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ON THE TRAVERSABILITY OF NEW COMPOSITION OF TOTAL GRAPHS

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Abstract. The F-composition $F(G)[H]$ is a graph with the set of vertices $V(F(G)[H]) = (V(G) \cup E(G)) \times V(H)$ and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ of $F(G)[H]$ are adjacent if and only if $[u_1 = v_1 \in V(G)$ and $(u_2, v_2) \in E(H)]$ or $[(u_1, v_1) \in E(F(G))]$. Here $F(G)$ be one of the symbols $\Gamma_T(G)$, or $\Delta_T(G)$. In this paper, we study the Eulerian and Hamiltonian properties of the resulting graphs.

Keywords: Euler graph, Hamiltonian graph, composition of two graphs, Gallai total graph, anti-Gallai total graph.

AMS Subject Classification: 05C45, 05C76

1. Introduction

A *graph* $G = (V, E)$ is an ordered pair of set of vertices and edges, where edges are unordered pair of vertices. Also G is said to be a (n, m) graph if $|V| = n$ and $|E| = m$. A graph is *simple* if it has neither self loop nor multiple edges. A *finite graph* is a graph with finite number of vertices and edges. Two vertices (edges) are said to be *adjacent* if they have a common edge (vertex). If a vertex v lies on an edge e, then they are said to be *incident* to each other. The *degree* $d_G(v)$ of a vertex $v \in V$ is the number of edges incident at v. A *regular graph* is the graph in which every vertex of the graph has same degree. Let $G = (V, E)$ be a graph with $|V| = n$, then the *adjacency matrix* $A(G)$ of G is defined as $A(G) = [a_{ij}]_{n\times n}$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j,\ i \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

A graph G is called *Euler graph* if there exists a closed walk in G with no repeated edge and all the edges are traversed exactly once. A closed path is called a *cycle*. A cycle containing all the vertices of the graph is called a *spanning cycle*. A graph containing a

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spanning cycle is called a *Hamiltonian graph*. A graph G is called *null graph* if the edge set of the graph G is empty.

Definition 1.1. *The Gallai graph $\Gamma(G)$ of a graph G is the graph in which $V(\Gamma(G)) = E(G)$ and two distinct edges of G are adjacent in $\Gamma(G)$ if they are adjacent in G, but do not span a triangle in G.*

Definition 1.2. *The anti-Gallai graph $\Delta(G)$ of a graph G is the graph in which $V(\Delta(G)) = E(G)$ and two distinct edges of G are adjacent in $\Delta(G)$ if they are adjacent in G and lie on a same triangle in G.*

These structures were employed by Gallai [4] in his research of comparability graphs. This notion was suggested by Sun [15] and was further used by him to study two class of perfect graphs. Lakshmanan et al. [9] and Palathingal et al. [14] investigated H-free properties of Gallai and anti-Gallai graph of H. Several other characteristics of Gallai and anti-Gallai graphs have been discussed (see [2, 3, 10, 11, 13]).

Definition 1.3. *The Gallai total graph $\Gamma_T(G)$ of G is the graph, where $V(\Gamma_T(G)) = V \cup E$ and $uv \in E(\Gamma_T(G))$ if and only if*

(i) *u and v are adjacent vertices in G, or*
(ii) *u is incident to v or v is incident to u in G, or*
(iii) *u and v are adjacent edges in G which do not span a triangle in G.*

Definition 1.4. *The anti-Gallai total graph $\Delta_T(G)$ of G is the graph, where $V(\Delta_T(G)) = V \cup E$ and $uv \in E(\Delta_T(G))$ if and only if*

(i) *u and v are adjacent vertices in G, or*
(ii) *u is incident to v or v is incident to u in G, or*
(iii) *u and v are adjacent edges in G and lie on a same triangle in G.*

Goyal et al. [5] introduced Gallai-total and anti-Gallai total graphs and discussed the traversability of these graphs. Characteristics of Gallai and anti-Gallai total simplicial complexes were researched by Liaquat [12] and Abbas et al. [1] respectively. Figure 1 shows the Gallai total graph $\Gamma_T(G)$ and anti-Gallai total graph $\Delta_T(G)$ of a graph G.

Definition 1.5. *The composition $G[H]$ of two graphs G and H, as defined by F. Harary* [7, 8]*, with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph with vertex set $V_1 \times V_2$ and $u = (u_1, u_2)$ is adjacent with $v = (v_1, v_2)$ whenever,*

(i) *$u_1 = v_1$ and $u_2v_2 \in E(H)$, or*
(ii) *$u_1v_1 \in E(G)$.*

Figure 2 shows the composition $G[H]$ of two graphs G and H.

Definition 1.6. *The F-composition $F(G)[H]$ of two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is a graph with vertex set $V(F(G)[H]) = (V(G) \cup E(G)) \times V(H)$ and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ of $F(G[H])$ are adjacent if and only if*

(i) *$u_1 = v_1 \in V(G)$ and $u_2v_2 \in E(H)$, or*
(ii) *$u_1v_1 \in E(F(G))$.*

It was introduced by Goyal et al. [6] and they also gave some results on the wiener indices of different graph operations of F-composition. Further, we use two graph operators, say, Gallai total graph $\Gamma_T(G)$ and anti-Gallai total graph $\Delta_T(G)$ as $F(G)$. Figure 3 illustrates the definitions of Γ-composition $H' = \Gamma_T(G)[H]$ and Δ-composition $H'' = \Delta_T(G)[H]$ of graphs G and H.
Remark: Degree of $v \in V(\Gamma_T(G)[H])$ is denoted by $d_\Gamma(v)$ and degree of $v \in V(\Delta_T(G)[H])$ is denoted by $d_\Delta(v)$.

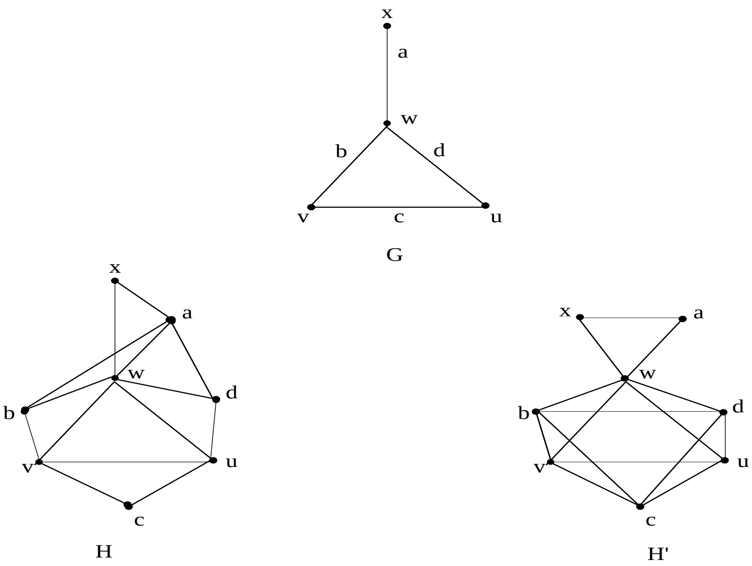

FIGURE 1. A graph G, its Gallai total graph $H = \Gamma_T(G)$ and its anti-Gallai total graph $H' = \Delta_T(G)$

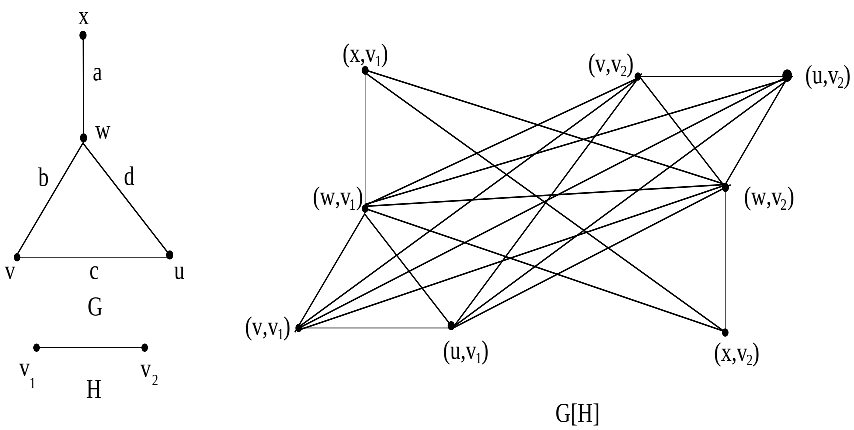

FIGURE 2. Composition $G[H]$ of two graphs G and H

2. Eulerian Γ-Composition of Graphs

Proposition 2.1. *Let $G = (n, m)$ and $H = (p, q)$ be two non-empty graphs and $\Gamma_T(G)[H]$ be Γ-composition of the graphs G and H, then*

$$d_\Gamma(u, v) = \begin{cases} 2pd_G(u) + d_H(v); & \text{if } u \in V(G), \\ p[d_G(v_1) + d_G(v_2) - 2t]; & \text{if } u = v_1v_2 \in E(G), \end{cases}$$

where t denotes the number of triangles containing u in G.

Proof. Let $\Gamma_T(G)[H]$ be the corresponding Γ-composition of G and H,

(i) Since the graph G is a subgraph of $\Gamma_T(G)$ and also each edge incident to u in G is adjacent to corresponding vertex u in $\Gamma_T(G)$. Therefore, degree of u in $\Gamma_T(G) = 2d_G(u)$, for $u \in V(G)$, and $\Gamma_T(G)[H]$ consists p copies of $\Gamma_T(G)$, so we

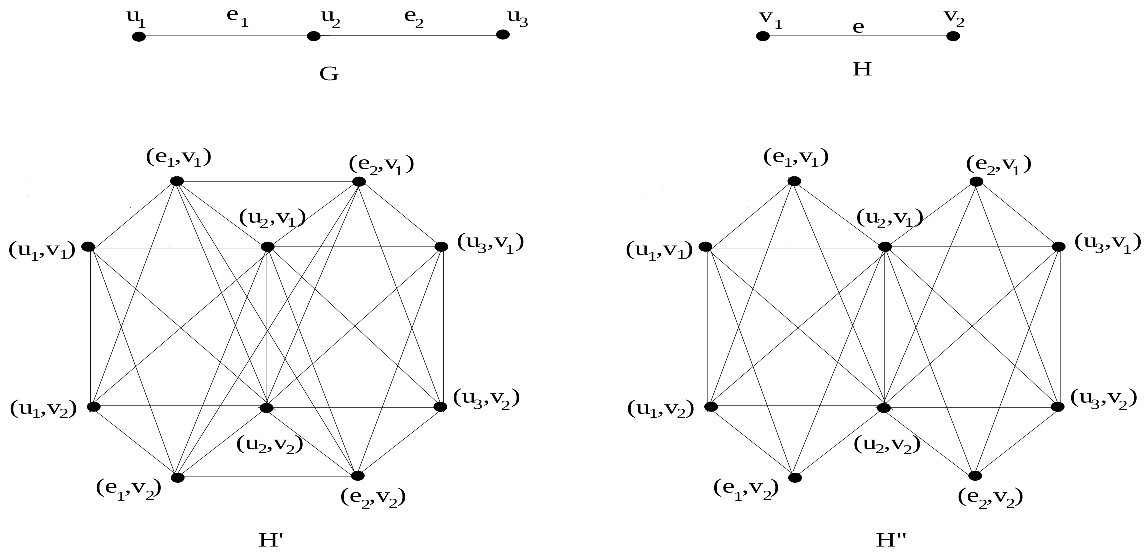

FIGURE 3. Two graphs G and H, their Γ-composition $H' = \Gamma_T(G)[H]$ and Δ-composition $H'' = \Delta_T(G)[H]$

get $d_\Gamma(u, v) = 2pd_G(u)$. By the first condition in the definition of $\Gamma_T(G)[H]$, the vertices of $\Gamma_T(G)[H]$ are adjacent corresponding to the vertices of G if the vertices of H are adjacent, contributing $d_H(v)$ to the degree of $\Gamma_T(G)[H]$. Therefore, $d_\Gamma(u, v) = 2pd_G(u) + d_H(v)$.

(ii) If $u = v_1v_2 \in E(G)$, then the corresponding vertex u of $\Gamma_T(G)$ is adjacent to all the edges which are adjacent to u, but do not lie on a same triangle with u in G. It implies that they contribute the degree $(d_G(v_1) - 1) + (d_G(v_2) - 1) - 2t$ (if u is the edge of a triangle, then it is not adjacent to those two edges of G which span a triangle with u in G) and u is also adjacent in $\Gamma_T(G)$ to the vertices by which it is incident in G, so 2 more degrees are also contributed. Therefore degree of the vertex corresponding to an edge of G in $\Gamma_T(G)$ is equal to $d_G(v_1)+d_G(v_2)-2t$. Also $\Gamma_T(G)[H]$ consists p copies of $\Gamma_T(G)$. Therefore, $d_\Gamma(u, v) = p[d_G(v_1)+d_G(v_2)-2t]$.

□

Lemma 2.1. [5] *The number of triangles in a graph G is equal to $\frac{tr(A^3)}{6}$, where A is the adjacency matrix of G.*

Proposition 2.2. *Let $G = (n, m)$ and $H = (p, q)$ be two graphs, then*

$$|E(\Gamma_T(G)[H])| = nq + p^2[2m - \tfrac{tr(A^3)}{2} + \tfrac{1}{2}\sum_{i=1}^{n}(d_G(u_i))^2],$$

where A is the adjacency matrix of G.

Proof. Let $G = (n, m)$ and $H = (p, q)$ be two graphs. Then, by Handshake Lemma on $\Gamma_T(G)[H]$ we have, Total degree of $\Gamma_T(G)[H]$

$$= \sum_{u\in V(G)}\sum_{v\in V(H)} d_\Gamma(u,v) + \sum_{u=v_iv_j\in E(G)}\sum_{v\in V(H)} d_\Gamma(u,v)$$

$$= \sum_{u\in V(G)}\sum_{v\in V(H)} [2pd_G(u)+d_H(v)] + \sum_{v_iv_j\in E(G)}\sum_{v\in V(H)} p[d_G(v_1)+d_G(v_2)-2t_{ij}]$$

(where t_{ij} is no. of triangles in G containing the v_iv_j edge)

$$= 2p\sum_{u\in V(G)}\sum_{v\in V(H)} d_G(u) + \sum_{u\in V(G)}\sum_{v\in V(H)} d_H(v) + p\sum_{v_iv_j\in E(G)}\sum_{v\in V(H)} [d_G(v_1)+d_G(v_2)]$$

$$- 2p\sum_{v_iv_j\in E(G)}\sum_{v\in V(H)} t_{ij}$$

$$= 2p^2\sum_{u\in V(G)} d_G(u) + n\sum_{v\in V(H)} d_H(v) + p^2\sum_{i=1}^{n}(d_G(u_i))^2 - 2p^2(3\times \text{total no. of triangles in } G)$$

$$= 2p^2\times 2m + n\times 2q + p^2\sum_{i=1}^{n}(d_G(u_i))^2 - 2p^2\left(3\frac{tr(A^3)}{6}\right)$$

$$= 2\left[nq + p^2\left(2m - 3\frac{tr(A^3)}{6} + \frac{1}{2}\sum_{i=1}^{n}(d_G(u_i))^2\right)\right]$$

Then by Handshake Lemma, the total number of edges in $\Gamma_T(G)[H]$,

$$|E(\Gamma_T(G)[H])| = nq + p^2[2m - \tfrac{tr(A^3)}{2} + \tfrac{1}{2}\sum_{i=1}^{n}(d_G(u_i))^2].$$

□

A graph is *trianglefree*, if it does not contain a triangle.

Lemma 2.2. [5] *The Gallai total graph $\Gamma_T(G)$ of a graph G is regular if and only if G is regular and triangle free.*

Proposition 2.3. *The Γ-composition $\Gamma_T(G)[H]$ of the graphs G and H is regular if and only if H is a null graph, G is regular and triangle free.*

Proof. Let H be a null graph, G be a regular and triangle free graph. Now, we have to show that $\Gamma_T(G)[H]$ is regular. Since G is regular i.e. degree of each vertex is same. Also, H is a null graph. It implies that vertices of $\Gamma_T(G)[H]$ corresponding to the vertices of G are of same degree (by the Proposition 2.1). Also it is given that G is triangle free. It follows that every vertex of $\Gamma_T(G)[H]$ corresponding to the edges of G has degree $p[d_G(u)+d_G(v)]$ (by the Proposition 2.1), where u and v are the end vertices of the edge and this is equal to $2pd_G(u)$, being G is regular. Therefore, degree of each vertex of $\Gamma_T(G)[H]$ is same. Hence, $\Gamma_T(G)[H]$ is regular.

Conversely, suppose that $\Gamma_T(G)[H]$ is regular. Now, we have to show that H is a null graph, G is regular and triangle free. Let, on contrary, G is neither regular nor triangle free and H is not null. If G is not regular, then degree of every vertex of $\Gamma_T(G)[H]$ corresponding to the vertices of G is not same, which is a contradiction to our fact that $\Gamma_T(G)[H]$ is regular. Hence G is regular. Now, if G is not triangle free and H is not null, then degree of every vertex of $\Gamma_T(G)[H]$ corresponding to the edges of G and degree of every vertex of $\Gamma_T(G)[H]$ corresponding to the vertices of G is not same, which is again a contradiction to our fact that $\Gamma_T(G)[H]$ is regular. Thus, G is triangle free and H is a null graph. Hence, H is a null graph, G is regular and triangle free. □

Lemma 2.3. [5] *The Gallai total graph $\Gamma_T(G)$ of G is connected if and only if G is connected.*

Proposition 2.4. *The Γ-composition $\Gamma_T(G)[H]$ of the graphs G and H is connected if and only if G is connected.*

Proof. Let G be a connected graph. Then $\Gamma_T(G)$ is connected (by Lemma 2.3). Since, $\Gamma_T(G)[H]$ consists p copies of $\Gamma_T(G)$, where $p = |V(H)|$. Therefore, all these p copies are connected components of $\Gamma_T(G)[H]$. Now, it remains to show that these components are connected to each other. By the second condition in the definition of Γ-composition, there exists atleast one vertex (u, v) in any component of $\Gamma_T(G)[H]$ adjacent to a vertex (a, b) of another component whenever $ua \in E(\Gamma_T(G))$, which proves the result.

Conversely, suppose $\Gamma_T(G)[H]$ is connected. Now, we have to show that G is connected. Let us assume that G is not connected. Then, there exists at least one pair of vertices, say (u, v) and (a, b) which has no path between them. This implies that there are no two vertices of $\Gamma_T(G)[H]$ which have a path between them. This implies that $\Gamma_T(G)[H]$ is disconnected, which is a contradiction to our assumption. Thus, G is connected. □

Corollary 2.1. *The $\Gamma_T^n(G)[H]$ of G and H is connected if and only if G is connected for all $n \geq 1$.*

Parity of a vertex means parity of its degree, i.e. degree of the vertex is either even or odd.

Theorem 2.1. *Let $G = (n, m)$ be a connected graph and $H = (p, q)$ be any non-empty graph, then Γ-composition $\Gamma_T(G)[H]$ of the graphs G and H is Eulerian if and only if H is Eulerian and either all the vertices of G are of the same parity or number of vertices in H is even.*

Proof. Let $\Gamma_T(G)[H]$ be an Eulerian graph. Then, the degree of every vertex $\Gamma_T(G)[H]$ is even. By Proposition 2.1, we can say that $d_H(v)$ is even which implies H is Eulerian and either p is even or all the vertices of G are of same parity in G.

Conversely, suppose all the vertices of G are of the same parity and degree of each vertex of H is even, then by the Proposition 2.1, the degree of vertices of $\Gamma_T(G)[H]$ corresponding to the vertices of G is even. If number of vertices of H is even and degree of each vertex in H is even, then by the Proposition 2.1, the degree of all the vertices of $\Gamma_T(G)[H]$ corresponding to the edges of G is even. Therefore, $\Gamma_T(G)[H]$ is Eulerian. Hence the theorem. □

Corollary 2.2. *If G and H are Eulerian then $\Gamma_T(G)[H]$ is also Eulerian, but converse is not true.*

Counter example of converse: The Γ-composition graph $H' = \Gamma_T(G)[H]$ of the graphs G and H is Eulerian, but G is not an Eulerian graph as shown in Figure 4.

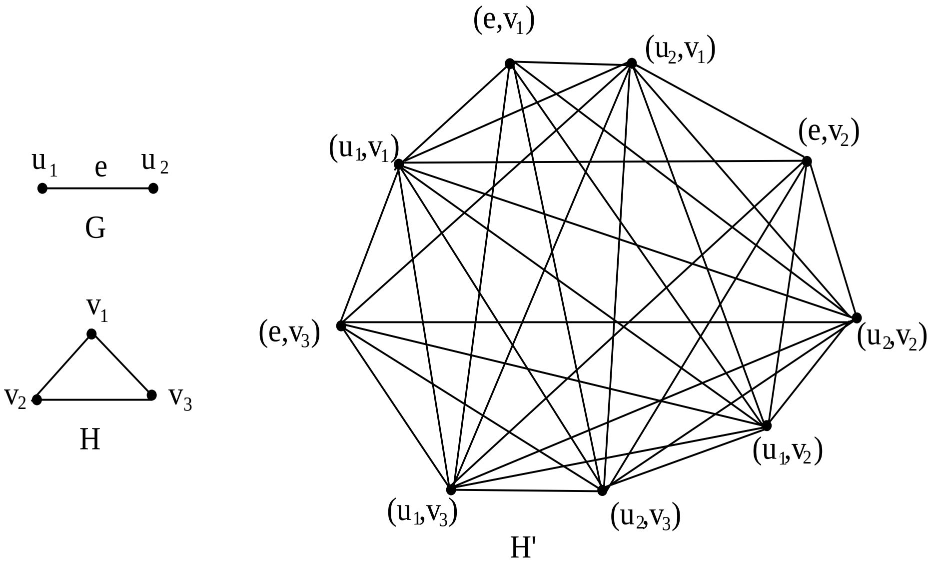

FIGURE 4. Eulerian Γ-composition graph $H' = \Gamma_T(G)[H]$ of a non-Eulerian graph G.

3. Eulerian Δ-Composition of graphs

Proposition 3.1. *Let $G = (n, m)$ and $H = (p, q)$ be two non-empty graphs and $\Delta_T(G)[H]$ be Δ-composition of the graphs G and H, then*

$$d_\Delta(u, v) = \begin{cases} 2pd_G(u) + d_H(v); & \text{if } u \in V(G), \\ 2p(t+1); & \text{if } u = v_1v_2 \in E(G), \end{cases}$$

where t denotes the number of triangles containing u in G.

Proof. Let $\Delta_T(G)[H]$ be the corresponding Δ-composition of G and H,

(*i*) Since the graph G is a subgraph of $\Delta_T(G)$ and also each edge incident to u in G is adjacent to corresponding vertex u in $\Delta_T(G)$. Therefore, degree of u in $\Delta_T(G) = 2d_G(u)$, for $u \in V(G)$, and $\Delta_T(G)[H]$ consists p copies of $\Delta_T(G)$, so we get $d_\Delta(u, v) = 2pd_G(u)$. And by the first condition in the definition of $\Delta_T(G)[H]$, the vertices of $\Delta_T(G)[H]$ are adjacent corresponding to the vertices of G if the vertices of H are adjacent, contributing $d_H(v)$ to the degree of $\Delta_T(G)[H]$. Therefore, $d_\Delta(u, v) = 2pd_G(u) + d_H(v)$.

(*ii*) If $v = v_1v_2 \in E(G)$, then the corresponding vertex v of $\Delta_T(G)$ is adjacent to all the edges which are adjacent to v, and lie on a same triangle with v in G. It implies that they contribute the degree $2t$ (if v is the edge of a triangle, then it is adjacent to those two edges of G which span triangle with v in G) and v is also adjacent in $\Delta_T(G)$ to the vertices by which it is incident in G, so 2 more degrees are also contributed, therefore, $d_\Delta(v) = 2t + 2$ and $\Delta_T(G)[H]$ consists p copies of $\Delta_T(G)$. Therefore, $d_\Delta(u, v) = 2p(t + 1)$.

□

Proposition 3.2. *Let $G = (n, m)$ and $H = (p, q)$ be two graphs, then*

$$|E(\Delta_T(G)[H])| = nq + 3p^2[m + \tfrac{tr(A^3)}{6}],$$

where A is the adjacency matrix of G.

Proof. Let $G = (n, m)$ and $H = (p, q)$ be two graphs. Then by Handshake Lemma on $\Delta_T(G)[H]$ we have,
Total degree of $\Delta_T(G)[H]$

$$\begin{aligned}
&= \sum_{u\in V(G)}\sum_{v\in V(H)} d_\Delta(u,v) + \sum_{u=v_iv_j\in E(G)}\sum_{v\in V(H)} d_\Delta(u,v)\\
&= \sum_{u\in V(G)}\sum_{v\in V(H)} [2pd_G(u) + d_H(v)] + \sum_{v_iv_j\in E(G)}\sum_{v\in V(H)} p[2t_{ij}+2]\\
&\quad (\text{ where } t_{ij} \text{ is no. of triangles in } G \text{ containing the } v_iv_j \text{ edge})\\
&= 2p\sum_{u\in V(G)}\sum_{v\in V(H)} d_G(u) + \sum_{u\in V(G)}\sum_{v\in V(H)} d_H(v) + 2p\sum_{v_iv_j\in E(G)}\sum_{v\in V(H)} t_{ij}\\
&+ 2p\sum_{v_iv_j\in E(G)}\sum_{v\in V(H)} 1\\
&= 2p^2\sum_{u\in V(G)} d_G(u) + n\sum_{v\in V(H)} d_H(v) + 2p^2(3\times \text{total no. of triangles in } G) + 2mp^2\\
&= 2p^2\times 2m + n\times 2q + 2p^2(3\frac{tr(A^3)}{6}) + 2mp^2\\
&= 2[nq + 3p^2(m + \frac{tr(A^3)}{6})]
\end{aligned}$$

Then by Handshake Lemma, the total number of edges in $\Delta_T(G)[H]$,

$$|E(\Delta_T(G)[H])| = nq + 3p^2[m + \tfrac{tr(A^3)}{6}].$$

□

A graph is called *l-triangular* if each edge of G lies on l number of triangles in G.

Lemma 3.1. [5] *The anti-Gallai total graph $\Delta_T(G)$ is regular if and only if G is l-triangular and $(l+1)$-regular.*

Proposition 3.3. *The Δ-composition $\Delta_T(G)[H]$ of the graphs G and H is regular if and only if H is a null graph, G is (l+1)-regular and l-triangular.*

Proof. Let H be a null graph, G be a $(l+1)$-regular and l-triangular graph. Now, we have to show that $\Delta_T(G)[H]$ is regular. Since G is $(l+1)$-regular, degree of each vertex is $l+1$. Also, H is a null graph. It implies that vertices of $\Delta_T(G)[H]$ corresponding to the vertices of G are of same degree $2p(l+1)$(by the Proposition 3.1). Also it is given that G is l-triangular. It follows that every vertex of $\Delta_T(G)[H]$ corresponding to the edges of G has degree $2p(l+1)$ (by the Proposition 3.1), being G is l-triangular. Therefore, degree of each vertex of $\Delta_T(G)[H]$ is same. Hence, $\Delta_T(G)[H]$ is regular.

Conversely, suppose that $\Delta_T(G)[H]$ is regular. Now, we have to show that H is a null graph, G is $(l+1)$-regular and l-triangular. Let, on contrary, G is neither $(l+1)$-regular nor l-triangular and H is not null. If G is not $(l+1)$-regular, then degree of every vertex of $\Delta_T(G)[H]$ corresponding to the vertices of G is not same, which is a contradiction to our fact that $\Delta_T(G)[H]$ is regular. Hence G is $(l+1)$-regular. Now, if G is not l-triangular and H is not null, then degree of a vertex of $\Delta_T(G)[H]$ corresponding to the edges of G and degree of a vertex of $\Delta_T(G)[H]$ corresponding to the vertices of G are not same, which is again a contradiction to our fact that $\Delta_T(G)[H]$ is regular. Thus, H is a null graph and G is l-triangular. Hence, H is a null graph, G is $(l+1)$-regular and l-triangular. □

Lemma 3.2. [5] *The anti-Gallai total graph $\Delta_T(G)$ of G is connected if and only if G is connected.*

Proposition 3.4. *The Δ-composition $\Delta_T(G)[H]$ of the graphs G and H is connected if and only if G is connected.*

Proof. Let G be a connected graph. Then $\Delta_T(G)$ is connected (by Lemma 3.2). Since, $\Delta_T(G)[H]$ consists p copies of $\Delta_T(G)$, where $p = |V(H)|$. Therefore, all these p copies are connected components of $\Delta_T(G)[H]$. Now, it remains to show that these components are connected to each other. By the second condition in the definition of Δ-composition, there exists atleast one vertex (u, v) in any component of $\Delta_T(G)[H]$ adjacent to a vertex (a, b) of another component whenever $ua \in E(\Delta_T(G))$, which proves the result.

Conversely, suppose $\Delta_T(G)[H]$ is connected. Now, we have to show that G is connected. Let us assume that G is not connected. Then, there exists at least one pair of vertices, say (u, v) and (a, b) which has no path between them. This implies there are no two vertices of $\Delta_T(G)[H]$ which have a path between them. This implies that $\Delta_T(G)[H]$ is disconnected, which is a contradiction to our assumption. Thus, G is connected. □

Corollary 3.1. *The $\Delta_T^n(G)[H]$ of G and H is connected if and only if G is connected for all $n \geq 1$.*

Theorem 3.1. *Let $G = (n, m)$ be a connected graph and $H = (p, q)$ be any non-empty graph, then Δ-composition $\Delta_T(G)[H]$ of the graphs G and H is Eulerian if and only if H is Eulerian.*

Proof. Let $\Delta_T(G)[H]$ be an Eulerian graph. Then, the degree of every vertex of $\Delta T(G)[H]$ is even. By Proposition 3.1, we can say that $d_H(v)$ is even which implies H is Eulerian.

Conversely, suppose H is Eulerian, then degree of every vertex of H is even. By Proposition 3.1, we can say that degree of all the vertices of $\Delta_T(G)[H]$ is even. This implies $\Delta_T(G)[H]$ is Eulerian. Hence, the theorem. □

Corollary 3.2. *If G and H are Eulerian graphs then $\Delta_T G[H]$ is Eulerian, but converse is not true.*

Counter example of converse: The Δ-composition graph $H' = \Delta_T(G)[H]$ of the graphs G and H is Eulerian, but G is not an Eulerian graph as shown in Figure 5.

4. Hamiltonian Γ-composition and Δ-composition of graphs

Lemma 4.1. [5] *The Gallai total graph $\Gamma_T(G)$ of a non-trivial graph G is Hamiltonian if and only if the set of all elements of G can be ordered in such a way that consecutive elements are neighbour as are the first and last elements, but the two edges are not consecutive elements, if both the edges are of same triangle.*

Theorem 4.1. *The Γ-composition $\Gamma_T(G)[H]$ of the graphs G and H is Hamiltonian if and only if Gallai total graph $\Gamma_T(G)$ of G is Hamiltonian.*

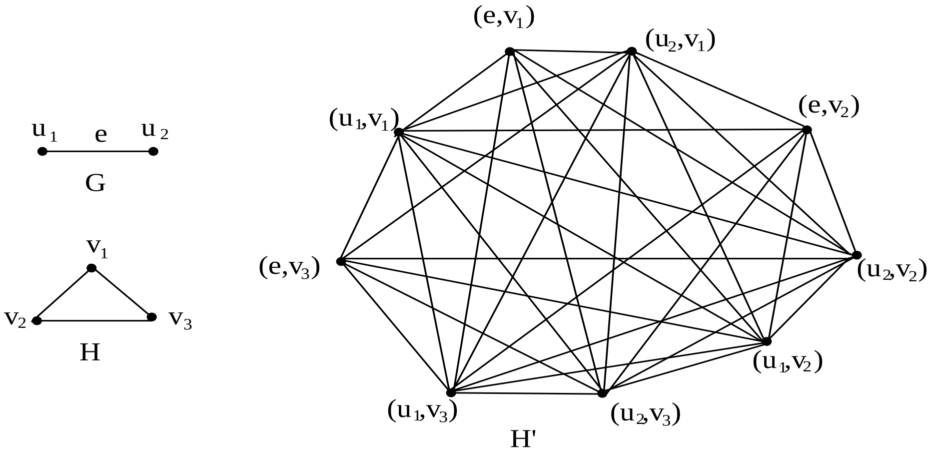

FIGURE 5. Eulerian Δ-composition graph $H' = \Delta_T(G)[H]$ of a non-Eulerian graph G.

Proof. Let G and H be two graphs. Also $\Gamma_T(G)$ is Hamiltonian so by the Lemma 4.1, all elements of G can be ordered in such a way that consecutive elements are neighbour as are the first and last elements, but the two edges are not consecutive elements if both the edges are of same triangle. Let $a_i's$ are the elements of G and $v_i \in V(H)$ Then for getting a closed Hamiltonian path for $\Gamma_T(G)[H]$ the following steps are followed:
STEP 1: Firstly find an open Hamiltonian path for first component of $\Gamma_T(G)[H]$, say, $(a_0, v_1), (a_1, v_1), ..., (a_n, v_1)$.
STEP 2: Same path will be constructed for all remaining components.
STEP 3: We get (a_n, v_1) adjacent to (a_0, v_2); (a_n, v_2) adjacent to (a_0, v_3); (a_n, v_3) adjacent to (a_0, v_4) and so on (a_n, v_{p-1}) adjacent to (a_0, v_p) (by the definition of $\Gamma_T(G)[H]$). Also, (a_n, v_p) adjacent to (a_0, v_1). In this way, we get a Hamiltonian cycle for $\Gamma_T(G)[H]$.
Hence, $\Gamma_T(G)[H]$ is Hamiltonian.

Conversely, let $\Gamma_T(G)[H]$ be a Hamiltonian graph. This follows that there exists a Hamlitonian cycle,

$$C = (c_0, c_1, c_2, \ldots c_x = c_0)$$

such that $c_0 = (a_0, v_1),\ c_1 = (a_1, v_1), \ldots c_n = (a_n, v_1),\ c_{n+1} = (a_1, v_2), \ldots c_{2n} = (a_n, v_2), \ldots c_{x-n} - (a_0, v_p), \ldots c_{x-1} = (a_n, v_p)$,
where $a_i's$ are elements of G and $v_i \in V(H)$.
We get (a_n, v_1) is adjacent to (a_0, v_2), (a_n, v_2) is adjacent to (a_0, v_3), and so on (a_n, v_{p-1}) is adjacent to (a_0, v_p), also (a_n, v_p) is adjacent to (a_0, v_1) (by definition of $\Gamma_T(G)[H]$).
This gives us an ordering $a_0, a_1, a_2, \ldots a_{n+1} = a_0$ of elements of G in such a way that consecutive elements are neighbour as are the first and last elements, but the two edges are not consecutive elements, if both the edges are of same triangle. So by Lemma 4.1, $\Gamma_T(G)$ is Hamiltonian. Hence, the theorem. □

Lemma 4.2. [5] *The anti-Gallai total graph $\Delta_T(G)$ of a non-trivial graph G is Hamiltonian if and only if the set of all elements of G can be ordered in such a way that consecutive elements are neighbour as are the first and last elements, but the two edges are not consecutive elements, if both the edges are not of same triangle.*

Theorem 4.2. *The Δ-composition $\Delta_T(G)[H]$ of the graphs G and H is Hamiltonian if and only anti-Gallai total graph $\Delta_T(G)$ of G is Hamiltonian.*

Proof. Let G and H be two graphs. Also $\Delta_T(G)$ is Hamiltonian so by the Lemma 4.2, all elements of G can be ordered in such a way that consecutive elements are neighbour as are the first and last elements, but the two edges are not consecutive elements if both the edges are not of same triangle. Let $a_i's$ are the elements of G and $v_i \in V(H)$. Then for getting a closed Hamiltonian path for $\Delta_T(G)[H]$ the following steps are followed:
STEP 1: Firstly find an open Hamiltonian path for first component of $\Delta_T(G)[H]$, say, $(a_0, v_1), (a_1, v_1), ..., (a_n, v_1)$.
STEP 2: Same path will be constructed for all remaining components.
STEP 3: We get (a_n, v_1) adjacent to (a_0, v_2); (a_n, v_2) adjacent to (a_0, v_3); (a_n, v_3) adjacent to (a_0, v_4) and so on (a_n, v_{p-1}) adjacent to (a_0, v_p) (by the definition of $\Delta_T(G)[H]$). Also, (a_n, v_p) adjacent to (a_0, v_1). In this way, we get a Hamiltonian cycle for $\Delta_T(G)[H]$. Hence, $\Delta_T(G)[H]$ is Hamiltonian.

Conversely, let $\Delta_T(G)[H]$ be a Hamiltonian graph. This follows that there exists a Hamlitonian cycle,

$$C = (c_0, c_1, c_2, \dots c_x = c_0)$$

such that $c_0 = (a_0, v_1), c_1 = (a_1, v_1), \dots c_n = (a_n, v_1), c_{n+1} = (a_1, v_2), \dots c_{2n} = (a_n, v_2), \dots$ $c_{x-n} - (a_0, v_p), \dots c_{x-1} = (a_n, v_p)$,
where $a_i's$ are elements of G and $v_i \in V(H)$.
We get (a_n, v_1) is adjacent to (a_0, v_2), (a_n, v_2) is adjacent to (a_0, v_3), and so on (a_n, v_{p-1}) is adjacent to (a_0, v_p), also (a_n, v_p) is adjacent to (a_0, v_1) (by definition of $\Delta_T(G)[H]$). This gives us an ordering $a_0, a_1, a_2, \dots a_{n+1} = a_0$ of elements of G in such a way that consecutive elements are neighbour as are the first and last elements, but the two edges are not consecutive elements, if both the edges are not of same triangle. So by Lemma 4.2, $\Delta_T(G)$ is Hamiltonian. Hence, the theorem. □

5. Conclusions

In this paper, we derive Eulerian and Hamiltonian properties of Γ-composition and Δ-composition of graphs.

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