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NEIGHBORHOOD CONNECTIVITY INDEX OF A VAGUE GRAPH

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Abstract. Neighborhood connectivity index (NCI) in graphs is a fundamental issue in fuzzy graph (FG) theory that has wide applications in the real world. Hence, in this paper, we investigate NCI in vague graphs(VG) with several examples. Also, one of the motives of this research is to introduce several bounds and index values of structures like trees, cycles and complete VGs are obtained. This article discusses about a parameter in VGs theory termed as NCI. Finally, an application is presented.

Keywords:Fuzzy graph, vague graph, connectivity index, Neighborhood connectivity index.

AMS Subject Classification: 83-02, 99A00

1. Introduction

Graph theory has always played an important role in mathematics. While discussing the relationship between objects, it focuses on things that have an undeniable membership. Zadeh [1] introduced the subject of a fuzzy set (FS) in 1995. Rosenfeld [2] proposed the subject of FGs. The definitions of FGs from the Zadeh fuzzy relations in 1973 was presented by Kaufmann [3]. Some of these product operations on FGs were presented by Mordeson and Peng [4]. Gau and Buehrer [5] proposed the concept of vague set (VS) by replacing the value of an element in a set with a subinterval of [0, 1]. Moreover, a VG can concentrate on determining the uncertainties coupled with the inconsistent and indeterminate information of any real-world problems where FGs may not lead to adequate results. Ramakrishna in [6] proposed a new concept of VGs, belonging to the FGs family, have good capabilities when faced with problems that cannot be expressed by FGs. The notion of a VG is a new mathematical attitude to model the ambiguity and uncertainty in decision-making issues. Kosari et al. [7] introduced a VG structure and Rao et al. [8] studied the certain properties of VGs. Ghorai et al. expressed [9] the regular product VGs and product vague line graph. Kosari et al. [10, 11] studied on VG and certain properties of domination in product VGs. Akram et al. [12 − 14] introduced the vague hypergraphs and regularity in vague intersection graphs. New result of domination on VGs was presented

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by Borzooei $[15-17]$. Connectivity index (CI) problem can also be used to model many real-world situations in the fields of circuit design, telecommunications, network flow, and so on. Binu et al. $[18-20]$ studied CI, cyclic CI, and Wiener index, and some of those are extended to bipolar graphs too. Sebastian et al. [21] expressed the generalized FG connectivity parameters. Mathew and Sunitha [22] defined several types of edge in FGs. Connectivity is an important concept in FGs. The stability of a FG is determined by the strength of connectedness of each pair of its vertices. Mordeson et al. [23] introduced a new notion of neighborhood connectivity (NC) in FGs. Jiang et al. [24] studied on CI in VGs. Naeem et al. [25] investigated CI of intuitionistic FGs. Bhutani et al. [26] brought up the concept of strong edges and strong paths in FGs. New results on FGs are proposed in $[27-36]$. NCI is a new parameter is related to connection studied in this work. NCI is one of the most important topics that has many applications in detection of competition condition in parallel programming.

1.1. **Methodology and Importance of Neighborhood connectivity index.** NCI is very much useful in molecular chemistry and used in spectral graph theory, network theory, and several field of mathematics. NCI has a vital role in real-world problems especially in Internet routing and transport network flow. VGs allow to describe two aspects of information using membership and nonmembership degrees under uncertainties. Keeping in view the importance of NCIs in real life problems and comprehension of FGs, we aim to develop CIs in the environment of VGs. CI is always considered as a cornerstone in VG. A new parameter related to CI is studied in this work. Since selection best location is relevant in most of the modern networks, the concepts of this paper can be used in a wide variety of problems. This can also be used in analyzing the effectiveness of scheduling and routing in different areas. We introduce new type of CIs, namely, NCI in the frame of VGs.

1.2. **Research gaps and motivation of study.** Different types of topological index of a graph has many applications and many results are available for FGs. But in many practical applications it is seen that many situations cannot be modeled using FGs. In these cases, to handle such a situation, those topological indices are needed to define in a VG. The VGs can amplify flexibility to model complex real-time problems better than an FG. the concept of NCI is one of the most important features of VGs that have many applications in real problems. The NCI plays a significant role in modeling various problems such as networking, transportation, and precise location detection.

1.3. **Contribution of this study.** In this work, we have explored a new index, namely the NCI of FGs towards VGs. These indices play the crucial role in modeling real-world problems . In the beginning, we have studied the CI and NCI of VGs and we discussed the essential preliminaries related to the work.Finally, an application was presented.

2. Preliminaries

In this section, we present some preliminary results which will be used throughout the paper.

Definition 2.1. [34] *A graph $\mathfrak{G}^*$ is a pair (X, E) where X is called the vertex set and $E \subseteq X \times X$ is called the edge set.*

Definition 2.2. [34] *An FG $\mathfrak{G} = (\phi, \psi)$ is a pair of function $\phi : X \to [0, 1]$ and $\psi : X \times X \to [0, 1]$ such that, for all $m, n \in X$,*

$$\psi(mn) \leq \min\{\phi(m), \phi(n)\},$$

Definition 2.3. [35] *A path $\mathfrak{P}$ of length l in an FG $\mathfrak{G} = (\phi, \psi)$ is a sequence of distinct vertices $x_0, x_1, x_2, ..., x_l$ such that $\psi(x_{k-1}x_k) > 0, k = 1, 2, 3, ..., l$. The degree of membership of a weakest edge is defined as its strength. The strength of connectedness between two vertices m and n, is defined as the maximum of the strength of all paths between m and n is denoted by $\psi^{\infty}(m, n)$ or $CONN_{\mathfrak{G}}(m, n)$.*

An edge mn is called to be a strong edge (SE) if $\psi^{\infty}(m, n) = \psi(mn)$. If $\psi(mn) = 0$ for each $n \in X$, then m is named an isolated vertex. If mn is an SE, then its weight is at least as great as the strength of the connectedness of its end vertices when it is deleted. Note that, $CONN_{\mathfrak{G}-mn}(m, n)$ is the strength of the connectedness between m and n in an FG obtained from $\mathfrak{G}$ by deleting the edge mn.

Definition 2.4. [36] *Assume mn is an edge in an FG $\mathfrak{G}$, then*
The edge mn is $\alpha-$ strong if $CONN_{\mathfrak{G}-mn}(m, n) < \psi(mn)$.
The edge mn is $\beta-$ strong if $CONN_{\mathfrak{G}-mn}(m, n) = \psi(mn)$.
The edge mn is $\delta-$ strong if $CONN_{\mathfrak{G}-mn}(m, n) > \psi(mn)$.
Therefore, an edge mn is an SE if it is either $\alpha-$strong or $\beta-$strong.

Two vertices m and n in an FG $\mathfrak{G}$ are called adjacent if $\psi(mn) > 0$, and m and n are named neighbors. The collection of all neighbors of m is denoted by $N(m)$. An edge mn of an FG is named effective if $\psi(mn) = \phi(m) \wedge \phi(n)$. Thus, m and n are named effective neighbors (EN). The set of all $\mathfrak{EN}$ of m is named $\mathfrak{EN}$ of m and is shown by $\mathfrak{EN}(m)$. Also, n is named strong neighbor(SN) of m if edge mn is strong. The set of all SN of m is named the open SN of m and is denoted by $\mathfrak{N}_s(m)$. The closed SN $\mathfrak{N}_s[m]$ is defined as $\mathfrak{N}_s[m] = \mathfrak{N}_s(m) \cup \{m\}$.

Definition 2.5. [5] *A vague set (VS) $\mathfrak{D}$ is a pair $(\mathfrak{D}^t, \mathfrak{D}^f)$ on a set X, where $\mathfrak{D}^t$ and $\mathfrak{D}^f$ are real- valued functions which can be defined from X to $[0, 1]$, so that, $\mathfrak{D}^t(s) + \mathfrak{D}^f(s) \leq 1, \quad \forall s \in X$.*

Definition 2.6. [6] *A pair $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$ is called a VG on graph $\mathfrak{G}^* = (X, E)$, where $\mathfrak{D} = (\mathfrak{D}^t, \mathfrak{D}^f)$ is a VS on X and $\mathfrak{Z} = (\mathfrak{Z}^t, \mathfrak{Z}^f)$ is a VS on E such that,*

$$\mathfrak{Z}^t(sy) \leq \min\{\mathfrak{D}^t(s), \mathfrak{D}^t(y)\},$$

$$\mathfrak{Z}^f(sy) \geq \max\{\mathfrak{D}^f(s), \mathfrak{D}^f(y)\},$$

for all $s, y \in X$. Note that $\mathfrak{Z}$ is called vague relation on $\mathfrak{D}$. A VG $\mathfrak{G}$ is named strong if

$$\mathfrak{Z}^t(sy) = \min\{\mathfrak{D}^t(s), \mathfrak{D}^t(y)\},$$

$$\mathfrak{Z}^f(sy) = \max\{\mathfrak{D}^f(s), \mathfrak{D}^f(y)\},$$

for all $sy \in E$.

Definition 2.7. [34] *Assume $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$ is a VG on $G^* = (X, E)$. The degree of a vertex s is denoted as $\mathfrak{d}(s) = (\mathfrak{d}^t(s), \mathfrak{d}^f(s))$, where*

$$\mathfrak{d}^t(s) = \sum_{s \neq y, y \in X} \mathfrak{Z}^t(sy) \quad , \quad \mathfrak{d}^f(s) = \sum_{s \neq y, y \in X} \mathfrak{Z}^f(sy).$$

The order of $\mathfrak{G}$ is defined as

$$O(\mathfrak{G}) = \left(\sum_{s\in X} \mathfrak{D}^t(s), \sum_{s\in X} \mathfrak{D}^f(s) \right).$$

Definition 2.8. [24] *Suppose $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$ is a VG on a graph $G^* = (X, E)$. Then*
(i) A VG $\mathfrak{G}' = (\mathfrak{D}', \mathfrak{Z}')$ on $G'^* = (X, E')$ is called a partial vague subgraph (PVSG) of $\mathfrak{G}$ if
$\mathfrak{D}'^t(s) \le \mathfrak{D}^t(s)$ *and* $\mathfrak{D}'^f(s) \ge \mathfrak{D}^f(s)$ *for all* $s \in X$,
$\mathfrak{Z}'^t(sy) \le \mathfrak{Z}^t(sy)$ *and* $\mathfrak{Z}'^f(sy) \ge \mathfrak{Z}^f(sy)$ *for all* $sy \in E$.
(ii) A VG $\mathfrak{G}' = (\mathfrak{D}', \mathfrak{Z}')$ is called a vague subgraph (VSG) of $\mathfrak{G}$ if
$\mathfrak{D}'^t(s) = \mathfrak{D}^t(s)$ *and* $\mathfrak{D}'^f(s) = \mathfrak{D}^f(s)$ *for all* $s \in X$,
$\mathfrak{Z}'^t(sy) = \mathfrak{Z}^t(sy)$ *and* $\mathfrak{Z}'^f(sy) = \mathfrak{Z}^f(sy)$ *for all* $sy \in E'$.

Definition 2.9. [36] *Suppose $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$ is a VG. Then*
(i)A path $\mathfrak{P} : s = s_0, s_1, s_2, ..., s_{l-1}, s_l = y$ in $\mathfrak{G}$ is a sequence of distinct vertices where $\mathfrak{Z}^t(s_{k-1}s_k) > 0$, $\mathfrak{Z}^f(s_{k-1}s_k) < 1$, $k = 1, 2, 3, ..., l$. The length of $\mathfrak{P}$ is l.
(ii) If $\mathfrak{P} : s = s_0, s_1, s_2, ..., s_{l-1}, s_l = y$ is a path between s and y of length l, then $(\mathfrak{Z}^t(sy))^l = \sup\{\mathfrak{Z}^t(ss_1) \wedge \mathfrak{Z}^t(s_1s_2) \wedge ... \wedge \mathfrak{Z}^t(s_{l-1}y)\}$ and $(\mathfrak{Z}^f(sy))^l = \inf\{\mathfrak{Z}^f(ss_1) \vee \mathfrak{Z}^f(s_1s_2) \vee ... \vee \mathfrak{Z}^f(s_{l-1}y)\}$.

$CONN_{\mathfrak{G}}(s, y) = (CONN^t_{\mathfrak{G}}(s, y), CONN^f_{\mathfrak{G}}(s, y)) = (\mathfrak{Z}^{t^\infty}(sy), \mathfrak{Z}^{f^\infty}(sy))$ is named the strength of connectedness between any two vertices s and y in $\mathfrak{G}$ where $CONN^t_{\mathfrak{G}}(s, y) = \sup\{(\mathfrak{Z}^t(sy))^l\}$ and $CONN^f_{\mathfrak{G}}(s, y) = \inf\{(f_{\mathfrak{Z}}(sy))^l\}$, $l = 1, 2, ...n$. A VG $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$ is named a connected VG , if $\mathfrak{Z}^t(sy) > 0$ and $\mathfrak{Z}^f(sy) > 0$, for every $sy \in E$. A VG $\mathfrak{G}$ is a complete VG, if $\mathfrak{Z}^t(sy) = \min\{\mathfrak{D}^t(s), \mathfrak{D}^t(y)\}$ and $\mathfrak{Z}^f(sy) = \max\{\mathfrak{D}^f(s), \mathfrak{D}^f(y)\}$ for every $s, y \in X$.

Definition 2.10. [24] *Assume mn is an edge in VG $\mathfrak{G}$, then*
The edge mn is $\alpha-$ strong if
$CONN^t_{\mathfrak{G}-mn}(m, n) < \mathfrak{Z}^t(mn)$ *and* $CONN^f_{\mathfrak{G}-mn}(m, n) > \mathfrak{Z}^f(mn)$
The edge mn is $\beta-$ strong if
$CONN^t_{\mathfrak{G}-mn}(m, n) = \mathfrak{Z}^t(mn)$ *and* $CONN^f_{\mathfrak{G}-mn}(m, n) = \mathfrak{Z}^f(mn)$
The edge mn is $\delta-$ strong if
$CONN^t_{\mathfrak{G}-mn}(m, n) > \mathfrak{Z}^t(mn)$ *and* $CONN^f_{\mathfrak{G}-mn}(m, n) < \mathfrak{Z}^f(mn)$.
Therefore, an edge mn is an SE if it is either $\alpha-$strong or $\beta-$strong.

Definition 2.11. [24] *The Connectivity Index ($\mathfrak{CI}$) of a VG $\mathfrak{G}$ denoted by $\mathfrak{CI}$ VG ($\mathfrak{G}$) is defined as,*

$$\mathfrak{CI}(\mathfrak{G}) = (\mathfrak{CI}^t(\mathfrak{G}), \mathfrak{CI}^f(\mathfrak{G})) =$$
$$\left(\sum_{s,y\in X} \mathfrak{D}^t(s)\mathfrak{D}^t(y).CONN^t_{\mathfrak{G}}(s, y), \sum_{s,y\in X} \mathfrak{D}^f(s)\mathfrak{D}^f(y).CONN^f_{\mathfrak{G}}(s, y) \right)$$

where $\mathfrak{CI}^t(\mathfrak{G})$ and $\mathfrak{CI}^f(\mathfrak{G})$ are the true-$\mathfrak{CI}$ and false-$\mathfrak{CI}$ of $\mathfrak{G}$.

Definition 2.12. [24] *Suppose $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$ is a VG on $\mathfrak{G}^* = (\mathfrak{D}^*, \mathfrak{Z}^*)$ where $\mathfrak{Z}^* = \{sy|\mathfrak{Z}^t(sy) > 0 \vee \mathfrak{Z}^f(sy) < 1\}$. Then the VG $\mathfrak{G}$ is a cycle if and only if $\mathfrak{G}^*$ is a cycle.*

Definition 2.13. [24]*The complement of a VG $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$ is a VG $\bar{\mathfrak{G}} = (\bar{\mathfrak{D}}, \bar{\mathfrak{Z}})$ where $\mathfrak{D} = \bar{\mathfrak{D}}$ and $\bar{\mathfrak{Z}}$ is described as follows.*

$$\bar{\mathfrak{Z}}^t(sy) = \mathfrak{D}^t(s) \wedge \mathfrak{D}^t(y) - \mathfrak{Z}^t(sy)$$

$$\bar{\mathfrak{Z}}^f(sy) = \mathfrak{D}^f(s) \vee \mathfrak{D}^f(y) + (1 - \mathfrak{Z}^f(sy)).$$

In Table 1, we show the essential notations.

TABLE 1. Some essential notations.

Notation	Meaning
FS	Fuzzy Set
FG	Fuzzy Graph
VS	Vague Set
VG	Vague Graph
VSG	vague subgraph
PVSG	Partial vague subgraph
CI	Connectivity Index
NCI	Neighborhood Connectivity Index

3. Neighborhood connectivity index in vague graphs

Definition 3.1. *The Neighborhood Connectivity Index ($\mathfrak{NCI}$) of a VG $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$ is defined as,*

$$\mathfrak{NCI}(\mathfrak{G}) = \left(\sum_{s \in X(\mathfrak{G})} d(s)e^t(s), \sum_{s \in X(\mathfrak{G})} d(s)e^f(s) \right)$$

where $d(s)$ is the cardinality of $\mathfrak{N}(s)$ and $e^t(s) = \vee\{\mathfrak{Z}^t(sy) : y \in \mathfrak{N}(s)\}$ and $e^f(s) = \wedge\{\mathfrak{Z}^f(sy) : y \in \mathfrak{N}(s)\}$ with $\mathfrak{N}(s) = \{y : \psi(sy) > 0, s, y \in X\}$. $e^t(s)$ and $e^f(s)$ is termed as the potential of the vertex s.

Example 3.1. *Consider the VG $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$ shown in Figure 1, where $X = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ and $E = \{a_1a_2, a_2a_3, a_1a_4, a_3a_4, a_4a_6, a_4a_5, a_5a_6, a_2a_4\}$.*

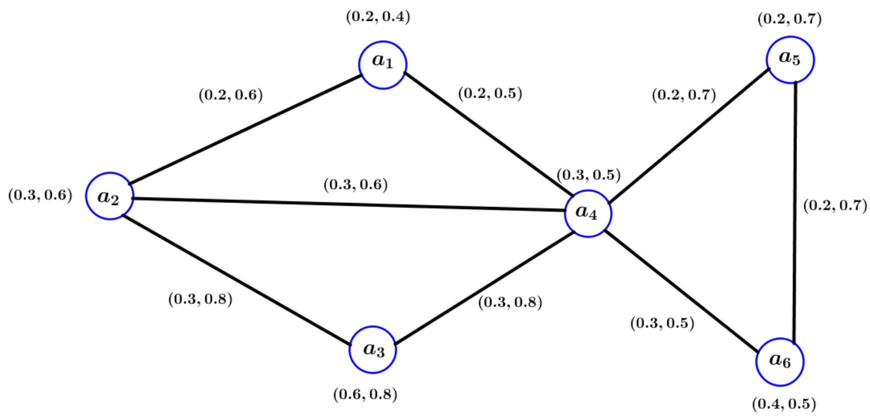

FIGURE 1. A VG $\mathfrak{G}$

We can find $d(a_1) = 2, e^t(a_1) = 0.2, e^f(a_1) = 0.5$. Similarly, we can find for the rest of the vertices, as is shown in Table 2.

Therefore, $\mathfrak{NCI} = (4.4, 9.3)$.

TABLE 2

Vertex	*d(x)*	$e^t(x)$	$e^f(x)$	$d(x)e^f(x)$	$d(x)e^f(x)$
a_1	*2*	*0.2*	*0.5*	*0.4*	*1*
a_2	*3*	*0.3*	*0.6*	*0.9*	*1.8*
a_3	*2*	*0.3*	*0.8*	*0.6*	*1.6*
a_4	*5*	*0.3*	*0.5*	*1.5*	*2.5*
a_5	*2*	*0.2*	*0.7*	*0.4*	*1.4*
a_6	*2*	*0.3*	*0.5*	*0.6*	*1*
$\mathfrak{NCI}$				*4.4*	*9.3*

Remark 3.1. $\mathfrak{NCI}$ *of VG is zero if and only if the Cardinality of its edge set is zero.*

Proposition 3.1. *If* $H = (\mathfrak{D}', \mathfrak{Z}')$ *is a PVSG of* $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$*, then*

$$\mathfrak{NCI}^t(H) \leq \mathfrak{NCI}^t(\mathfrak{G})$$

$$\mathfrak{NCI}^f(H) \geq \mathfrak{NCI}^f(\mathfrak{G})$$

Proof. Suppose $H = (\mathfrak{D}', \mathfrak{Z}')$ is a PVSG of $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$ with vertex set $X = \{s_1, s_2, s_3, ..., s_n\}$. Let s be an arbitary vertex in X' vertex set of H. Then ${\mathfrak{Z}'}^t(ss_i) \leq \mathfrak{Z}^t(ss_i)$ and ${\mathfrak{Z}'}^f(ss_i) \geq \mathfrak{Z}^f(ss_i)$ for all other vertices $s_i \in X'$. Therefore, $\vee_i\{{\mathfrak{Z}'}^t(ss_i)\} \leq \vee_i\{\mathfrak{Z}^t(ss_i)\}$ and $\wedge_i\{{\mathfrak{Z}'}^f(ss_i)\} \geq \wedge_i\{\mathfrak{Z}^f(ss_i)\}$. Also, $d_H(s) \leq d_{\mathfrak{G}}(s)$. Therefore,

$$\mathfrak{NCI}^t(H) = \sum_{s_i} d_H(s_i)(\vee_i\{{\mathfrak{Z}'}^t(ss_i)\}) \leq \sum_{s_i} d_{\mathfrak{G}}(s_i)(\vee_i\{\mathfrak{Z}^t(ss_i)\}) = \mathfrak{NCI}^t(\mathfrak{G})$$

Also,

$$\mathfrak{NCI}^f(H) = \sum_{s_i} d_H(s_i)(\wedge_i\{{\mathfrak{Z}'}^f(ss_i)\}) \geq \sum_{s_i} d_{\mathfrak{G}}(s_i)(\wedge_i\{\mathfrak{Z}^f(ss_i)\}) = \mathfrak{NCI}^f(\mathfrak{G})$$

□

Remark 3.2. *For a VSG* $H = (\mathfrak{D}', \mathfrak{Z}')$ *of a VG* $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$*,*

$$\mathfrak{NCI}^t(H) \leq \mathfrak{NCI}^t(\mathfrak{G}) \quad , \quad \mathfrak{NCI}^f(H) \leq \mathfrak{NCI}^f(\mathfrak{G}).$$

Example 3.2. *Consider the VG* $H = (\mathfrak{D}', \mathfrak{Z}')$ *shown in Figure 2, where* $X' = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ *and* $E' = \{a_1a_2, a_2a_3, a_1a_4, a_3a_4, a_4a_6, a_4a_5, a_5a_6, a_2a_4\}$*. We can see H is a PVSG of* $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$*, mentioned in Example 3.2.*

The cardinality of neighborhood for vertices is shown in Table 3, and also, we can calculate $\mathfrak{NCI}(H) = (3.3, 7)$*.*

TABLE 3

Vertex	*d(x)*	$e^t(x)$	$e^f(x)$	$d(x)e^f(x)$	$d(x)e^f(x)$
a_1	*2*	*0.1*	*0.2*	*0.4*	*1*
a_2	*3*	*0.3*	*0.6*	*0.9*	*1.8*
a_3	*2*	*0.3*	*0.8*	*0.6*	*1.6*
a_4	*5*	*0.3*	*0.5*	*1.5*	*2.5*
a_5	*2*	*0.2*	*0.7*	*0.4*	*1.4*
a_6	*2*	*0.2*	*0.6*	*0.4*	*1.2*
$\mathfrak{NCI}$				*4*	*9.5*

Therefore, $\mathfrak{NCI}^t(H) = 4 \leq \mathfrak{NCI}^t(\mathfrak{G}) = 4.4$ *and* $\mathfrak{NCI}^f(H) = 9.5 \geq \mathfrak{NCI}^f(\mathfrak{G}) = 9.3$.

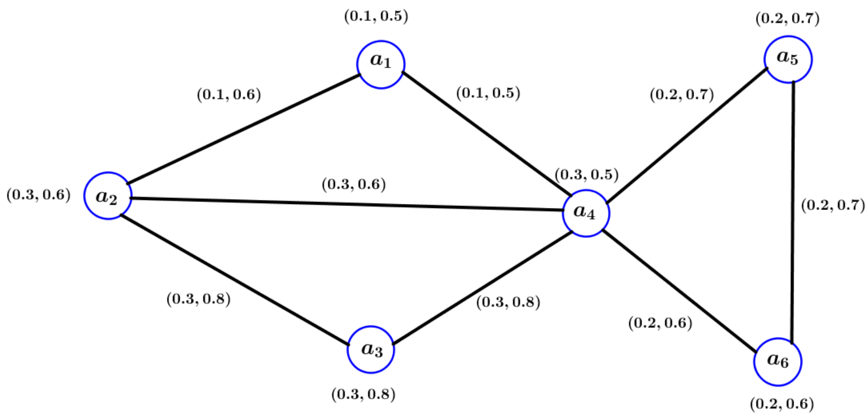

FIGURE 2. A VSG H of $\mathfrak{G}$

Proposition 3.2. *For a VG* $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$ *on* X *with* $|X| = n > 0$. *We have*

$$0 \leq \mathfrak{NCI}^t(\mathfrak{G}) \leq n(n-1) \quad , \quad 0 \leq \mathfrak{NCI}^f(\mathfrak{G}) \leq n(n-1)$$

Proof. For every $s \in X$, $0 \leq d(s) \leq n-1$, $0 \leq e^t(s) \leq 1$, $0 \leq e^f(s) \leq 1$. Thus, $0 \leq d(s)e^t(s) \leq n-1$, $0 \leq d(s)e^f(s) \leq n-1$. It follows $0 \leq \mathfrak{NCI}^t(\mathfrak{G}) = \sum_{s\in X} d(s)e^t(s) \leq n(n-1)$, and also $0 \leq \mathfrak{NCI}^f(\mathfrak{G}) = \sum_{s\in X} d(s)e^f(s) \leq n(n-1)$. □

Proposition 3.3. *Suppose* $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$ *is a connected VG with n edges. Then,*

$$2nk \leq \mathfrak{NCI}^t(\mathfrak{G}) \leq 2nl$$

$$2nk' \leq \mathfrak{NCI}^f(\mathfrak{G}) \leq 2nl'$$

where, $k = \wedge\{e^t(s), s \in X\}, l = \vee\{e^t(s), s \in X\}$ *and* $k' = \wedge\{e^f(s), s \in X\}, l' = \vee\{e^f(s), s \in X\}$.

Proof. Let $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$ be a VG with n edges. Then,

$$\mathfrak{NCI}^t(\mathfrak{G}) = \sum_{s\in X} e^t(s)d(s) \leq \sum_{s\in X} ld(s) = l \sum_{s\in X} d(s) = l \times 2n = 2nl$$

$$\mathfrak{NCI}^t(\mathfrak{G}) = \sum_{s\in X} e^t(s)d(s) \geq \sum_{s\in X} kd(s) = k \sum_{s\in X} d(s) = k \times 2n = 2nk$$

Therefore,

$$2nk \leq \mathfrak{NCI}^t(\mathfrak{G}) \leq 2nl$$

Also,

$$\mathfrak{NCI}^f(\mathfrak{G}) = \sum_{s\in X} e^f(s)d(s) \leq \sum_{s\in X} l'd(s) = l' \sum_{s\in X} d(s) = l' \times 2n = 2nl'$$

$$\mathfrak{NCI}^f(\mathfrak{G}) = \sum_{s\in X} e^f(s)d(s) \geq \sum_{s\in X} k'd(s) = k' \sum_{s\in X} d(s) = k' \times 2n = 2nk'$$

Therefore,

$$2nk' \leq \mathfrak{NCI}^f(\mathfrak{G}) \leq 2nl'$$

□

Proposition 3.4. *Suppose $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$ is a complete VG (CVG) with $X = \{s_1, s_2, s_3, ..., s_n\}$ such that $k_1^t \leq k_2^t \leq ... \leq k_n^t$ and $k_1^f \geq k_2^f \geq ... \geq k_n^f$, where $k_i^t = \mathfrak{D}^t(s_i)$ and $k_i^f = \mathfrak{D}^f(s_i), 1 \leq i \leq n$. Then,*

$$\mathfrak{NCI}^t(\mathfrak{G}) = (n-1)(k_1^t + k_2^t + ... + k_{n-2}^t + k_{n-1}^t + k_{n-1}^t)$$
$$\mathfrak{NCI}^f(\mathfrak{G}) = (n-1)(k_1^f + k_2^f + ... + k_{n-2}^f + k_{n-1}^f + k_{n-1}^f)$$

Proof. Suppose $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$ is a VG. We know that, for a CVG $\mathfrak{Z}^t(s_i s_j) > 0$ and $\mathfrak{Z}^f(s_i s_j) > 0$, for all $s_i s_j \in X$. Therefore, $d(s_i) = n-1$ for all s_i , $1 \leq i \leq n$. Now, we can check the potential of vertices. While considering s_1 we see that it is the vertex with minimum membership value, and maximal non-membership value. Therefore, $e^t(s_1) = k_1^t, e^f(s_1) = k_1^f$. Consider the vertices $s_i, 1 < i < n$. Here,

$$\mathfrak{Z}^t(s_m s_i) \leq k_i^t \quad , \mathfrak{Z}^f(s_m s_i) \leq k_m^f$$

for all $m < i$, and

$$\mathfrak{Z}^t(s_m s_i) = k_i^t \quad , \mathfrak{Z}^f(s_m s_i) = k_i^f$$

for all $m > i$. Therefore, $e^t(s_i) = k_i^t, e^f(s_i) = k_i^f, 2 \leq i \leq n-1$. At last, we consider the vertex s_n. We can see that

$$\mathfrak{Z}^t(s_i s_n) \leq k_{n-1}^t, 1 \leq i \leq n-1$$
$$\mathfrak{Z}^F(s_i s_n) \geq k_{n-1}^t, 1 \leq i \leq n-1$$

Since, $\mathfrak{Z}^t(s_{n-1}s_n) \leq k_{n-1}^t$ and $\mathfrak{Z}^f(s_{n-1}s_n) \leq k_{n-1}^f$, $e^t(s_n) = k_{n-1}^t, e^f(s_n) = k_{n-1}^f$. With summing up all those values, we have

$$\mathfrak{NCI}^t(\mathfrak{G}) = (n-1)(k_1^t + k_2^t + ... + k_{n-2}^t + k_{n-1}^t + k_{n-1}^t)$$
$$\mathfrak{NCI}^f(\mathfrak{G}) = (n-1)(k_1^f + k_2^f + ... + k_{n-2}^f + k_{n-1}^f + k_{n-1}^f)$$

□

Proposition 3.5. *$\mathfrak{NCI}$ of two isomorphic VGs are equal.*

Proof. Suppose h is a bijection between the isomorphic VGs $\mathfrak{G}_1 = (\mathfrak{D}_1, \mathfrak{Z}_1)$ on X_1 and $\mathfrak{G}_2 = (\mathfrak{D}_2, \mathfrak{Z}_2)$ on X_2. Since weights of the edges and vertices are preserved by an isomorphism, $\mathfrak{N}_{\mathfrak{G}_1}(s) = \mathfrak{N}_{\mathfrak{G}_2}(h(s))$ which implies $d_{\mathfrak{G}_1}(s) = d_{\mathfrak{G}_2}(h(s))$ for $s \in X$. Similarly,

$$\mathfrak{Z}_1^t(sy) = \mathfrak{Z}_2^t(h(s)h(y))$$
$$\mathfrak{Z}_1^f(sy) = \mathfrak{Z}_2^f(h(s)h(y))$$

for all $s, y \in X$. So, $e_{\mathfrak{G}_1}^t(s) = e_{\mathfrak{G}_2}^t(h(s))$ and $e_{\mathfrak{G}_1}^f(s) = e_{\mathfrak{G}_2}^f(h(s))$. It follows

$$\mathfrak{NCI}^t(\mathfrak{G}_1) = \sum_{s \in X_1} d_{\mathfrak{G}_1}(s) e_{\mathfrak{G}_1}^t(s) = \sum_{h(s) \in X_2} d_{\mathfrak{G}_2}(h(s)) e_{\mathfrak{G}_1}^t(h(s)) = \mathfrak{NCI}^t(\mathfrak{G}_2)$$

$$\mathfrak{NCI}^f(\mathfrak{G}_1) = \sum_{s \in X_1} d_{\mathfrak{G}_1}(s) e_{\mathfrak{G}_1}^f(s) = \sum_{h(s) \in X_2} d_{\mathfrak{G}_2}(h(s)) e_{\mathfrak{G}_1}^f(h(s)) = \mathfrak{NCI}^f(\mathfrak{G}_2).$$

□

Theorem 3.1. *Suppose $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$ is a VG cycle with $|X| = n$ for which every α- strong edge is of strength k_1, k_2 and every β- strong edge is of constant strength, then*

$$\mathfrak{NCI}^t(\mathfrak{G}) = 2nk_1 \quad , \quad \mathfrak{NCI}^t(\mathfrak{G}) = 2nk_2$$

Proof. Suppose $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$ is a VG cycle with $|X| = n$. Since $\mathfrak{G}^* = (X, E)$ is a VG cycle, $d(s) = 2$, for every $s \in X$. Also, from the assumption it follows that k_1 is greater than the constant strength of β- strong edges, that implies $e^t(s) = k_1$, and k_2 is a lesser that the constant strength of β- strong edges, that implies $e^f(s) = k_2$ for every $s \in X$. Therefor,

$$\mathfrak{NCI}^t(\mathfrak{G}) = \sum_{i=1}^{n} 2k_1 = 2nk_1 \quad , \mathfrak{NCI}^f(\mathfrak{G}) = \sum_{i=1}^{n} 2k_2 = 2nk_2.$$

□

Theorem 3.2. *For a given $n \in N$ and $k_1, k_2 \in R^+$, with $k_1 \leq 2n$ and $k_2 \leq 2n$, there exists a VG $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$ of $\mathfrak{NCI}^t k_1$ and $\mathfrak{NCI}^f k_2$ with $|E| = n$.*

Proof. Suppose $|E| = n$. Construct a VG $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$ such that $\mathfrak{D}^t(s_i) \geq \frac{k_1}{2n}, \mathfrak{D}^f(s_i) \geq \frac{k_2}{2n}$ for every $s_i \in X, \mathfrak{D}^t(s_i s_j) = \frac{k_1}{2n}, \mathfrak{D}^f(s_i s_j) = \frac{k_2}{2n}$ for every $s_i s_j \in E$. Now , we can check $\mathfrak{NCI}$ of the constructed graph. Here, $e^t(s_i) = \frac{k_1}{2n}, e^f(s_i) = \frac{k_2}{2n}$ for all $s_i \in X$. Thus,

$$\mathfrak{NCI}^t(\mathfrak{G}) = \sum_{s_i \in X} d(s_i)\frac{k_1}{2n} = \frac{k_1}{2n} \sum_{s_i \in X} d(s_i) = \frac{k_1}{2n} \times 2n = k_1$$

$$\mathfrak{NCI}^f(\mathfrak{G}) = \sum_{s_i \in X} d(s_i)\frac{k_2}{2n} = \frac{k_2}{2n} \sum_{s_i \in X} d(s_i) = \frac{k_2}{2n} \times 2n = k_2$$

Hence, our constructed graph is a VG of $\mathfrak{NCI}$ s with $|E| = n$.

□

Example 3.3. *Suppose $|E| = 5, k_1 = 6, k_2 = 8$. Clearly, $6 \leq 10, 8 \leq 10$. We can find a VG $\mathfrak{G}$ shown in Figure 3, where $X = \{a, b, c, m, n\}$ and $E = \{ab, ac, bc, cm, cn\}$.*

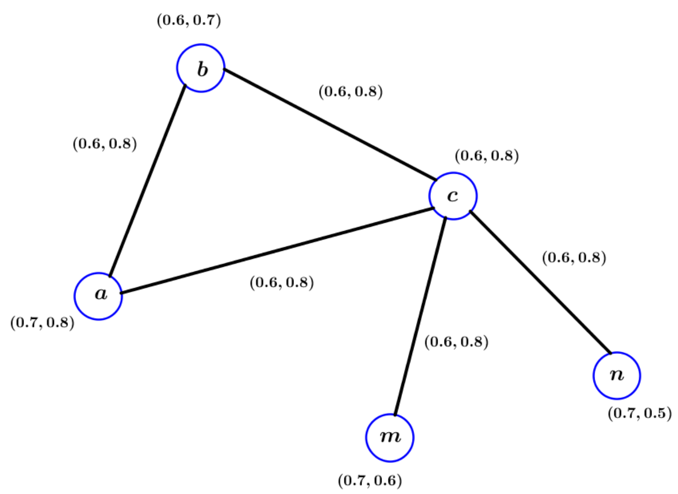

FIGURE 3. A VG $\mathfrak{G}$

The cardinality of the neighborhood for vertices is shown in Table 4, and we can calculated $\mathfrak{NCI} = (6, 8)$.

Also, by using Theorem 3.15 we can calculate,

$$\mathfrak{NCI}^t(G) = \frac{6}{10} \times 10 = 6 \quad , \quad \mathfrak{NCI}^f(G) = \frac{8}{10} \times 10 = 8$$

TABLE 4

Vertex	$d(x)$	$e^t(x)$	$e^f(x)$	$d(x)e^t(x)$	$d(x)e^f(x)$
a	2	0.6	0.8	1.2	1.6
b	2	0.6	0.8	1.2	1.6
c	4	0.6	0.8	2.4	3.2
m	1	0.6	0.8	0.6	0.8
n	1	0.6	0.8	0.6	0.8
$\mathfrak{NCI}$				6	8

Theorem 3.3. *Suppose a VG $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$ with $|X| = n \geq 4$, $\mathfrak{D}^t(s_i) = k_1$ and $\mathfrak{D}^f(s_i) = k_2$, for every $s_i \in X$. Consider $\mathfrak{G}^c = (\mathfrak{D}^c, \mathfrak{Z}^c)$ is a complement of VG $\mathfrak{G}$.*

$$\mathfrak{NCI}^t(\mathfrak{G}^c) - \mathfrak{NCI}^t(\mathfrak{G}) \geq n^2k_1 - 5nk_1$$

$$\mathfrak{NCI}^f(\mathfrak{G}^c) - \mathfrak{NCI}^f(\mathfrak{G}) \leq n^2k_2 - 5nk_2$$

Proof. Suppose $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$ is VG cycle. The neighborhood of every vertex in the vague cycle has two vertices. Therefore, $d(s) = 2$. The potential of every vertex will always be less than k_1. Since every vertex has strength k_1. Therefore, $e^t(s) \leq k_1$. Thus,

$$\mathfrak{NCI}^t(\mathfrak{G}) = \sum_{s\in X} d(s)e^t(s) = 2\sum_{s\in X} e^t(s) \leq 2nk_1 \quad ,(1)$$

Now, suppose the complement $\mathfrak{G}^c = (\mathfrak{D}^c, \mathfrak{Z}^c)$ of the graph $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$. Each vertex can have a neighborhood of cardinality greater than $n - 3$. ie $d(s) \geq n - 3$ for every $s \in X$. Since all the edges other than those lying on the cycle has strength k_1, and all others have strength less than k_1, we can write $e^t(s) = k_1$. Therefore,

$$\mathfrak{NCI}^t(\mathfrak{G}^c) = \sum_{s\in X} d(s)e^t(s) = k_1\sum_{s\in X} d(s) \geq nk_1(n-3) = n^2k_1 - 3nk_1 \quad ,(2)$$

By using equation 1 and 2 we have,

$$\mathfrak{NCI}^t(\mathfrak{G}^c) - \mathfrak{NCI}^t(\mathfrak{G}) \geq n^2k_1 - 3nk_1 - 2nk_1 = n^2k_1 - 5nk_1$$

Similarity, suppose $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$ is VG cycle. Therefore, $d(s) = 2$. The potential of every vertex will always be greater than k_2. Since every vertex has strength k_2. Therefore, $e^f(s) \geq k_2$. Thus,

$$\mathfrak{NCI}^f(\mathfrak{G}) = \sum_{s\in X} d(s)e^f(s) = 2\sum_{s\in X} e^f(s) \geq 2nk_2 \quad ,(3)$$

$\mathfrak{G}^c = (\mathfrak{D}^c, \mathfrak{Z}^c)$ is a complement of $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$, we have $d(s) \geq n - 3$ for every $s \in X$. Since all the edges other than those lying on the cycle has strength k_2, and all others have strength greater than k_2, we can write $e^2(s) = k_2$. Therefore,

$$\mathfrak{NCI}^f(\mathfrak{G}^c) = \sum_{s\in X} d(s)e^f(s) = k_2\sum_{s\in X} d(s) \geq nk_2(n-3) = n^2k_2 - 3nk_2 \quad ,(4)$$

□

Definition 3.2. *Two sets of vertices are called a twinning vertex sets of cardinality k if each set has cardinality k and $\mathfrak{NCI}$ of the graph obtained after removing each set is same.*

3.1. **Algorithm.** In this part, by using a following algorithm, we find the $\mathfrak{NCI}$ with n vertices. Suppose $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$ is a VG with n vertices $X = \{s_1, s_2, ..., s_n\}$.
1. Construct the matrix $\mathfrak{A} = [m_{ij}]$ with $m_{ij} = \mathfrak{Z}^t(s_is_j)$ for membership and $m_{ij} = \mathfrak{Z}^f(s_is_j)$ for non-membership.
2. If the largest membership value in every row of the matrix is e_i^t and the least non-membership value in every row of the matrix is e_i^f. We find them.
3. If the number of non-zero entres in every row of the matrix is d_i. We find it.
4. Then, $\mathfrak{NCI}^t = \sum_{i=1}^n e_i^t \times d_i$ and $\mathfrak{NCI}^f = \sum_{i=1}^n e_i^f \times d_i$.

Example 3.4. *Suppose* $\mathfrak{G} = (\mathfrak{D}, \mathfrak{Z})$ *is a VG in Figure 4 with* $X = \{a, b, c, m, n\}$ *and* $E = \{ab, bc, ac, am, an, mn\}$.

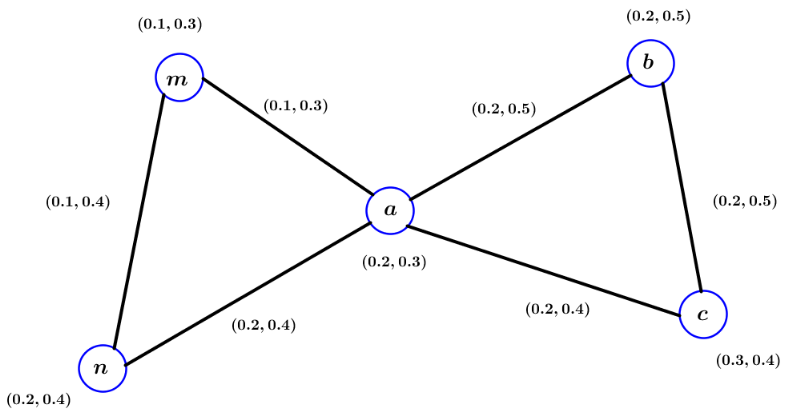

FIGURE 4. A VG $\mathfrak{G}$

We have the matrix of the VG,

$$\mathfrak{A}^t(\mathfrak{G}) = \begin{bmatrix} 0 & 0.2 & 0.2 & 0.1 & 0.2 \\ 0.2 & 0 & 0.2 & 0 & 0 \\ 0.2 & 0.2 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0 & 0.1 \\ 0.2 & 0 & 0 & 0.1 & 0 \end{bmatrix}$$

We find the largest membership value in every row of the matrix. The $\mathfrak{NCI}^t$ *is calculated by summing the product of largest value of every row and number of non zero entries in every row.*
$\mathfrak{NCI}^t = 0.2 \times 4 + 0.2 \times 2 + 0.2 \times 2 + 0.1 \times 2 + 0.2 \times 2 = 2.2$

$$\mathfrak{A}^f(\mathfrak{G}) = \begin{bmatrix} 0 & 0.5 & 0.4 & 0.3 & 0.4 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0.4 & 0.5 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0 & 0.4 \\ 0.4 & 0 & 0 & 0.4 & 0 \end{bmatrix}$$

We find the least non- membership value in every row of the matrix. The $\mathfrak{NCI}^f$ *is calculated by summing the product of least value of every row and number of non zero*

entries in every row.
$\mathfrak{NCI}^f = 0.3 \times 4 + 0.5 \times 2 + 0.4 \times 2 + 0.3 \times 2 + 0.4 \times 2 = 4.4$

4. Application

Agriculture is an important part of the economy having a significant role in exports of any country. The findings present that Export Price Index, GDP and exchange rates have a positive effect and exchange rate fluctuations and population have a negative effect on agricultural exports. So, it is recommended that, for improving agricultural exports, GDP and exchange rate should be increased. In addition, by decreasing the fluctuation in exchange rate, the negative effects will be controlled.

We have six suitable regions for growing crops and using agricultural products in terms of weather conditions and availability of required facilities. We are going to export the products grown in these areas to other countries. These areas are connected due to favorable conditions for agriculture. First, we construct the VG $\mathfrak{G}$ from this data. The vertices represent the regions and the edges represent the connection. The weight of vertices and edges is shown in Figure 5.

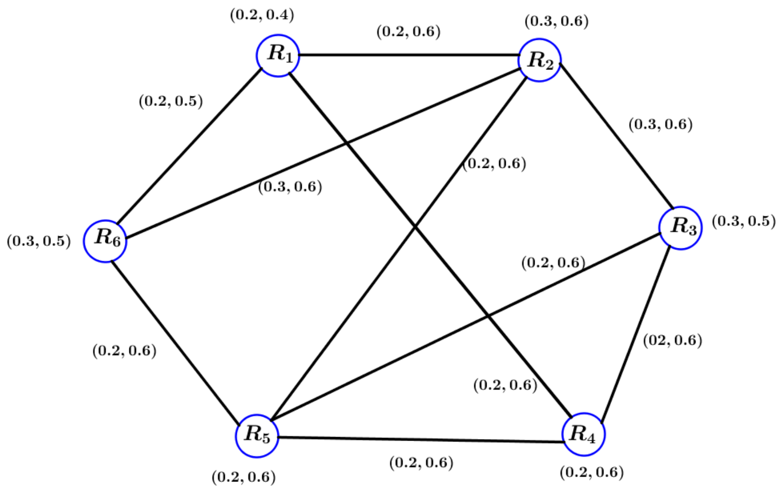

Figure 5. A VG $\mathfrak{G}$

Now, we write adjacency matrix of $\mathfrak{G}$ for analyzed data, Now by using the algorithm which we mentioned previously, we calculate $\mathfrak{NCI}$ of $\mathfrak{G}$.

$$\mathfrak{A}^t(\mathfrak{G}) = \begin{bmatrix} 0 & 0.2 & 0 & 0.2 & 0 & 0.2 \\ 0.2 & 0 & 0.3 & 0 & 0.2 & 0.3 \\ 0 & 0.3 & 0 & 0.2 & 0.2 & 0 \\ 0.2 & 0 & 0.2 & 0 & 0.2 & 0 \\ 0 & 0.2 & 0.2 & 0.2 & 0 & 0.2 \\ 0.2 & 0.3 & 0 & 0 & 0.2 & 0 \end{bmatrix}$$

$$\mathfrak{A}^f(\mathfrak{G}) = \begin{bmatrix} 0 & 0.6 & 0 & 0.6 & 0 & 0.5 \\ 0.6 & 0 & 0.6 & 0 & 0.6 & 0.6 \\ 0 & 0.6 & 0 & 0.6 & 0.6 & 0 \\ 0.6 & 0 & 0.6 & 0 & 0.6 & 0 \\ 0 & 0.6 & 0.6 & 0.6 & 0 & 0.6 \\ 0.5 & 0.6 & 0 & 0 & 0.6 & 0 \end{bmatrix}$$

By using the above algorithm, we have
$\mathfrak{NCI}^t(\mathfrak{G}) = 0.2 \times 3 + 0.3 \times 4 + 0.3 \times 3 + 0.2 \times 3 + 0.2 \times 4 + 0.3 \times 3 = 5$ $\mathfrak{NCI}^f(\mathfrak{G}) = 0.5 \times 3 + 0.6 \times 4 + 0.6 \times 3 + 0.6 \times 3 + 0.6 \times 4 + 0.5 \times 3 = 11.4$
The cardinality of neighborhood for all vertices is shown in Table 5.

Table 5

Vertex	$d(x)$	$e^t(x)$	$e^f(x)$	$d(x)e^t(x)$	$d(x)e^f(x)$
R_1	3	0.2	0.5	0.6	1.5
R_2	4	0.3	0.6	1.2	2.4
R_3	3	0.3	0.6	0.9	1.8
R_4	3	0.2	0.6	0.6	1.8
R_5	4	0.2	0.6	0.8	2.4
R_6	3	0.3	0.5	0.9	1.5

Here, we construct the adjacency matrix of $\mathfrak{G} - \{R_1, R_3\}$.

$$\mathfrak{A}^t(\mathfrak{G} - \{R_1, R_3\}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0.2 & 0 & 0.2 & 0 & 0.2 \\ 0 & 0.3 & 0 & 0 & 0.2 & 0 \end{bmatrix}$$

$$\mathfrak{A}^f(\mathfrak{G} - \{R_1, R_3\}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0.6 & 0 & 0.6 & 0 & 0.6 \\ 0 & 0.6 & 0 & 0 & 0.6 & 0 \end{bmatrix}$$

Here, we obtaine the $\mathfrak{NCI}^t(\mathfrak{G} - \{R_1, R_3\}) = 2$ and $\mathfrak{NCI}^f(\mathfrak{G} - \{R_1, R_3\}) = 4.8$
Also, we construct the adjacency matrix of $\mathfrak{G} - \{R_5, R_6\}$.

$$\mathfrak{A}^t(\mathfrak{G} - \{R_5, R_6\}) = \begin{bmatrix} 0 & 0.2 & 0 & 0.2 & 0 & 0 \\ 0.2 & 0 & 0.3 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0.2 & 0 & 0 \\ 0.2 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathfrak{A}^f(\mathfrak{G}-\{R_5,R_6\})=\begin{bmatrix}0&0.6&0&0.6&0&0\\0.6&0&0.6&0&0&0\\0&0.6&0&0.6&0&0\\0.6&0&0.6&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$$

Here, we obtaine the $\mathfrak{NCI}^t(\mathfrak{G}-\{R_5,R_6\})=2$ and $\mathfrak{NCI}^f(\mathfrak{G}-\{R_5,R_6\})=4.8$.

Therefore, we can see that $\{R_1,R_3\},\{R_5,R_6\}$ are twinning vertex sets of cardinality two with $\mathfrak{NCI}=(2,4.8)$

In this application, the findings indicate that regions $\{R_1,R_3\}$ and $\{R_5,R_6\}$ play an important role in increasing the GDP and exchange rate to improve the export of agricultural products. By removing these areas, the amount of exports will decrease.

5. Conclusions

A VG, an extension of the basic notion of an FG, can be employed to deal with deeper aspects of uncertainty and imprecision for which the use of FGs would not fully succeed. The VGs can amplify flexibility to model complex real-time problems better than an FG. One of the most important features of VGs that have many applications in real problems is the concept of NCI. The CI plays a significant role in modeling various problems such as networking, transportation, and precise location detection. An NCI is a new parameter related to connection studied in this work. NCI has many applications in different fields. Therefore, in this paper, we examined NCI on VGs. Finally, an application of NCI has been introduced. In the future, we intend to broaden the scope of our research to include topological indices and the notion of NCI on vague influence graphs. There is a scope for extensive theoretical and practical analysis of these topics. We also aim to introduce some other topological indices in VGs and investigate their applications.

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