\left(\mathcal{S}_{\alpha,\beta}^{\gamma,k}f(z), \mathcal{S}_{\alpha+1,\beta}^{\gamma,k}f(z), \mathcal{S}_{\alpha+2,\beta}^{\gamma,k}f(z), \mathcal{S}_{\alpha+3,\beta}^{\gamma,k}f(z), \mathcal{S}_{\alpha+4,\beta}^{\gamma,k}f(z); z\right)\right| < \mathtt{M},$$

*then*

$$\left|\mathcal{S}_{\alpha,\beta}^{\gamma,k}f(z)\right| < \mathtt{M} \quad (z \in \Delta).$$

**Corollary 2.4.** *Let $\Re(\alpha) \geq \frac{1-n}{2}, n \geq 3$ and $\mathtt{M} > 0$. If $f \in \mathcal{A}$ satisfies $\left|\mathcal{S}_{\alpha+1,\beta}^{\gamma,k}f(z)\right| < \mathtt{M}$, then*

$$\left|\mathcal{S}_{\alpha,\beta}^{\gamma,k}f(z)\right| < \mathtt{M} \quad (z \in \Delta).$$

*Proof.* This follows from Corollary 2.3 by taking $\phi(\mathtt{a}, \mathtt{b}, \mathtt{x}, \mathtt{y}, \mathtt{w}; z) = \mathtt{b} = \frac{\mathtt{n}+(\alpha-1)}{\alpha}\mathtt{M}e^{i\theta}$. □

**Theorem 2.5.** *Assume that $n \geq 3$, $\mathtt{M} > 0$. If $f \in \mathcal{A}$ satisfies the conditions $\left|\mathcal{S}_{\alpha+2,\beta}^{\gamma,k}f(z)\right| \leq n^2M$ and*

$$\left|\alpha(\alpha+1)(\alpha+2)(\alpha+3)\mathcal{S}_{\alpha+4,\beta}^{\gamma,k}f(z) - \alpha(\alpha+1)(\alpha+2)\mathcal{S}_{\alpha+3,\beta}^{\gamma,k}f(z)\right| < h(z),$$

*then*

$$\left|\mathcal{S}_{\alpha,\beta}^{\gamma,k}f(z)\right| < \mathtt{M} \quad (z \in \Delta).$$

*Proof.* Assume that $\phi(\mathtt{a}, \mathtt{b}, \mathtt{x}, \mathtt{y}, \mathtt{w}; z) = \alpha(\alpha+1)(\alpha+2)(\alpha+3)\mathtt{w} - \alpha(\alpha+1)(\alpha+2)^2\mathtt{y}$, $\Omega = h(\Delta)$ such that

$$h(z) = \left(|2\alpha + 3\alpha^2 + \alpha^3| + 3|(2 + 3\alpha + \alpha^2)|\right) 3\mathtt{M}z.$$

Now, by applying Theorem (2.4), we show that $\phi \in \Phi[\Omega, \mathtt{M}]$. Because

$$\begin{aligned}&\left|\phi\left(\mathtt{M}e^{i\theta}, \frac{n+(\alpha-1)}{\alpha}\mathtt{M}e^{i\theta}, \frac{\mathtt{L}+[2\alpha\mathtt{n}+\alpha(\alpha-1)]\mathtt{M}e^{i\theta}}{\alpha(\alpha+1)}, \frac{\mathtt{N}+(\alpha+1)[3\mathtt{L}+3\alpha\mathtt{n}+\alpha(\alpha-1)]\mathtt{M}e^{i\theta}}{\alpha(\alpha+1)(\alpha+2)},\right.\right.\\ &\quad\left.\left.\frac{\mathtt{X}+4(\alpha+2)\mathtt{N}+6(\alpha+1)(\alpha+2)\mathtt{L}}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} + \frac{[4\alpha(\alpha+1)(\alpha+2)\mathtt{n}+\alpha(\alpha^2-1)(\alpha+2)]\mathtt{M}e^{i\theta}}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}; z\right)\right|\\ &\quad= |\phi(\mathtt{a}, \mathtt{b}, \mathtt{x}, \mathtt{y}, \mathtt{w}; z)|.\end{aligned}$$

Since,

$$\begin{aligned}&|\phi(\mathtt{a}, \mathtt{b}, \mathtt{x}, \mathtt{y}, \mathtt{w}; z)| = \alpha(\alpha+1)(\alpha+2)(\alpha+3)\mathtt{w} - \alpha(\alpha+1)(\alpha+2)^2\mathtt{y}\\ &= |\mathtt{A} + (6+3\alpha)\mathtt{N} + (3\alpha^2+9\alpha+6)\mathtt{L} + (\alpha^3+3\alpha^2+2\alpha)n\mathtt{M}e^{i\theta}|\\ &= \left|\mathtt{A}e^{-i\theta} + (6+3\alpha)\mathtt{N}e^{-i\theta} + (3\alpha^2+9\alpha+6)\mathtt{L}e^{-i\theta} + (\alpha^3+3\alpha^2+2\alpha)n\mathtt{M}\right|\\ &\geq \Re(\mathtt{A}e^{-i\theta}) + |(6+3\alpha)|\Re(\mathtt{N}e^{-i\theta}) + |(3\alpha^2+9\alpha+6)|\Re(\mathtt{L}e^{-i\theta}) + |(\alpha^3+3\alpha^2+2\alpha)|n\mathtt{M}\\ &\geq |(\alpha^3+3\alpha^2+2\alpha)|n\mathtt{M} + |(3\alpha^2+9\alpha+6)|n(n-1)\mathtt{M}\\ &\geq \left(|2\alpha+3\alpha^2+\alpha^3| + 3|(2+3\alpha+\alpha^2)|\right) 3\mathtt{M},\end{aligned}$$

such that $\Re(\mathtt{A}e^{-i\theta}) \geq 0, \Re(\mathtt{N}e^{-i\theta}) \geq 0$ and $\Re(\mathtt{L}e^{-i\theta}) \geq (n-1)n\mathtt{M}$ for all $\theta \in \mathbb{R}, z \in \Delta$ and $n \geq 3$. The proof is complete. □

**Definition 2.3.** *If $\Omega \subseteq \mathbb{C}$ and $\varrho \in \mathcal{Q}_1 \cap \mathcal{H}_1$. Let $\Phi_1[\Omega, \varrho]$ be the family of admissible functions which consists of functions $\phi : \mathbb{C}^5 \times \Delta \longrightarrow \mathbb{C}$ that satisfy the condition of admissibility:*

$$\phi(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}; z) \notin \Omega,$$

*whenever*

$$\boldsymbol{a} = \varrho(\zeta),\ \boldsymbol{b} = \frac{m\zeta\varrho'(\zeta) + \alpha\varrho(z)}{\alpha},\ \Re\left\{\frac{(\alpha+1)\boldsymbol{x} + \boldsymbol{b} - (\alpha+2)\boldsymbol{a}}{\boldsymbol{b} - \boldsymbol{a}} - 2(\alpha+1)\right\} \geq m\Re\left\{1 + \frac{\zeta\varrho''(\zeta)}{\varrho'(\zeta)}\right\},$$

$$\Re\left\{\frac{(\alpha+1)(\alpha+2)[\boldsymbol{y} - 3\boldsymbol{x} + 2\boldsymbol{a}]}{\boldsymbol{b} - \boldsymbol{a}} + 3(\alpha+1)(\alpha+2)\right\} \geq m^2\Re\left\{\frac{\zeta^2\varrho'''(\zeta)}{\varrho'(\zeta)}\right\} \text{ and}$$

$$\Re\left\{\frac{(\alpha+1)(\alpha+2)(\alpha+3)[\boldsymbol{w} - 4\boldsymbol{y} + 6\boldsymbol{x} - 2\boldsymbol{a}]}{\boldsymbol{b} - \boldsymbol{a}} - 4(\alpha+1)(\alpha+2)(\alpha+3)\right\} \geq m^3\Re\left\{\frac{\zeta^3\varrho''''(\zeta)}{\varrho'(\zeta)}\right\},$$

*where $z \in \Delta, \zeta \in \partial\Delta \setminus E(\varrho)$ and $m \geq 3$.*

**Theorem 2.6.** *Assume that $\Omega \subseteq \mathbb{C}$ and $\phi \in \Phi_1[\Omega, \varrho]$. If $f \in \mathcal{A}$ and $\varrho \in \mathcal{Q}_1$ satisfy the following conditions:*

$$\Re\left(\frac{\zeta^2\varrho'''(\zeta)}{\varrho'(\zeta)}\right) \geq 0, \qquad \left|\left(\frac{\mathcal{S}^{\gamma,k}_{\alpha+2,\beta}f(z)}{z\varrho'(z)}\right)\right| \leq m^2 \tag{21}$$

*and*

$$\phi\left(\frac{\mathcal{S}^{\gamma,k}_{\alpha,\beta}f(z)}{z}, \frac{\mathcal{S}^{\gamma,k}_{\alpha+1,\beta}f(z)}{z}, \frac{\mathcal{S}^{\gamma,k}_{\alpha+2,\beta}f(z)}{z}, \frac{\mathcal{S}^{\gamma,k}_{\alpha+3,\beta}f(z)}{z}, \frac{\mathcal{S}^{\gamma,k}_{\alpha+4,\beta}f(z)}{z}; z\right) \subset \Omega, \tag{22}$$

*then*

$$\frac{\mathcal{S}^{\gamma,k}_{\alpha,\beta}f(z)}{z} \prec \varrho(z) \quad (z \in \Delta).$$

*Proof.* Define the function $g(\omega)$ by

$$\frac{\mathcal{S}^{\gamma,k}_{\alpha,\beta}f(z)}{z} = g(z) \quad (\omega \in \Delta). \tag{23}$$

Differentiating (23) with respect to $z$ and by using the recurrence relation (5), we have

$$\frac{\mathcal{S}^{\gamma,k}_{\alpha+1,\beta}f(z)}{z} = \frac{zg'(z) + \alpha g(z)}{\alpha}. \tag{24}$$

Again, differentiating (24) with respect to $z$ and by making use of (5), we obtain

$$\frac{\mathcal{S}^{\gamma,k}_{\alpha+2,\beta}f(z)}{z} = \frac{z^2g''(z) + 2(\alpha+1)zg'(z) + \alpha(\alpha+1)g(z)}{\alpha(\alpha+1)}. \tag{25}$$

Further computations show that

$$\frac{\mathcal{S}^{\gamma,k}_{\alpha+3,\beta}f(z)}{z} = \frac{z^3g'''(z) + 3(\alpha+2)z^2g''(z)}{\alpha(\alpha+1)(\alpha+2)} + \frac{3(\alpha+1)(\alpha+2)zg'(z) + \alpha(\alpha+1)(\alpha+2)g(z)}{\alpha(\alpha+1)(\alpha+2)}, \tag{26}$$

and

$$\frac{\mathcal{S}_{\alpha+4,\beta}^{\gamma,k}f(z)}{z} = \frac{z^4g''''(z) + (\alpha+3)[4z^3g'''(z) + 6(\alpha+2)z^2g''(z)]}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} + \frac{4(\alpha+1)(\alpha+2)(\alpha+3)zg'(z) + \alpha(\alpha+1)(\alpha+2)(\alpha+3)g(z)}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}. \quad (27)$$

Let

$$\mathtt{a} = \mathtt{r}, \quad \mathtt{b} = \frac{\mathtt{s} + \alpha\mathtt{r}}{\alpha}, \quad \mathtt{x} = \frac{\mathtt{t} + 2(\alpha+1)\mathtt{s} + \alpha(\alpha+1)\mathtt{r}}{\alpha(\alpha+1)},$$

$$\mathtt{y} = \frac{\mathtt{u} + 3(\alpha+2)\mathtt{t} + 3(\alpha+1)(\alpha+2)\mathtt{s} + \alpha(\alpha+1)(\alpha+2)\mathtt{r}}{\alpha(\alpha+1)(\alpha+2)} \quad \text{and}$$

$$\mathtt{w} = \frac{\mathtt{v} + 4(\alpha+3)\mathtt{u} + 6(\alpha+2)(\alpha+3)\mathtt{t} + 4(\alpha+1)(\alpha+2)(\alpha+3)\mathtt{s}}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} + \mathtt{r}.$$

We now define the transformation $\psi(\mathtt{r},\mathtt{s},\mathtt{t},\mathtt{u},\mathtt{v};z) : \mathbb{C}^5 \times \Delta \longrightarrow \mathbb{C}$ by

$$\psi(\mathtt{r},\mathtt{s},\mathtt{t},\mathtt{u},\mathtt{v};z) = \phi(\mathtt{a},\mathtt{b},\mathtt{x},\mathtt{y},\mathtt{w};z) = \phi\left(\mathtt{r}, \frac{\mathtt{s}+\alpha\mathtt{r}}{\alpha}, \frac{\mathtt{t}+2(\alpha+1)\mathtt{s}}{\alpha(\alpha+1)} + \mathtt{r}, \right. \quad (28)$$

$$\left. \frac{\mathtt{u} + (\alpha+2)[3\mathtt{t} + 3(\alpha+1)\mathtt{s}]}{\alpha(\alpha+1)(\alpha+2)} + \mathtt{r}, \frac{\mathtt{v} + (\alpha+3)[4\mathtt{u} + (\alpha+2)\{6\mathtt{t} + 4(\alpha+1)\mathtt{s}\}]}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} + \mathtt{r}; z\right).$$

Making use of the equations (23) to (27), we find from (28) that

$$\psi\left(g(z), zg'(z), z^2g''(z), z^3g'''(z), z^4g''''(z); z\right) = \phi\left(\frac{\mathcal{S}_{\alpha,\beta}^{\gamma,k}f(z)}{z}, \frac{\mathcal{S}_{\alpha+1,\beta}^{\gamma,k}f(z)}{z}, \frac{\mathcal{S}_{\alpha+2,\beta}^{\gamma,k}f(z)}{z}, \frac{\mathcal{S}_{\alpha+3,\beta}^{\gamma,k}f(z)}{z}, \frac{\mathcal{S}_{\alpha+4,\beta}^{\gamma,k}f(z)}{z}; z\right).$$

Therefore, (22) transforms into

$$\psi\left(g(z), zg'(z), z^2g''(z), z^3g'''(z), z^4g''''(z); z\right) \in \Omega.$$

We observe that

$$\frac{\mathtt{t}}{\mathtt{s}} + 1 = \frac{\mathtt{b} + (\alpha+1)\mathtt{x} - (\alpha+2)\mathtt{a}}{\mathtt{b}-\mathtt{a}} - 2(\alpha+1),$$

$$\frac{\mathtt{u}}{\mathtt{s}} = \frac{(\alpha+1)(\alpha+2)[\mathtt{y} - 3\mathtt{x} + 2\mathtt{a}]}{\mathtt{b}-\mathtt{a}} + 3(\alpha+1)(\alpha+2)$$

and

$$\frac{\mathtt{v}}{\mathtt{s}} = \frac{(\alpha+1)(\alpha+2)(\alpha+3)[\mathtt{w} - 4\mathtt{y} + 6\mathtt{x} - 2\mathtt{a}]}{\mathtt{b}-\mathtt{a}} - 4(\alpha+1)(\alpha+2)(\alpha+3).$$

Hence, the admissibility condition for $\phi \in \Phi_1[\Omega, \varrho]$ of Definition 2.3 is equivalent to the admissibility condition for the function $\psi \in \Psi_j[\Omega, \varrho]$. Thus, by Lemma 1.1, we have

$$g(z) = \frac{\mathcal{S}_{\alpha,\beta}^{\gamma,k}f(z)}{z} \prec \varrho(z) \quad (z \in \Delta)$$

which evidently completes the proof of Theorem 2.6. □

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\Delta)$ for some conformal mapping $h$ from $\Delta$ into the domain $\Omega$. We denote the class $\Phi[h(\Delta), \varrho]$ by $\Phi[h, \varrho]$. Next result is an immediate consequences of Theorem 2.6.

**Theorem 2.7.** *Let $\phi \in \Phi[h, \varrho]$. If $f \in \mathcal{A}$ and $\varrho \in \mathcal{Q}_1$ satisfies (21) and*

$$\phi\left(\frac{\mathcal{S}_{\alpha,\beta}^{\gamma,k}f(z)}{z}, \frac{\mathcal{S}_{\alpha+1,\beta}^{\gamma,k}f(z)}{z}, \frac{\mathcal{S}_{\alpha+2,\beta}^{\gamma,k}f(z)}{z}, \frac{\mathcal{S}_{\alpha+3,\beta}^{\gamma,k}f(z)}{z}, \frac{\mathcal{S}_{\alpha+4,\beta}^{\gamma,k}f(z)}{z}; z\right) \prec h(z),$$

*then*

$$\frac{\mathcal{S}_{\alpha,\beta}^{\gamma,k}f(z)}{z} \prec \varrho(z) \quad (z \in \Delta).$$

In the particular case $\varrho(z) = 1 + Mz, M > 0$, the class of admissible functions $\Phi[\Omega, \varrho]$ becomes $\Phi[\Omega, M]$.

**Definition 2.4.** *Let $\Omega \subseteq \mathbb{C}$ and $M > 0$. The family of admissible functions $\Phi[\Omega, M]$ consists of the functions $\phi : \mathbb{C}^5 \times \Delta \longrightarrow \mathbb{C}$, which satisfy the following admissibility condition*

$$\phi\left(1 + Me^{i\theta}, \frac{\alpha + (n+\alpha)}{\alpha}Me^{i\theta}, \frac{L + \alpha(\alpha+1) + (\alpha+1)[2n+\alpha]Me^{i\theta}}{\alpha(\alpha+1)},\right.$$
$$\frac{N + 3(\alpha+2)L + \alpha(\alpha+1)(\alpha+2) + (\alpha+1)(\alpha+2)[3n+\alpha]Me^{i\theta}}{\alpha(\alpha+1)(\alpha+2)},$$
$$\left.\frac{X + (\alpha+3)[4N + 6(\alpha+2)L + \alpha(\alpha+1)(\alpha+2)]}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} + \frac{[4n+\alpha]Me^{i\theta}}{\alpha}; z\right) \notin \Omega$$

*whenever $\omega \in \Delta, \Re\left\{Le^{-i\theta}\right\} \geq (n-1)nM, \Re\left\{Ne^{-i\theta}\right\} \geq 0$ and $\Re\left\{Xe^{-i\theta}\right\} \geq 0$ for every $\theta \in \mathbb{R}$ and $n \geq 3$.*

**Corollary 2.5.** *Let $\phi \in \Phi[M]$. If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{\mathcal{S}_{\alpha+2,\beta}^{\gamma,k}f(z)}{z}\right| \leq n^2 M \quad (n \geq 3;\ M > 0)$$

*and*

$$\phi\left(\frac{\mathcal{S}_{\alpha,\beta}^{\gamma,k}f(z)}{z}, \frac{\mathcal{S}_{\alpha+1,\beta}^{\gamma,k}f(z)}{z}, \frac{\mathcal{S}_{\alpha+2,\beta}^{\gamma,k}f(z)}{z}, \frac{\mathcal{S}_{\alpha+3,\beta}^{\gamma,k}f(z)}{z}, \frac{\mathcal{S}_{\alpha+4,\beta}^{\gamma,k}f(z)}{z}; z\right) \in \Omega,$$

*then*

$$\left|\frac{\mathcal{S}_{\alpha,\beta}^{\gamma,k}f(z)}{z} - 1\right| < M \quad (z \in \Delta).$$

**Remark 2.1.** *For different choices of parameters, we can obtain the corresponding results for the operators $\mathbb{J}_p f(z)$, $\mathbb{I}_p f(z)$ and $\mathbb{S}_p f(z)$, which are defined by (6), (7) and (8), respectively.*

## 3. Conclusions

In the present paper, we have derived several fourth-order differential subordination results for analytic functions in the open unit disk $\Delta$ by using the operator $\mathcal{S}_{\alpha,c}^{k;d}$ which is defined by (4) means of the convolution, involving the normalized form of the three-parameter family $\omega_{b,d,p}(z)$ of the generalized Bessel functions of the first kind, which is defined by (2). Our results have been obtained by considering suitable classes of admissible functions. The results obtained in this paper could inspire future work to get fourth-order differential subordinations involving different types of linear and nonlinear operators.

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