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PICTURE FUZZY IDEALS IN NEAR RINGS UNDER GROUP ACTIONS

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ABSTRACT. A useful technique for examining the symmetry and automorphism characteristics of near rings is group action. In this paper, we study picture fuzzy (PF) ideals of near rings, which are extension of both fuzzy and intuitionistic fuzzy ideals, under group action. Also, we study properties of picture fuzzy ideals under $\mathfrak{G}$-homomorphism. In order to develop an innovative structure for picture fuzzy sets on near rings under group actions, we concentrate on merging the theories of picture fuzzy sets on near rings. The purpose of this manuscript is to use picture fuzzy sets to deal with various theories in near rings. We then provide appropriate definitions for the operations of picture fuzzy ideals over a near ring, including product, composition and intersection as well as study properties of images and inverse images of picture fuzzy ideals under group actions.

Keywords: Picture fuzzy ideals, Group action, $\mathfrak{G}$-homomorphism.

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1. Introduction

In order to counteract ambiguity in daily life, Zadeh [1] extended the idea of classical set theory by introducing the fuzzy set. Many direct and indirect generalizations of the fuzzy set have been developed and effectively used in the majority of real-world problems. Pattern recognition, decision-making issues, clustering analysis, and medical diagnostics are just a few of the real-world applications where the fuzzy set theory has been researched. Unfortunately, the failure of the fuzzy set theory was caused by inadequate knowledge regarding the function's negative membership degree. In order to solve this issue, Atanassov [2] gave the concept of intuitionistic fuzzy (hereinafter referred to as IF) set which included the negative membership degree of the function in fuzzy set theory in such a way that sum of the positive membership degree and negative membership degree must not exceed by 1. Furthermore, in [3, 4], Atanassov defined some new operations on IF set and studied their properties. There are several issues that cannot be represented by IF set theory in real life. For instance, in a voting scenario, human opinions may include additional responses of the following options: yes, no, abstain, and refusal. In order to fill these gaps, Cuong and Kreinovich [5] added the neutral function to the IFS theory in 2013. In 2014, Cuong [6]

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examined certain picture fuzzy (hereinafter referred to as PF) set features and proposed PF set distance measurements. In [7], Extension principle for PF set has been defined, and some of its characteristics have been studied as well as examples have been used to demonstrate picture fuzzy arithmetic based on the extension principle. Parimal et al. [8] defined PF ideals on near rings and studied properties of PF ideals with examples. The intersection, union, algebraic sum, complement, scalar multiplication, algebraic product and exponentiation operations are some of the algebraic operations which were studied in [9]. In 2022, Zhang et al. [10] formalised the connection of PF set operations with those on fuzzy sets and IF sets by extending more fundamental operations from classical sets to PF sets. They also reexamined the features of PF relations, picture fuzzy rough sets and PF soft sets and developed new explicit formulations for the Zadeh's Extension Principle for PF sets using these new operations. In 2023, Ali et al. [11] introduced and studied the group action on fuzzy ideals of a near ring .

In this paper we define group action on a PF ideals of near ring $\mathfrak{N}$ and study properties of intersection, direct product of picture fuzzy ideals under group actions. We also extend some results of [?] in the setting of picture fuzzy ideals.

2. Preliminaries

Definition 2.1. *A picture fuzzy (PF) set $\mathfrak{A}$ on a nonempty set $\mathfrak{S}$ is defined as*

$$\mathfrak{A} = \{(s, \zeta_{\mathfrak{A}}(s), \tau_{\mathfrak{A}}(s), \chi_{\mathfrak{A}}(s))\},$$

where $\zeta_{\mathfrak{A}}, \tau_{\mathfrak{A}}, \chi_{\mathfrak{A}} : \mathfrak{S} \to [0,1]$ are called positive, abstinence and negative membership functions, respectively. $\zeta_{\mathfrak{A}}, \tau_{\mathfrak{A}}$ and $\chi_{\mathfrak{A}}$ satisfy $0 \leq \zeta_{\mathfrak{A}}(x) + \tau_{\mathfrak{A}}(s) + \chi_{\mathfrak{A}}(s) \leq 1$. Moreover, $1 - (\zeta_{\mathfrak{A}}(x) + \tau_{\mathfrak{A}}(s) + \chi_{\mathfrak{A}}(s))$ is known as degree of refusal.

Definition 2.2. *Let $\mathfrak{G}$ be a group acts on a near ring $\mathfrak{N}$. Then action of $\mathfrak{G}$ on PF set $\mathfrak{A}$ is as follows:*

$$\mathfrak{A}^{\mathfrak{g}} = \{(\mathfrak{s}, \zeta_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}), \tau_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}), \chi_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}))\},$$

where $\zeta_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}) = \zeta_{\mathfrak{A}}(\mathfrak{s}^{\mathfrak{g}}), \tau_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}) = \tau_{\mathfrak{A}}(\mathfrak{s}^{\mathfrak{g}})$ and $\chi_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}) = \chi_{\mathfrak{A}}(\mathfrak{s}^{\mathfrak{g}})$.

Definition 2.3. [5] *Let $\mathfrak{P}$ and $\mathfrak{Q}$ be PF sets of $\mathfrak{N}$. Then union of $\mathfrak{P}$ and $\mathfrak{Q}$ is defined as*

$$\mathfrak{P} \cup \mathfrak{Q} = \{(\mathfrak{s}, \zeta_{\mathfrak{P}\cup\mathfrak{Q}}(\mathfrak{s}), \tau_{\mathfrak{P}\cup\mathfrak{Q}}(\mathfrak{s}), \chi_{\mathfrak{P}\cup\mathfrak{Q}}(\mathfrak{s}))\},$$

where

$$\zeta_{\mathfrak{P}\cup\mathfrak{Q}}(\mathfrak{s}) = \zeta_{\mathfrak{P}}(\mathfrak{s}) \vee \zeta_{\mathfrak{Q}}(\mathfrak{s}),$$
$$\tau_{\mathfrak{P}\cup\mathfrak{Q}}(\mathfrak{s}) = \tau_{\mathfrak{P}}(\mathfrak{s}) \wedge \tau_{\mathfrak{Q}}(\mathfrak{s}),$$
$$\chi_{\mathfrak{P}\cup\mathfrak{Q}}(\mathfrak{s}) = \chi_{\mathfrak{P}}(\mathfrak{s}) \wedge \chi_{\mathfrak{Q}}(\mathfrak{s}),$$

for all $\mathfrak{s} \in \mathfrak{N}$.
Intersection of $\mathfrak{P}$ and $\mathfrak{Q}$ is defined as

$$\mathfrak{P} \cap \mathfrak{Q} = \{(\mathfrak{s}, \zeta_{\mathfrak{P}\cap\mathfrak{Q}}(\mathfrak{s}), \tau_{\mathfrak{P}\cap\mathfrak{Q}}(\mathfrak{s}), \chi_{\mathfrak{P}\cap\mathfrak{Q}}(\mathfrak{s}))\},$$

where

$$\zeta_{\mathfrak{P}\cap\mathfrak{Q}}(\mathfrak{s}) = \zeta_{\mathfrak{P}}(\mathfrak{s}) \wedge \zeta_{\mathfrak{Q}}(\mathfrak{s}),$$
$$\tau_{\mathfrak{P}\cap\mathfrak{Q}}(\mathfrak{s}) = \tau_{\mathfrak{P}}(\mathfrak{s}) \wedge \tau_{\mathfrak{Q}}(\mathfrak{s}),$$
$$\chi_{\mathfrak{P}\cap\mathfrak{Q}}(\mathfrak{s}) = \chi_{\mathfrak{P}}(\mathfrak{s}) \vee \chi_{\mathfrak{Q}}(\mathfrak{s}).$$

3. Picture Fuzzy Ideals

Definition 3.1. *A PF set $\mathfrak{I}$ is said to be PF ideal of $\mathfrak{N}$ if*

(i) $\zeta_{\mathfrak{I}}(\mathfrak{s}_1-\mathfrak{s}_2) \geq \zeta_{\mathfrak{I}}(\mathfrak{s}_1)\wedge\zeta_{\mathfrak{I}}(\mathfrak{s}_2)$, $\tau_{\mathfrak{I}}(\mathfrak{s}_1-\mathfrak{s}_2) \leq \tau_{\mathfrak{I}}(\mathfrak{s}_1)\vee\tau_{\mathfrak{I}}(\mathfrak{s}_2)$, $\chi_{\mathfrak{I}}(\mathfrak{s}_1-\mathfrak{s}_2) \leq \chi_{\mathfrak{I}}(\mathfrak{s}_1)\vee\chi_{\mathfrak{I}}(\mathfrak{s}_2)$,

(ii) $\zeta_{\mathfrak{I}}(\mathfrak{s}_1+\mathfrak{s}_2-\mathfrak{s}_1) \geq \zeta_{\mathfrak{I}}(\mathfrak{s}_2)$, $\tau_{\mathfrak{I}}(\mathfrak{s}_1+\mathfrak{s}_2-\mathfrak{s}_1) \leq \tau_{\mathfrak{I}}(\mathfrak{s}_2)$, $\chi_{\mathfrak{I}}(\mathfrak{s}_1+\mathfrak{s}_2-\mathfrak{s}_1) \leq \chi_{\mathfrak{I}}(\mathfrak{s}_2)$,

(iii) $\zeta_{\mathfrak{I}}(\mathfrak{s}_1\mathfrak{s}_2) \geq \zeta_{\mathfrak{I}}(\mathfrak{s}_2)$, $\tau_{\mathfrak{I}}(\mathfrak{s}_1\mathfrak{s}_2) \leq \tau_{\mathfrak{I}}(\mathfrak{s}_2)$, $\chi_{\mathfrak{I}}(\mathfrak{s}_1\mathfrak{s}_2) \leq \chi_{\mathfrak{I}}(\mathfrak{s}_2)$,

(iv) $\zeta_{\mathfrak{I}}((\mathfrak{s}_1+\mathfrak{s}_2)\mathfrak{s}_3-\mathfrak{s}_1\mathfrak{s}_3) \geq \zeta_{\mathfrak{I}}(\mathfrak{s}_2)$, $\tau_{\mathfrak{I}}((\mathfrak{s}_1+\mathfrak{s}_2)\mathfrak{s}_3-\mathfrak{s}_1\mathfrak{s}_3) \leq \tau_{\mathfrak{I}}(\mathfrak{s}_2)$, $\chi_{\mathfrak{I}}((\mathfrak{s}_1+\mathfrak{s}_2)\mathfrak{s}_3-\mathfrak{s}_1\mathfrak{s}_3) \leq \chi_{\mathfrak{I}}(\mathfrak{s}_2)$,

where $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_2 \in \mathfrak{N}$.

Example 3.1. *Let $\mathbb{Z}$ be a near ring under usual addition and multiplication defined as $\mathfrak{m}\mathfrak{n} = \mathfrak{n}$ for all $\mathfrak{m}, \mathfrak{n} \in \mathbb{Z}$. We define a picture fuzzy set $\mathfrak{A} = (\zeta_{\mathfrak{A}}, \tau_{\mathfrak{A}}, \chi_{\mathfrak{A}})$ as follows: For all $\mathfrak{s} \in \mathfrak{N}$,*

$$\zeta_{\mathfrak{A}}(\mathfrak{s}) = \begin{cases} 0.8 & \text{if } \mathfrak{s} \in 4\mathbb{Z} \\ 0.2 & \text{otherwise,} \end{cases} \quad \tau_{\mathfrak{A}}(\mathfrak{s}) = \begin{cases} 0.1 & \text{if } \mathfrak{s} \in 4\mathbb{Z} \\ 0.35 & \text{otherwise,} \end{cases} \quad \chi_{\mathfrak{A}}(\mathfrak{s}) = \begin{cases} 0.05 & \text{if } \mathfrak{s} \in 4\mathbb{Z} \\ 0.4 & \text{otherwise.} \end{cases}$$

It can be easily seen that $\mathfrak{A}$ is a picture fuzzy ideal of $\mathfrak{N}$.

Proposition 3.1. *If $\mathfrak{A}$ is a PF ideal of $\mathfrak{N}$, then $\mathfrak{A}^{\mathfrak{g}}$ is also a PF ideal of $\mathfrak{N}$.*

Proof. Let $\mathfrak{A} = \{(\mathfrak{s}, \zeta_{\mathfrak{A}}(\mathfrak{s}), \tau_{\mathfrak{A}}(\mathfrak{s}), \chi_{\mathfrak{A}}(\mathfrak{s}))\}$ be a PF ideal of $\mathfrak{N}$. Then for $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3 \in \mathfrak{N}$,

(i)

$$\begin{aligned} \zeta_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}_1-\mathfrak{s}_2) = \zeta_{\mathfrak{A}}(\mathfrak{s}_1-\mathfrak{s}_2)^{\mathfrak{g}} &= \zeta_{\mathfrak{A}}({\mathfrak{s}_1}^{\mathfrak{g}}-{\mathfrak{s}_2}^{\mathfrak{g}}) \\ &\geq \zeta_{\mathfrak{A}}(\mathfrak{s}_1)^{\mathfrak{g}} \wedge \zeta_{\mathfrak{A}}(\mathfrak{s}_2)^{\mathfrak{g}} \\ &= \zeta_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}_1) \wedge \zeta_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}_2), \end{aligned} \tag{1}$$

$$\begin{aligned} \tau_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}_1-\mathfrak{s}_2) = \tau_{\mathfrak{A}}(\mathfrak{s}_1-\mathfrak{s}_2)^{\mathfrak{g}} &= \tau_{\mathfrak{A}}({\mathfrak{s}_1}^{\mathfrak{g}}-{\mathfrak{s}_2}^{\mathfrak{g}}) \\ &\leq \tau_{\mathfrak{A}}(\mathfrak{s}_1)^{\mathfrak{g}} \wedge \tau_{\mathfrak{A}}(\mathfrak{s}_2)^{\mathfrak{g}} \\ &= \tau_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}_1) \wedge \tau_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}_2). \end{aligned} \tag{2}$$

In the similar manner,

$$\chi_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}_1-\mathfrak{s}_2) \leq \chi_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}_1) \wedge \chi_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}_2). \tag{3}$$

(ii)

$$\begin{aligned} \zeta_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}_1+\mathfrak{s}_2-\mathfrak{s}_1) = \zeta_{\mathfrak{A}}(\mathfrak{s}_1+\mathfrak{s}_2-\mathfrak{s}_1)^{\mathfrak{g}} &= \zeta_{\mathfrak{A}}({\mathfrak{s}_1}^{\mathfrak{g}}+{\mathfrak{s}_2}^{\mathfrak{g}}-{\mathfrak{s}_1}^{\mathfrak{g}}) \\ &\geq \zeta_{\mathfrak{A}}(\mathfrak{s}_2)^{\mathfrak{g}} \\ &= \zeta_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}_2), \end{aligned} \tag{4}$$

$$\begin{aligned} \tau_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}_1+\mathfrak{s}_2-\mathfrak{s}_1) = \tau_{\mathfrak{A}}(\mathfrak{s}_1+\mathfrak{s}_2-\mathfrak{s}_1)^{\mathfrak{g}} &= \tau_{\mathfrak{A}}({\mathfrak{s}_1}^{\mathfrak{g}}+{\mathfrak{s}_2}^{\mathfrak{g}}-{\mathfrak{s}_1}^{\mathfrak{g}}) \\ &\leq \tau_{\mathfrak{A}}(\mathfrak{s}_2)^{\mathfrak{g}} \\ &= \tau_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}_2). \end{aligned} \tag{5}$$

Similarly,

$$\chi_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}_1+\mathfrak{s}_2-\mathfrak{s}_1) \leq \chi_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}_2). \tag{6}$$

(iii)

$$\begin{aligned} \zeta_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}_1\mathfrak{s}_2) = \zeta_{\mathfrak{A}}(\mathfrak{s}_1\mathfrak{s}_2)^{\mathfrak{g}} &= \zeta_{\mathfrak{A}}({\mathfrak{s}_1}^{\mathfrak{g}}{\mathfrak{s}_2}^{\mathfrak{g}}) \\ &\geq \zeta_{\mathfrak{A}}(\mathfrak{s}_2)^{\mathfrak{g}} \\ &= \zeta_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}_2), \end{aligned} \tag{7}$$

$$\begin{aligned}\tau_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}_1\mathfrak{s}_2) = \tau_{\mathfrak{A}}(\mathfrak{s}_1\mathfrak{s}_2)^{\mathfrak{g}} &= \tau_{\mathfrak{A}}(\mathfrak{s}_1{}^{\mathfrak{g}}\mathfrak{s}_2{}^{\mathfrak{g}})\\ &\leq \tau_{\mathfrak{A}}(\mathfrak{s}_2)^{\mathfrak{g}}\\ &= \tau_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}_2).\end{aligned} \tag{8}$$

Likewise equation (8), we have

$$\chi_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}_1\mathfrak{s}_2) \leq \chi_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}_2). \tag{9}$$

(iv)

$$\begin{aligned}\zeta_{\mathfrak{A}^{\mathfrak{g}}}((\mathfrak{s}_1+\mathfrak{s}_2)\mathfrak{s}_3-\mathfrak{s}_1\mathfrak{s}_3) &= \zeta_{\mathfrak{A}}((\mathfrak{s}_1+\mathfrak{s}_2)\mathfrak{s}_3-\mathfrak{s}_1\mathfrak{s}_3)^{\mathfrak{g}}\\ &= \zeta_{\mathfrak{A}}((\mathfrak{s}_1{}^{\mathfrak{g}}+\mathfrak{s}_2{}^{\mathfrak{g}})\mathfrak{s}_3{}^{\mathfrak{g}}-\mathfrak{s}_1{}^{\mathfrak{g}}\mathfrak{s}_3{}^{\mathfrak{g}})\\ &\geq \zeta_{\mathfrak{A}}(\mathfrak{s}_2)^{\mathfrak{g}}\\ &= \zeta_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}_2),\end{aligned} \tag{10}$$

$$\begin{aligned}\tau_{\mathfrak{A}^{\mathfrak{g}}}((\mathfrak{s}_1+\mathfrak{s}_2)\mathfrak{s}_3-\mathfrak{s}_1\mathfrak{s}_3) &= \tau_{\mathfrak{A}}((\mathfrak{s}_1+\mathfrak{s}_2)\mathfrak{s}_3-\mathfrak{s}_1\mathfrak{s}_3)^{\mathfrak{g}}\\ &= \tau_{\mathfrak{A}}((\mathfrak{s}_1{}^{\mathfrak{g}}+\mathfrak{s}_2{}^{\mathfrak{g}})\mathfrak{s}_3{}^{\mathfrak{g}}-\mathfrak{s}_1{}^{\mathfrak{g}}\mathfrak{s}_3{}^{\mathfrak{g}})\\ &\leq \tau_{\mathfrak{A}}(\mathfrak{s}_2)^{\mathfrak{g}}\\ &= \tau_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}_2).\end{aligned} \tag{11}$$

Analogously,

$$\chi_{\mathfrak{A}^{\mathfrak{g}}}((\mathfrak{s}_1+\mathfrak{s}_2)\mathfrak{s}_3-\mathfrak{s}_1\mathfrak{s}_3) \leq \chi_{\mathfrak{A}^{\mathfrak{g}}}(\mathfrak{s}_2). \tag{12}$$

Equations (1)-(12) imply that $\mathfrak{A}^{\mathfrak{g}}$ is a PF ideal. □

Theorem 3.1. *If $\mathfrak{P}$ and $\mathfrak{Q}$ are any PF ideals of $\mathfrak{N}$, then $(\mathfrak{P}\cap\mathfrak{Q})^{\mathfrak{g}}$ is a PF ideal of $\mathfrak{N}$.*

Proof. Let $\mathfrak{P}$ and $\mathfrak{Q}$ be two PF ideals of $\mathfrak{N}$. Then for $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3 \in \mathfrak{N}$,

(i)

$$\begin{aligned}\zeta_{(\mathfrak{P}\cap\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{s}_1-\mathfrak{s}_2) = \zeta_{\mathfrak{P}\cap\mathfrak{Q}}(\mathfrak{s}_1-\mathfrak{s}_2)^{\mathfrak{g}} &= \zeta_{\mathfrak{P}\cap\mathfrak{Q}}(\mathfrak{s}_1{}^{\mathfrak{g}}-\mathfrak{s}_2{}^{\mathfrak{g}})\\ &= \zeta_{\mathfrak{P}}(\mathfrak{s}_1{}^{\mathfrak{g}}-\mathfrak{s}_2{}^{\mathfrak{g}}) \wedge \zeta_{\mathfrak{Q}}(\mathfrak{s}_1{}^{\mathfrak{g}}-\mathfrak{s}_2{}^{\mathfrak{g}})\\ &\geq \{\zeta_{\mathfrak{P}}(\mathfrak{s}_1)^{\mathfrak{g}} \wedge \zeta_{\mathfrak{P}}(\mathfrak{s}_2)^{\mathfrak{g}}\} \wedge \{\zeta_{\mathfrak{Q}}(\mathfrak{s}_1)^{\mathfrak{g}} \wedge \zeta_{\mathfrak{Q}}(\mathfrak{s}_2)^{\mathfrak{g}}\}\\ &= \{\zeta_{\mathfrak{P}}(\mathfrak{s}_1)^{\mathfrak{g}} \wedge \zeta_{\mathfrak{Q}}(\mathfrak{s}_1)^{\mathfrak{g}}\} \wedge \{\zeta_{\mathfrak{P}}(\mathfrak{s}_2)^{\mathfrak{g}} \wedge \zeta_{\mathfrak{Q}}(\mathfrak{s}_2)^{\mathfrak{g}}\}\\ &= \zeta_{\mathfrak{P}\cap\mathfrak{Q}}(\mathfrak{s}_1)^{\mathfrak{g}} \wedge \zeta_{\mathfrak{P}\cap\mathfrak{Q}}(\mathfrak{s}_2)^{\mathfrak{g}}\\ &= \zeta_{(\mathfrak{P}\cap\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{s}_1) \wedge \zeta_{(\mathfrak{P}\cap\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{s}_2),\end{aligned} \tag{13}$$

$$\begin{aligned}\tau_{(\mathfrak{P}\cap\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{s}_1-\mathfrak{s}_2) = \tau_{\mathfrak{P}\cap\mathfrak{Q}}(\mathfrak{s}_1-\mathfrak{s}_2)^{\mathfrak{g}} &= \tau_{\mathfrak{P}\cap\mathfrak{Q}}(\mathfrak{s}_1{}^{\mathfrak{g}}-\mathfrak{s}_2{}^{\mathfrak{g}})\\ &= \tau_{\mathfrak{P}}(\mathfrak{s}_1{}^{\mathfrak{g}}-\mathfrak{s}_2{}^{\mathfrak{g}}) \vee \tau_{\mathfrak{Q}}(\mathfrak{s}_1{}^{\mathfrak{g}}-\mathfrak{s}_2{}^{\mathfrak{g}})\\ &\leq \{\tau_{\mathfrak{P}}(\mathfrak{s}_1)^{\mathfrak{g}} \vee \tau_{\mathfrak{P}}(\mathfrak{s}_2)^{\mathfrak{g}}\} \vee \{\tau_{\mathfrak{Q}}(\mathfrak{s}_1)^{\mathfrak{g}} \vee \tau_{\mathfrak{Q}}(\mathfrak{s}_2)^{\mathfrak{g}}\}\\ &= \{\tau_{\mathfrak{P}}(\mathfrak{s}_1)^{\mathfrak{g}} \vee \tau_{\mathfrak{Q}}(\mathfrak{s}_1)^{\mathfrak{g}}\} \vee \{\tau_{\mathfrak{P}}(\mathfrak{s}_2)^{\mathfrak{g}} \vee \tau_{\mathfrak{Q}}(\mathfrak{s}_2)^{\mathfrak{g}}\}\\ &= \tau_{\mathfrak{P}\cap\mathfrak{Q}}(\mathfrak{s}_1)^{\mathfrak{g}} \vee \tau_{\mathfrak{P}\cap\mathfrak{Q}}(\mathfrak{s}_2)^{\mathfrak{g}}\\ &= \tau_{(\mathfrak{P}\cap\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{s}_1) \vee \tau_{(\mathfrak{P}\cap\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{s}_2).\end{aligned} \tag{14}$$

As proof of $\tau_{(\mathfrak{P}\cap\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{s}_1-\mathfrak{s}_2) \leq \tau_{(\mathfrak{P}\cap\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{s}_1) \vee \tau_{(\mathfrak{P}\cap\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{s}_2)$, we get

$$\chi_{(\mathfrak{P}\cap\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{s}_1-\mathfrak{s}_2) \leq \chi_{(\mathfrak{P}\cap\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{s}_1) \vee \chi_{(\mathfrak{P}\cap\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{s}_2). \tag{15}$$

(ii)

$$\begin{aligned}\zeta_{(\mathfrak{P}\cap\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{s}_1+\mathfrak{s}_2-\mathfrak{s}_1) = \zeta_{\mathfrak{P}\cap\mathfrak{Q}}(\mathfrak{s}_1+\mathfrak{s}_2-\mathfrak{s}_1)^{\mathfrak{g}} &= \zeta_{\mathfrak{P}\cap\mathfrak{Q}}(\mathfrak{s}_1{}^{\mathfrak{g}}+\mathfrak{s}_2{}^{\mathfrak{g}}-\mathfrak{s}_1{}^{\mathfrak{g}})\\ &= \zeta_{\mathfrak{P}}(\mathfrak{s}_1{}^{\mathfrak{g}}+\mathfrak{s}_2{}^{\mathfrak{g}}-\mathfrak{s}_1{}^{\mathfrak{g}}) \wedge \zeta_{\mathfrak{Q}}(\mathfrak{s}_1{}^{\mathfrak{g}}+\mathfrak{s}_2{}^{\mathfrak{g}}\\ &\quad -\mathfrak{s}_1{}^{\mathfrak{g}})\\ &\geq \zeta_{\mathfrak{P}}(\mathfrak{s}_2)^{\mathfrak{g}} \wedge \zeta_{\mathfrak{Q}}(\mathfrak{s}_2)^{\mathfrak{g}}\\ &= \zeta_{(\mathfrak{P}\cap\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{s}_2), \end{aligned} \tag{16}$$

$$\begin{aligned}\tau_{(\mathfrak{P}\cap\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{s}_1+\mathfrak{s}_2-\mathfrak{s}_1) = \tau_{\mathfrak{P}\cap\mathfrak{Q}}(\mathfrak{s}_1+\mathfrak{s}_2-\mathfrak{s}_1)^{\mathfrak{g}} &= \tau_{\mathfrak{P}\cap\mathfrak{Q}}(\mathfrak{s}_1{}^{\mathfrak{g}}+\mathfrak{s}_2{}^{\mathfrak{g}}-\mathfrak{s}_1{}^{\mathfrak{g}})\\ &= \tau_{\mathfrak{P}}(\mathfrak{s}_1{}^{\mathfrak{g}}+\mathfrak{s}_2{}^{\mathfrak{g}}-\mathfrak{s}_1{}^{\mathfrak{g}}) \wedge \tau_{\mathfrak{Q}}(\mathfrak{s}_1{}^{\mathfrak{g}}+\mathfrak{s}_2{}^{\mathfrak{g}}\\ &\quad -\mathfrak{s}_1{}^{\mathfrak{g}})\\ &\leq \tau_{\mathfrak{P}}(\mathfrak{s}_2)^{\mathfrak{g}} \wedge \tau_{\mathfrak{Q}}(\mathfrak{s}_2)^{\mathfrak{g}}\\ &= \tau_{(\mathfrak{P}\cap\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{s}_2). \end{aligned} \tag{17}$$

Also, we can easily see that

$$\chi_{(\mathfrak{P}\cap\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{s}_1+\mathfrak{s}_2-\mathfrak{s}_1) \leq \chi_{(\mathfrak{P}\cap\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{s}_2), \tag{18}$$

(iii)

$$\begin{aligned}\zeta_{(\mathfrak{P}\cap\mathfrak{Q})^{\mathfrak{g}}}((\mathfrak{s}_1+\mathfrak{s}_2)\mathfrak{s}_3-\mathfrak{s}_1\mathfrak{s}_3) &= \zeta_{\mathfrak{P}\cap\mathfrak{Q}}((\mathfrak{s}_1+\mathfrak{s}_2)\mathfrak{s}_3-\mathfrak{s}_1\mathfrak{s}_3)^{\mathfrak{g}}\\ &= \zeta_{\mathfrak{P}\cap\mathfrak{Q}}((\mathfrak{s}_1{}^{\mathfrak{g}}+\mathfrak{s}_2{}^{\mathfrak{g}})\mathfrak{s}_3{}^{\mathfrak{g}}-\mathfrak{s}_1{}^{\mathfrak{g}}\mathfrak{s}_3{}^{\mathfrak{g}})\\ &= \zeta_{\mathfrak{P}}((\mathfrak{s}_1{}^{\mathfrak{g}}+\mathfrak{s}_2{}^{\mathfrak{g}})\mathfrak{s}_3{}^{\mathfrak{g}}-\mathfrak{s}_1{}^{\mathfrak{g}}\mathfrak{s}_3{}^{\mathfrak{g}}) \wedge \zeta_{\mathfrak{Q}}((\mathfrak{s}_1{}^{\mathfrak{g}}+\mathfrak{s}_2{}^{\mathfrak{g}})\mathfrak{s}_3{}^{\mathfrak{g}}-\mathfrak{s}_1{}^{\mathfrak{g}}\mathfrak{s}_3{}^{\mathfrak{g}})\\ &\geq \zeta_{\mathfrak{P}}(\mathfrak{s}_2)^{\mathfrak{g}} \wedge \zeta_{\mathfrak{Q}}(\mathfrak{s}_2)^{\mathfrak{g}}\\ &= \zeta_{(\mathfrak{P}\cap\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{s}_2), \end{aligned} \tag{19}$$

$$\begin{aligned}\tau_{(\mathfrak{P}\cap\mathfrak{Q})^{\mathfrak{g}}}((\mathfrak{s}_1+\mathfrak{s}_2)\mathfrak{s}_3-\mathfrak{s}_1\mathfrak{s}_3) &= \tau_{\mathfrak{P}\cap\mathfrak{Q}}((\mathfrak{s}_1+\mathfrak{s}_2)\mathfrak{s}_3-\mathfrak{s}_1\mathfrak{s}_3)^{\mathfrak{g}}\\ &= \tau_{\mathfrak{P}\cap\mathfrak{Q}}((\mathfrak{s}_1{}^{\mathfrak{g}}+\mathfrak{s}_2{}^{\mathfrak{g}})\mathfrak{s}_3{}^{\mathfrak{g}}-\mathfrak{s}_1{}^{\mathfrak{g}}\mathfrak{s}_3{}^{\mathfrak{g}})\\ &= \tau_{\mathfrak{P}}((\mathfrak{s}_1{}^{\mathfrak{g}}+\mathfrak{s}_2{}^{\mathfrak{g}})\mathfrak{s}_3{}^{\mathfrak{g}}-\mathfrak{s}_1{}^{\mathfrak{g}}\mathfrak{s}_3{}^{\mathfrak{g}}) \wedge \tau_{\mathfrak{Q}}((\mathfrak{s}_1{}^{\mathfrak{g}}+\mathfrak{s}_2{}^{\mathfrak{g}})\mathfrak{s}_3{}^{\mathfrak{g}}-\mathfrak{s}_1{}^{\mathfrak{g}}\mathfrak{s}_3{}^{\mathfrak{g}})\\ &\leq \tau_{\mathfrak{P}}(\mathfrak{s}_2)^{\mathfrak{g}} \wedge \tau_{\mathfrak{Q}}(\mathfrak{s}_2)^{\mathfrak{g}}\\ &= \tau_{(\mathfrak{P}\cap\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{s}_2). \end{aligned} \tag{20}$$

It is easy to see that

$$\chi_{(\mathfrak{P}\cap\mathfrak{Q})^{\mathfrak{g}}}((\mathfrak{s}_1+\mathfrak{s}_2)\mathfrak{s}_3-\mathfrak{s}_1\mathfrak{s}_3) \leq \chi_{(\mathfrak{P}\cap\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{s}_2). \tag{21}$$

Equations (13)-(21) imply that $(\mathfrak{P}\cap\mathfrak{Q})^{\mathfrak{g}}$ is a PF ideal. □

Definition 3.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be PF subsets of near rings $\mathfrak{N}_1$ and $\mathfrak{N}_2$, respectively. Then, direct product $\mathfrak{A}\times\mathfrak{B}:\mathfrak{N}_1\times\mathfrak{N}_2\to[0,1]$ is defined by*

$$\mathfrak{A}\times\mathfrak{B} = \{\langle(\mathfrak{a},\mathfrak{b}), \zeta_{(\mathfrak{A}\times\mathfrak{B})}(\mathfrak{a},\mathfrak{b}), \tau_{(\mathfrak{A}\times\mathfrak{B})}(\mathfrak{a},\mathfrak{b}), \chi_{(\mathfrak{A}\times\mathfrak{B})}(\mathfrak{a},\mathfrak{b})\rangle : \mathfrak{a}\in\mathfrak{A}, \mathfrak{b}\in\mathfrak{B}\},$$

where

$$\begin{aligned}\zeta_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{a},\mathfrak{b}) &= \zeta_{\mathfrak{A}}(\mathfrak{a}) \wedge \zeta_{\mathfrak{B}}(\mathfrak{b}),\\ \tau_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{a},\mathfrak{b}) &= \tau_{\mathfrak{A}}(\mathfrak{a}) \vee \tau_{\mathfrak{B}}(\mathfrak{b}),\\ \chi_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{a},\mathfrak{b}) &= \chi_{\mathfrak{A}}(\mathfrak{a}) \vee \chi_{\mathfrak{B}}(\mathfrak{b}).\end{aligned}$$

Definition 3.3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be PF ideals of near rings $\mathfrak{N}_1$ and $\mathfrak{N}_2$, respectively. Then, $\mathfrak{A}\times\mathfrak{B}$ is PF ideal if for $\mathfrak{a}_1,\mathfrak{a}_2\in\mathfrak{A}$ and $\mathfrak{b}_1,\mathfrak{b}_2\in\mathfrak{B}$ following conditions hold:*

$$\begin{aligned}
\zeta_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{a}_1,\mathfrak{a}_2)-(\mathfrak{b}_1,\mathfrak{b}_2)) &\geq \zeta_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{a}_1,\mathfrak{a}_2)\wedge\zeta_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{b}_1,\mathfrak{b}_2)\\
\tau_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{a}_1,\mathfrak{a}_2)-(\mathfrak{b}_1,\mathfrak{b}_2)) &\leq \tau_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{a}_1,\mathfrak{a}_2)\vee\tau_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{b}_1,\mathfrak{b}_2)\\
\chi_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{a}_1,\mathfrak{a}_2)-(\mathfrak{b}_1,\mathfrak{b}_2)) &\leq \chi_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{a}_1,\mathfrak{a}_2)\vee\chi_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{b}_1,\mathfrak{b}_2)\\
\zeta_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{b}_1,\mathfrak{b}_2)+(\mathfrak{a}_1,\mathfrak{a}_2)-(\mathfrak{b}_1,\mathfrak{b}_2)) &\geq \wedge\zeta_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{a}_1,\mathfrak{a}_2)\\
\tau_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{b}_1,\mathfrak{b}_2)+(\mathfrak{a}_1,\mathfrak{a}_2)-(\mathfrak{b}_1,\mathfrak{b}_2)) &\leq \vee\tau_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{a}_1,\mathfrak{a}_2)\\
\chi_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{b}_1,\mathfrak{b}_2)+(\mathfrak{a}_1,\mathfrak{a}_2)-(\mathfrak{b}_1,\mathfrak{b}_2)) &\leq \vee\chi_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{a}_1,\mathfrak{a}_2) \qquad (22)\\
\zeta_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{a}_1,\mathfrak{a}_2)(\mathfrak{b}_1,\mathfrak{b}_2)) &\geq \zeta_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{b}_1,\mathfrak{b}_2)\\
\tau_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{a}_1,\mathfrak{a}_2)(\mathfrak{b}_1,\mathfrak{b}_2)) &\leq \tau_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{b}_1,\mathfrak{b}_2)\\
\chi_{\mathfrak{A}\times\mathfrak{B}}((\mathfrak{a}_1,\mathfrak{a}_2)(\mathfrak{b}_1,\mathfrak{b}_2)) &\leq \chi_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{b}_1,\mathfrak{b}_2)\\
\zeta_{\mathfrak{A}\times\mathfrak{B}}\{((\mathfrak{a}_1,\mathfrak{a}_2)+(\mathfrak{c}_1,\mathfrak{c}_2))(\mathfrak{b}_1,\mathfrak{b}_2)-(\mathfrak{a}_1,\mathfrak{a}_2)(\mathfrak{b}_1,\mathfrak{b}_2)\} &\geq \zeta_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{c}_1,\mathfrak{c}_2)\\
\tau_{\mathfrak{A}\times\mathfrak{B}}\{((\mathfrak{a}_1,\mathfrak{a}_2)+(\mathfrak{c}_1,\mathfrak{c}_2))(\mathfrak{b}_1,\mathfrak{b}_2)-(\mathfrak{a}_1,\mathfrak{a}_2)(\mathfrak{b}_1,\mathfrak{b}_2)\} &\leq \tau_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{c}_1,\mathfrak{c}_2)\\
\chi_{\mathfrak{A}\times\mathfrak{B}}\{((\mathfrak{a}_1,\mathfrak{a}_2)+(\mathfrak{c}_1,\mathfrak{c}_2))(\mathfrak{b}_1,\mathfrak{b}_2)-(\mathfrak{a}_1,\mathfrak{a}_2)(\mathfrak{b}_1,\mathfrak{b}_2)\} &\leq \chi_{\mathfrak{A}\times\mathfrak{B}}(\mathfrak{c}_1,\mathfrak{c}_2).
\end{aligned}$$

Theorem 3.2. *If $\mathfrak{P}$ and $\mathfrak{Q}$ are any PF ideals of $\mathfrak{N}_1$ and $\mathfrak{N}_2$, respectively, then $(\mathfrak{P}\times\mathfrak{Q})^{\mathfrak{g}}$ is a PF ideal of $\mathfrak{N}_1\times\mathfrak{N}_2$.*

Proof. Let $\mathfrak{P}$ and $\mathfrak{Q}$ be PF ideals of $\mathfrak{N}_1$ and $\mathfrak{N}_2$ respectively and $(\mathfrak{a}_1,\mathfrak{a}_2),(\mathfrak{b}_1,\mathfrak{b}_2),(\mathfrak{c}_1,\mathfrak{c}_2)\in\mathfrak{N}_1\times\mathfrak{N}_2$. Then

$$\begin{aligned}
\zeta_{(\mathfrak{P}\times\mathfrak{Q})^{\mathfrak{g}}}((\mathfrak{a}_1,\mathfrak{a}_2)-(\mathfrak{b}_1,\mathfrak{b}_2)) &= \zeta_{\mathfrak{P}\times\mathfrak{Q}}((\mathfrak{a}_1^{\mathfrak{g}}-\mathfrak{b}_1^{\mathfrak{g}}),(\mathfrak{a}_2^{\mathfrak{g}}-\mathfrak{b}_2^{\mathfrak{g}}))\\
&= \zeta_{\mathfrak{P}}(\mathfrak{a}_1^{\mathfrak{g}}-\mathfrak{b}_1^{\mathfrak{g}})\wedge\zeta_{\mathfrak{Q}}(\mathfrak{a}_2^{\mathfrak{g}}-\mathfrak{b}_2^{\mathfrak{g}})\\
&\geq \{(\zeta_{\mathfrak{P}}(\mathfrak{a}_1^{\mathfrak{g}})\wedge\zeta_{\mathfrak{P}}(\mathfrak{b}_1^{\mathfrak{g}}))\wedge(\zeta_{\mathfrak{Q}}(\mathfrak{a}_2^{\mathfrak{g}})\wedge\zeta_{\mathfrak{Q}}(\mathfrak{b}_2^{\mathfrak{g}}))\}\\
&= \zeta_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{a}_1^{\mathfrak{g}},\mathfrak{a}_2^{\mathfrak{g}})\wedge\zeta_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{b}_1^{\mathfrak{g}},\mathfrak{b}_2^{\mathfrak{g}})\\
&= \zeta_{(\mathfrak{P}\times\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{a}_1,\mathfrak{a}_2)\wedge\zeta_{(\mathfrak{P}\times\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{b}_1,\mathfrak{b}_2), \qquad (23)\\
\tau_{(\mathfrak{P}\times\mathfrak{Q})^{\mathfrak{g}}}((\mathfrak{a}_1,\mathfrak{a}_2)-(\mathfrak{b}_1,\mathfrak{b}_2)) &= \tau_{\mathfrak{P}\times\mathfrak{Q}}((\mathfrak{a}_1^{\mathfrak{g}}-\mathfrak{b}_1^{\mathfrak{g}}),(\mathfrak{a}_2^{\mathfrak{g}}-\mathfrak{b}_2^{\mathfrak{g}}))\\
&= \tau_{\mathfrak{P}}(\mathfrak{a}_1^{\mathfrak{g}}-\mathfrak{b}_1^{\mathfrak{g}})\vee\tau_{\mathfrak{Q}}(\mathfrak{a}_2^{\mathfrak{g}}-\mathfrak{b}_2^{\mathfrak{g}})\\
&\leq \{(\tau_{\mathfrak{P}}(\mathfrak{a}_1^{\mathfrak{g}})\vee\tau_{\mathfrak{P}}(\mathfrak{b}_1^{\mathfrak{g}}))\vee(\tau_{\mathfrak{Q}}(\mathfrak{a}_2^{\mathfrak{g}})\vee\tau_{\mathfrak{Q}}(\mathfrak{b}_2^{\mathfrak{g}}))\}\\
&= \tau_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{a}_1^{\mathfrak{g}},\mathfrak{a}_2^{\mathfrak{g}})\vee\tau_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{b}_1^{\mathfrak{g}},\mathfrak{b}_2^{\mathfrak{g}})\\
&= \tau_{(\mathfrak{P}\times\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{a}_1,\mathfrak{a}_2)\vee\tau_{(\mathfrak{P}\times\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{b}_1,\mathfrak{b}_2). \qquad (24)
\end{aligned}$$

Furthermore, we can easily get that

$$\chi_{(\mathfrak{P}\times\mathfrak{Q})^{\mathfrak{g}}}((\mathfrak{a}_1,\mathfrak{a}_2)-(\mathfrak{b}_1,\mathfrak{b}_2)) \leq \chi_{(\mathfrak{P}\times\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{a}_1,\mathfrak{a}_2)\vee\chi_{(\mathfrak{P}\times\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{b}_1,\mathfrak{b}_2). \qquad (25)$$

Also,

$$\begin{aligned}
\zeta_{(\mathfrak{P}\times\mathfrak{Q})^{\mathfrak{g}}}((\mathfrak{b}_1,\mathfrak{b}_2)+(\mathfrak{a}_1,\mathfrak{a}_2)-(\mathfrak{b}_1,\mathfrak{b}_2)) &= \zeta_{\mathfrak{P}\times\mathfrak{Q}}((\mathfrak{b}_1^{\mathfrak{g}}+\mathfrak{a}_1^{\mathfrak{g}}-\mathfrak{b}_1^{\mathfrak{g}}),(\mathfrak{b}_2^{\mathfrak{g}}+\mathfrak{a}_2^{\mathfrak{g}}-\mathfrak{b}_2^{\mathfrak{g}}))\\
&= \zeta_{\mathfrak{P}}(\mathfrak{b}_1^{\mathfrak{g}}+\mathfrak{a}_1^{\mathfrak{g}}-\mathfrak{b}_1^{\mathfrak{g}})\wedge\zeta_{\mathfrak{Q}}(\mathfrak{b}_2^{\mathfrak{g}}+\mathfrak{a}_2^{\mathfrak{g}}-\mathfrak{b}_2^{\mathfrak{g}})\\
&\geq \zeta_{\mathfrak{P}}(\mathfrak{a}_1^{\mathfrak{g}})\wedge\zeta_{\mathfrak{Q}}(\mathfrak{a}_2^{\mathfrak{g}})\\
&= \zeta_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{a}_1^{\mathfrak{g}},\mathfrak{a}_2^{\mathfrak{g}})\\
&= \zeta_{(\mathfrak{P}\times\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{a}_1,\mathfrak{a}_2), \qquad (26)
\end{aligned}$$

$$\begin{aligned}\tau_{(\mathfrak{P}\times\mathfrak{Q})^{\mathfrak{g}}}((\mathfrak{b}_1,\mathfrak{b}_2)+(\mathfrak{a}_1,\mathfrak{a}_2)-(\mathfrak{b}_1,\mathfrak{b}_2)) &= \tau_{\mathfrak{P}\times\mathfrak{Q}}((\mathfrak{b}_1^{\mathfrak{g}}+\mathfrak{a}_1^{\mathfrak{g}}-\mathfrak{b}_1^{\mathfrak{g}}),(\mathfrak{b}_2^{\mathfrak{g}}+\mathfrak{a}_2^{\mathfrak{g}}-\mathfrak{b}_2^{\mathfrak{g}}))\\ &= \tau_{\mathfrak{P}}(\mathfrak{b}_1^{\mathfrak{g}}+\mathfrak{a}_1^{\mathfrak{g}}-\mathfrak{b}_1^{\mathfrak{g}}) \vee \tau_{\mathfrak{Q}}(\mathfrak{b}_2^{\mathfrak{g}}+\mathfrak{a}_2^{\mathfrak{g}}-\mathfrak{b}_2^{\mathfrak{g}})\\ &\leq \tau_{\mathfrak{P}}(\mathfrak{a}_1^{\mathfrak{g}}) \vee \zeta_{\mathfrak{Q}}(\mathfrak{a}_2^{\mathfrak{g}})\\ &= \tau_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{a}_1^{\mathfrak{g}},\mathfrak{a}_2^{\mathfrak{g}})\\ &= \tau_{(\mathfrak{P}\times\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{a}_1,\mathfrak{a}_2). \qquad (27)\end{aligned}$$

Similarly as equation (27), we get

$$\chi_{(\mathfrak{P}\times\mathfrak{Q})^{\mathfrak{g}}}((\mathfrak{b}_1,\mathfrak{b}_2)+(\mathfrak{a}_1,\mathfrak{a}_2)-(\mathfrak{b}_1,\mathfrak{b}_2)) \leq \chi_{(\mathfrak{P}\times\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{a}_1,\mathfrak{a}_2). \qquad (28)$$

Moreover,

$$\begin{aligned}\zeta_{(\mathfrak{P}\times\mathfrak{Q})^{\mathfrak{g}}}((\mathfrak{a}_1,\mathfrak{a}_2)(\mathfrak{b}_1,\mathfrak{b}_2)) &= \zeta_{\mathfrak{P}\times\mathfrak{Q}}((\mathfrak{a}_1^{\mathfrak{g}}\mathfrak{b}_1^{\mathfrak{g}}),(\mathfrak{a}_2^{\mathfrak{g}}\mathfrak{b}_2^{\mathfrak{g}}))\\ &= \zeta_{\mathfrak{P}}(\mathfrak{a}_1^{\mathfrak{g}}\mathfrak{b}_1^{\mathfrak{g}}) \wedge \zeta_{\mathfrak{Q}}(\mathfrak{a}_2^{\mathfrak{g}}\mathfrak{b}_2^{\mathfrak{g}})\\ &\geq (\zeta_{\mathfrak{P}}(\mathfrak{b}_1^{\mathfrak{g}}) \wedge \zeta_{\mathfrak{Q}}(\mathfrak{b}_2^{\mathfrak{g}}))\\ &= \zeta_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{b}_1^{\mathfrak{g}},\mathfrak{b}_2^{\mathfrak{g}})\\ &= \zeta_{(\mathfrak{P}\times\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{b}_1,\mathfrak{b}_2) \qquad (29)\\ \tau_{(\mathfrak{P}\times\mathfrak{Q})^{\mathfrak{g}}}((\mathfrak{a}_1,\mathfrak{a}_2)(\mathfrak{b}_1,\mathfrak{b}_2)) &= \tau_{\mathfrak{P}\times\mathfrak{Q}}((\mathfrak{a}_1^{\mathfrak{g}}\mathfrak{b}_1^{\mathfrak{g}}),(\mathfrak{a}_2^{\mathfrak{g}}\mathfrak{b}_2^{\mathfrak{g}}))\\ &= \tau_{\mathfrak{P}}(\mathfrak{a}_1^{\mathfrak{g}}\mathfrak{b}_1^{\mathfrak{g}}) \vee \tau_{\mathfrak{Q}}(\mathfrak{a}_2^{\mathfrak{g}}\mathfrak{b}_2^{\mathfrak{g}})\\ &\leq (\tau_{\mathfrak{P}}(\mathfrak{b}_1^{\mathfrak{g}}) \vee \tau_{\mathfrak{Q}}(\mathfrak{b}_2^{\mathfrak{g}}))\\ &= \tau_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{b}_1^{\mathfrak{g}},\mathfrak{b}_2^{\mathfrak{g}})\\ &= \tau_{(\mathfrak{P}\times\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{b}_1,\mathfrak{b}_2). \qquad (30)\end{aligned}$$

Also, we can get

$$\chi_{(\mathfrak{P}\times\mathfrak{Q})^{\mathfrak{g}}}((\mathfrak{a}_1,\mathfrak{a}_2)(\mathfrak{b}_1,\mathfrak{b}_2)) \leq \chi_{(\mathfrak{P}\times\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{b}_1,\mathfrak{b}_2). \qquad (31)$$

For $(\mathfrak{a}_1,\mathfrak{a}_2),(\mathfrak{b}_1,\mathfrak{b}_2),(\mathfrak{c}_1,\mathfrak{c}_2) \in (\mathfrak{N}_1 \times \mathfrak{N}_2)$,

$$\begin{aligned}\zeta_{(\mathfrak{A}\times\mathfrak{B})^{\mathfrak{g}}}\{((\mathfrak{a}_1,\mathfrak{a}_2)+(\mathfrak{c}_1,\mathfrak{c}_2))(\mathfrak{b}_1,\mathfrak{b}_2)-(\mathfrak{a}_1,\mathfrak{a}_2)(\mathfrak{b}_1,\mathfrak{b}_2)\} &= \zeta_{(\mathfrak{A}\times\mathfrak{B})}\{(\mathfrak{a}_1^{\mathfrak{g}}+\mathfrak{c}_1^{\mathfrak{g}})\mathfrak{b}_1^{\mathfrak{g}}-\\ &\quad \mathfrak{a}_1^{\mathfrak{g}}\mathfrak{b}_1^{\mathfrak{g}},(\mathfrak{a}_2^{\mathfrak{g}}+\mathfrak{c}_2^{\mathfrak{g}})\mathfrak{b}_2^{\mathfrak{g}}-\mathfrak{a}_2^{\mathfrak{g}}\mathfrak{b}_2^{\mathfrak{g}}\}\\ &= \zeta_{\mathfrak{P}}\{(\mathfrak{a}_1^{\mathfrak{g}}+\mathfrak{c}_1^{\mathfrak{g}})\mathfrak{b}_1^{\mathfrak{g}}-\mathfrak{a}_1^{\mathfrak{g}}\mathfrak{b}_1^{\mathfrak{g}}\}\wedge\\ &\quad \zeta_{\mathfrak{Q}}\{(\mathfrak{a}_2^{\mathfrak{g}}+\mathfrak{c}_2^{\mathfrak{g}})\mathfrak{b}_2^{\mathfrak{g}}-\mathfrak{a}_2^{\mathfrak{g}}\mathfrak{b}_2^{\mathfrak{g}}\}\\ &\geq \zeta_{\mathfrak{P}}(\mathfrak{c}_1^{\mathfrak{g}}) \wedge \zeta_{\mathfrak{Q}}(\mathfrak{c}_2^{\mathfrak{g}})\\ &= \zeta_{(\mathfrak{P}\times\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{c}_1,\mathfrak{c}_2) \qquad (32)\\ \tau_{(\mathfrak{A}\times\mathfrak{B})^{\mathfrak{g}}}\{((\mathfrak{a}_1,\mathfrak{a}_2)+(\mathfrak{c}_1,\mathfrak{c}_2))(\mathfrak{b}_1,\mathfrak{b}_2)-(\mathfrak{a}_1,\mathfrak{a}_2)(\mathfrak{b}_1,\mathfrak{b}_2)\} &= \tau_{(\mathfrak{A}\times\mathfrak{B})}\{(\mathfrak{a}_1^{\mathfrak{g}}+\mathfrak{c}_1^{\mathfrak{g}})\mathfrak{b}_1^{\mathfrak{g}}-\\ &\quad \mathfrak{a}_1^{\mathfrak{g}}\mathfrak{b}_1^{\mathfrak{g}},(\mathfrak{a}_2^{\mathfrak{g}}+\mathfrak{c}_2^{\mathfrak{g}})\mathfrak{b}_2^{\mathfrak{g}}-\mathfrak{a}_2^{\mathfrak{g}}\mathfrak{b}_2^{\mathfrak{g}}\}\\ &= \tau_{\mathfrak{P}}\{(\mathfrak{a}_1^{\mathfrak{g}}+\mathfrak{c}_1^{\mathfrak{g}})\mathfrak{b}_1^{\mathfrak{g}}-\mathfrak{a}_1^{\mathfrak{g}}\mathfrak{b}_1^{\mathfrak{g}}\}\vee\\ &\quad \tau_{\mathfrak{Q}}\{(\mathfrak{a}_2^{\mathfrak{g}}+\mathfrak{c}_2^{\mathfrak{g}})\mathfrak{b}_2^{\mathfrak{g}}-\mathfrak{a}_2^{\mathfrak{g}}\mathfrak{b}_2^{\mathfrak{g}}\}\\ &\leq \tau_{\mathfrak{P}}(\mathfrak{c}_1^{\mathfrak{g}}) \vee \tau_{\mathfrak{Q}}(\mathfrak{c}_2^{\mathfrak{g}})\\ &= \tau_{(\mathfrak{P}\times\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{c}_1,\mathfrak{c}_2) \qquad (33)\end{aligned}$$

and

$$\chi_{(\mathfrak{A}\times\mathfrak{B})^{\mathfrak{g}}}\{((\mathfrak{a}_1,\mathfrak{a}_2)+(\mathfrak{c}_1,\mathfrak{c}_2))(\mathfrak{b}_1,\mathfrak{b}_2)-(\mathfrak{a}_1,\mathfrak{a}_2)(\mathfrak{b}_1,\mathfrak{b}_2)\} \leq \chi_{(\mathfrak{P}\times\mathfrak{Q})^{\mathfrak{g}}}(\mathfrak{c}_1,\mathfrak{c}_2). \qquad (34)$$

□

4. $\mathfrak{G}$-homomorphism of PFI

Definition 4.1. *Let $f : \mathfrak{N}_1 \to \mathfrak{N}_2$ be a map from a near ring $\mathfrak{N}_1$ to a near ring $\mathfrak{N}_2$ and $\mathfrak{P}$, $\mathfrak{Q}$ are PF set of $\mathfrak{N}_1$ and $\mathfrak{N}_2$. Then, image of $\mathfrak{P}$ under f, i.e., $f(\mathfrak{P})$ is defined as $f(\mathfrak{P}) = \{\mathfrak{n}, f(\zeta_{\mathfrak{P}})(\mathfrak{n}), f(\tau_{\mathfrak{P}})(\mathfrak{n}), f(\chi_{\mathfrak{P}})(\mathfrak{n})\}$, where*

$$f(\zeta_{\mathfrak{P}})(\mathfrak{n}) = \begin{cases} \bigvee \left\{\zeta_{\mathfrak{P}}(\mathfrak{m}) \mid \mathfrak{m} \in f^{-1}(\mathfrak{n})\right\} & \textit{if } f^{-1}(\mathfrak{n}) \neq \phi \\ 0 & \textit{otherwise,} \end{cases}$$

$$f(\tau_{\mathfrak{P}})(\mathfrak{n}) = \begin{cases} \bigwedge \left\{\tau_{\mathfrak{P}}(\mathfrak{m}) \mid \mathfrak{m} \in f^{-1}(\mathfrak{n})\right\} & \textit{if } f^{-1}(\mathfrak{n}) \neq \phi \\ 0 & \textit{otherwise,} \end{cases}$$

$$f(\chi_{\mathfrak{P}})(\mathfrak{n}) = \begin{cases} \bigwedge \left\{\chi_{\mathfrak{P}}(\mathfrak{m}) \mid \mathfrak{m} \in f^{-1}(\mathfrak{n})\right\} & \textit{if } f^{-1}(\mathfrak{n}) \neq \phi \\ 0 & \textit{otherwise,} \end{cases}$$

where $f(\chi_{\mathfrak{P}})(\mathfrak{n}) = (1 - f(1 - \chi_{\mathfrak{P}}))(\mathfrak{n})$.

Definition 4.2. *Let $\mathfrak{P} = (\zeta_{\mathfrak{P}}, \tau_{\mathfrak{P}}, \chi_{\mathfrak{P}})$ and $\mathfrak{Q} = (\zeta_{\mathfrak{Q}}, \tau_{\mathfrak{Q}}, \chi_{\mathfrak{Q}})$ be any PF ideals of a near ring $\mathfrak{N}$. Then composition $\mathfrak{P} \odot \mathfrak{Q} = (\zeta_{\mathfrak{P}} \circ \zeta_{\mathfrak{Q}},\ \tau_{\mathfrak{P}} \circ \tau_{\mathfrak{Q}},\ \chi_{\mathfrak{P}} \bullet \chi_{\mathfrak{Q}})$ is defined as follows:*

$$(\zeta_{\mathfrak{P}} \circ \zeta_{\mathfrak{Q}})(\mathfrak{n}) = \begin{cases} \bigcup\limits_{\mathfrak{n}=\mathfrak{n}_1\mathfrak{n}_2} \zeta_{\mathfrak{P}}(\mathfrak{n}_1) \wedge \zeta_{\mathfrak{Q}}(\mathfrak{n}_2) & \textit{if } \mathfrak{n} = \mathfrak{n}_1\mathfrak{n}_2 \\ 0 & \textit{otherwise,} \end{cases}$$

$$(\tau_{\mathfrak{P}} \circ \tau_{\mathfrak{Q}})(\mathfrak{n}) = \begin{cases} \bigcup\limits_{\mathfrak{n}=\mathfrak{n}_1\mathfrak{n}_2} \tau_{\mathfrak{P}}(\mathfrak{n}_1) \wedge \tau_{\mathfrak{Q}}(\mathfrak{n}_2) & \textit{if } \mathfrak{n} = \mathfrak{n}_1\mathfrak{n}_2 \\ 0 & \textit{otherwise,} \end{cases}$$

$$(\chi_{\mathfrak{P}} \bullet \chi_{\mathfrak{Q}})(\mathfrak{n}) = \begin{cases} \bigcap\limits_{\mathfrak{n}=\mathfrak{n}_1\mathfrak{n}_2} \chi_{\mathfrak{P}}(\mathfrak{n}_1) \vee \chi_{\mathfrak{Q}}(\mathfrak{n}_2) & \textit{if } \mathfrak{n} = \mathfrak{n}_1\mathfrak{n}_2 \\ 0 & \textit{otherwise.} \end{cases}$$

Definition 4.3. *A homomorphism f from a near ring $\mathfrak{N}_1$ to a near ring $\mathfrak{N}_2$. i.e., $f : \mathfrak{N}_1 \to \mathfrak{N}_2$ is called $\mathfrak{G}$-homomorphism, if for all $g \in \mathfrak{G}, r \in \mathfrak{N}$, $f(g \cdot x) = g \cdot f(x)$, where group $\mathfrak{G}$ acts on both the near rings.*

Theorem 4.1. *If f is a homomorphism from a near ring $\mathfrak{N}_1$ onto a near ring $\mathfrak{N}_2$ and $\mathfrak{A}_1, \mathfrak{A}_2$ are PF ideals of $\mathfrak{N}_1$, and $\mathfrak{B}_1, \mathfrak{B}_2$ are PF ideals of $\mathfrak{N}_2$, then the following hold*

(i) $(f^{-1}(\mathfrak{B}_1) \cup f^{-1}(\mathfrak{B}_2))^{\mathfrak{G}} = (f^{-1}(\mathfrak{B}_1 \cup \mathfrak{B}_2))^{\mathfrak{G}}$
(ii) $(f(\mathfrak{A}_1) \odot f(\mathfrak{A}_2))^{\mathfrak{G}} \subseteq (f(\mathfrak{A}_1 \odot \mathfrak{A}_2))^{\mathfrak{G}}$
(iii) If f is $\mathfrak{G}$-homomorphism and $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$, then $(f^{-1}(\mathfrak{B}_1))^{\mathfrak{G}} \subseteq f^{-1}(\mathfrak{B}_2)^{\mathfrak{G}}$.

Proof. (*i*) Let $\mathfrak{n} \in \mathfrak{N}_1$. Then

$$\begin{aligned}
(f^{-1}(\zeta_{\mathfrak{B}_1}) \cup f^{-1}(\zeta_{\mathfrak{B}_2}))^{\mathfrak{G}}(\mathfrak{n}) &= \bigcap_{\mathfrak{g}\in\mathfrak{G}}(f^{-1}(\zeta_{\mathfrak{B}_1}) \cup f^{-1}(\zeta_{\mathfrak{B}_2}))^{\mathfrak{g}}(\mathfrak{n}) \\
&= \bigcap_{\mathfrak{g}\in\mathfrak{G}}(f^{-1}(\zeta_{\mathfrak{B}_1}) \cup f^{-1}(\zeta_{\mathfrak{B}_2}))(\mathfrak{n}^{\mathfrak{g}}) \\
&= \bigcap_{\mathfrak{g}\in\mathfrak{G}}\{(f^{-1}(\zeta_{\mathfrak{B}_1})(\mathfrak{n}^{\mathfrak{g}}) \vee f^{-1}(\zeta_{\mathfrak{B}_2}))(\mathfrak{n}^{\mathfrak{g}})\} \\
&= \bigcap_{\mathfrak{g}\in\mathfrak{G}}\{\zeta_{\mathfrak{B}_1}(f(\mathfrak{n}^{\mathfrak{g}})) \vee \zeta_{\mathfrak{B}_2}(f(\mathfrak{n}^{\mathfrak{g}}))\} \\
&= \bigcap_{\mathfrak{g}\in\mathfrak{G}}\{\zeta_{\mathfrak{B}_1\cup\mathfrak{B}_2}(f(\mathfrak{n}^{\mathfrak{g}}))\} \\
&= \bigcap_{\mathfrak{g}\in\mathfrak{G}}\{f^{-1}(\zeta_{\mathfrak{B}_1\cup\mathfrak{B}_2})(\mathfrak{n}^{\mathfrak{g}})\} \\
&= \bigcap_{\mathfrak{g}\in\mathfrak{G}}(f^{-1}(\zeta_{\mathfrak{B}_1\cup\mathfrak{B}_2}))^{\mathfrak{g}}(\mathfrak{n}) \\
&= (f^{-1}(\zeta_{\mathfrak{B}_1\cup\mathfrak{B}_2}))^{\mathfrak{G}}(\mathfrak{n}), \qquad (35)
\end{aligned}$$

$$\begin{aligned}
(f^{-1}(\tau_{\mathfrak{B}_1}) \cup f^{-1}(\tau_{\mathfrak{B}_2}))^{\mathfrak{G}}(\mathfrak{n}) &= \bigcap_{\mathfrak{g}\in\mathfrak{G}}(f^{-1}(\tau_{\mathfrak{B}_1}) \cup f^{-1}(\tau_{\mathfrak{B}_2}))^{\mathfrak{g}}(\mathfrak{n}) \\
&= \bigcap_{\mathfrak{g}\in\mathfrak{G}}(f^{-1}(\tau_{\mathfrak{B}_1}) \cup f^{-1}(\tau_{\mathfrak{B}_2}))(\mathfrak{n}^{\mathfrak{g}}) \\
&= \bigcap_{\mathfrak{g}\in\mathfrak{G}}\{(f^{-1}(\tau_{\mathfrak{B}_1})(\mathfrak{n}^{\mathfrak{g}}) \wedge f^{-1}(\tau_{\mathfrak{B}_2}))(\mathfrak{n}^{\mathfrak{g}})\} \\
&= \bigcap_{\mathfrak{g}\in\mathfrak{G}}\{\tau_{\mathfrak{B}_1}(f(\mathfrak{n}^{\mathfrak{g}})) \wedge \tau_{\mathfrak{B}_2}(f(\mathfrak{n}^{\mathfrak{g}}))\} \\
&= \bigcap_{\mathfrak{g}\in\mathfrak{G}}\{\tau_{\mathfrak{B}_1\cup\mathfrak{B}_2}(f(\mathfrak{n}^{\mathfrak{g}}))\} \\
&= \bigcap_{\mathfrak{g}\in\mathfrak{G}}\{f^{-1}(\tau_{\mathfrak{B}_1\cup\mathfrak{B}_2})(\mathfrak{n}^{\mathfrak{g}})\} \\
&= \bigcap_{\mathfrak{g}\in\mathfrak{G}}(f^{-1}(\tau_{\mathfrak{B}_1\cup\mathfrak{B}_2}))^{\mathfrak{g}}(\mathfrak{n}) \\
&= (f^{-1}(\tau_{\mathfrak{B}_1\cup\mathfrak{B}_2}))^{\mathfrak{G}}(\mathfrak{n}), \qquad (36)
\end{aligned}$$

$$\begin{aligned}
(f^{-1}(\chi_{\mathfrak{B}_1}) \cup f^{-1}(\chi_{\mathfrak{B}_2}))^{\mathfrak{G}}(\mathfrak{n}) &= \bigcap_{\mathfrak{g}\in\mathfrak{G}}(f^{-1}(\chi_{\mathfrak{B}_1}) \cup f^{-1}(\chi_{\mathfrak{B}_2}))^{\mathfrak{g}}(\mathfrak{n}) \\
&= \bigcap_{\mathfrak{g}\in\mathfrak{G}}(f^{-1}(\chi_{\mathfrak{B}_1}) \cup f^{-1}(\chi_{\mathfrak{B}_2}))(\mathfrak{n}^{\mathfrak{g}}) \\
&= \bigcap_{\mathfrak{g}\in\mathfrak{G}}\{(f^{-1}(\chi_{\mathfrak{B}_1})(\mathfrak{n}^{\mathfrak{g}}) \wedge f^{-1}(\chi_{\mathfrak{B}_2}))(\mathfrak{n}^{\mathfrak{g}})\} \\
&= \bigcap_{\mathfrak{g}\in\mathfrak{G}}\{\chi_{\mathfrak{B}_1}(f(\mathfrak{n}^{\mathfrak{g}})) \wedge \chi_{\mathfrak{B}_2}(f(\mathfrak{n}^{\mathfrak{g}}))\} \\
&= \bigcap_{\mathfrak{g}\in\mathfrak{G}}\{\chi_{\mathfrak{B}_1\cup\mathfrak{B}_2}(f(\mathfrak{n}^{\mathfrak{g}}))\} \\
&= \bigcap_{\mathfrak{g}\in\mathfrak{G}}\{f^{-1}(\chi_{\mathfrak{B}_1\cup\mathfrak{B}_2})(\mathfrak{n}^{\mathfrak{g}})\} \\
&= \bigcap_{\mathfrak{g}\in\mathfrak{G}}(f^{-1}(\chi_{\mathfrak{B}_1\cup\mathfrak{B}_2}))^{\mathfrak{g}}(\mathfrak{n}) \\
&= (f^{-1}(\chi_{\mathfrak{B}_1\cup\mathfrak{B}_2}))^{\mathfrak{G}}(\mathfrak{n}). \qquad (37)
\end{aligned}$$

(ii) Let $\mathfrak{m} \in \mathfrak{N}_2$. Then,

$$\begin{aligned}(f(\zeta_{\mathfrak{A}_1}) \circ f(\zeta_{\mathfrak{A}_2}))^{\mathfrak{G}}(\mathfrak{m}) &= \bigcap_{\mathfrak{g}\in\mathfrak{G}} (f(\zeta_{\mathfrak{A}_1}) \circ f(\zeta_{\mathfrak{A}_2}))(\mathfrak{m}^{\mathfrak{g}}) \\ &= \bigcap_{\mathfrak{g}\in\mathfrak{G}} [\sup_{\mathfrak{m}^{\mathfrak{g}}=\mathfrak{m}_1\mathfrak{m}_2} \{\min(f(\zeta_{\mathfrak{A}_1})(\mathfrak{m}_1), f(\zeta_{\mathfrak{A}_2})(\mathfrak{m}_2))\}] \\ &= \bigcap_{\mathfrak{g}\in\mathfrak{G}} [\sup_{\mathfrak{m}^{\mathfrak{g}}=\mathfrak{m}_1\mathfrak{m}_2} \{\min(\sup_{\mathfrak{a}_1\in f^{-1}(\mathfrak{m}_1)} \zeta_{\mathfrak{A}_1}(\mathfrak{a}_1), \sup_{\mathfrak{b}_1\in f^{-1}(\mathfrak{m}_2)} \zeta_{\mathfrak{A}_2}(\mathfrak{b}_1))\}].\end{aligned}$$

Also, for some $\mathfrak{a}_2 \in f^{-1}(\mathfrak{m}_1), \mathfrak{b}_2 \in f^{-1}(\mathfrak{m}_2)$, $\mathfrak{a}_2\mathfrak{b}_2 = \mathfrak{c}$, and $\mathfrak{m}^{\mathfrak{g}} = \mathfrak{m}_1\mathfrak{m}_2$,

$$\begin{aligned}\bigcap_{\mathfrak{g}\in\mathfrak{G}} [\sup_{\mathfrak{m}^{\mathfrak{g}}=\mathfrak{m}_1\mathfrak{m}_2} \{\min(\sup_{\mathfrak{a}_1\in f^{-1}(\mathfrak{m}_1)} \zeta_{\mathfrak{A}_1}(\mathfrak{a}_1), \sup_{\mathfrak{b}_1\in f^{-1}(\mathfrak{m}_2)} \zeta_{\mathfrak{A}_2}(\mathfrak{b}_1))\}] &\leq \bigcap_{\mathfrak{g}\in\mathfrak{G}} \{\min(\zeta_{\mathfrak{A}_1}(\mathfrak{a}_2), \zeta_{\mathfrak{A}_2}(\mathfrak{b}_2))\} \\ &\leq \bigcap_{\mathfrak{g}\in\mathfrak{G}} (\zeta_{\mathfrak{A}_1}\zeta_{\mathfrak{A}_2})(\mathfrak{a}_2\mathfrak{b}_2) \\ &\leq \bigcap_{\mathfrak{g}\in\mathfrak{G}} \sup_{\mathfrak{c}\in f^{-1}(\mathfrak{m}^{\mathfrak{g}})} (\zeta_{\mathfrak{A}_1}\zeta_{\mathfrak{A}_2})(\mathfrak{c}) \\ &= \bigcap_{\mathfrak{g}\in\mathfrak{G}} f(\zeta_{\mathfrak{A}_1}\zeta_{\mathfrak{A}_2})(\mathfrak{m}^{\mathfrak{g}}) \\ &= f(\zeta_{\mathfrak{A}_1}\zeta_{\mathfrak{A}_2})^{\mathfrak{G}}(\mathfrak{m}).\end{aligned}$$

Thus,

$$(f(\zeta_{\mathfrak{A}_1}) \circ f(\zeta_{\mathfrak{A}_2}))^{\mathfrak{G}} \subseteq f(\zeta_{\mathfrak{A}_1} \circ \zeta_{\mathfrak{A}_2})^{\mathfrak{G}}. \tag{38}$$

In the similar manner, we can show that

$$(f(\tau_{\mathfrak{A}_1}) \circ f(\tau_{\mathfrak{A}_2}))^{\mathfrak{G}} \subseteq f(\tau_{\mathfrak{A}_1} \circ \tau_{\mathfrak{A}_2})^{\mathfrak{G}}. \tag{39}$$

Furthermore, for $\mathfrak{m} \in \mathfrak{N}$,

$$\begin{aligned}(f(\chi_{\mathfrak{A}_1}) \bullet f(\chi_{\mathfrak{A}_2}))^{\mathfrak{G}}(\mathfrak{m}) &= \bigcap_{\mathfrak{g}\in\mathfrak{G}} (f(\chi_{\mathfrak{A}_1}) \bullet f(\chi_{\mathfrak{A}_2}))(\mathfrak{m}^{\mathfrak{g}}) \\ &= \bigcap_{\mathfrak{g}\in\mathfrak{G}} [\inf_{\mathfrak{m}^{\mathfrak{g}}=\mathfrak{m}_1\mathfrak{m}_2} \{\sup(f(\chi_{\mathfrak{A}_1})(\mathfrak{m}_1), f(\chi_{\mathfrak{A}_2})(\mathfrak{m}_2))\}] \\ &= \bigcap_{\mathfrak{g}\in\mathfrak{G}} [\inf_{\mathfrak{m}^{\mathfrak{g}}=\mathfrak{m}_1\mathfrak{m}_2} \{\sup(\inf_{\mathfrak{a}_1\in f^{-1}(\mathfrak{m}_1)} \chi_{\mathfrak{A}_1}(\mathfrak{a}_1), \inf_{\mathfrak{b}_1\in f^{-1}(\mathfrak{m}_2)} \chi_{\mathfrak{A}_2}(\mathfrak{b}_1))\}].\end{aligned}$$

Also for some $\mathfrak{a}_2 \in f^{-1}(\mathfrak{m}_1), \mathfrak{b}_2 \in f^{-1}(\mathfrak{m}_2)$, $\mathfrak{a}_2\mathfrak{b}_2 = \mathfrak{c}$, and $\mathfrak{m}^{\mathfrak{g}} = \mathfrak{m}_1\mathfrak{m}_2$,

$$\begin{aligned}\bigcap_{\mathfrak{g}\in\mathfrak{G}} [\inf_{\mathfrak{m}^{\mathfrak{g}}=\mathfrak{m}_1\mathfrak{m}_2} \{\sup(\inf_{\mathfrak{a}_1\in f^{-1}(\mathfrak{m}_1)} \chi_{\mathfrak{A}_1}(\mathfrak{a}_1), \inf_{\mathfrak{b}_1\in f^{-1}(\mathfrak{m}_2)} \chi_{\mathfrak{A}_2}(\mathfrak{b}_1))\}] &\geq \bigcap_{\mathfrak{g}\in\mathfrak{G}} \{\min(\chi_{\mathfrak{A}_1}(\mathfrak{a}_2), \chi_{\mathfrak{A}_2}(\mathfrak{b}_2))\} \\ &\geq \bigcap_{\mathfrak{g}\in\mathfrak{G}} (\chi_{\mathfrak{A}_1}\chi_{\mathfrak{A}_2})(\mathfrak{a}_2\mathfrak{b}_2) \\ &\geq \bigcap_{\mathfrak{g}\in\mathfrak{G}} \sup_{\mathfrak{c}\in f^{-1}(\mathfrak{m}^{\mathfrak{g}})} (\chi_{\mathfrak{A}_1}\chi_{\mathfrak{A}_2})(\mathfrak{c}) \\ &= \bigcap_{\mathfrak{g}\in\mathfrak{G}} f(\chi_{\mathfrak{A}_1}\chi_{\mathfrak{A}_2})(\mathfrak{m}^{\mathfrak{g}}) \\ &= f(\chi_{\mathfrak{A}_1}\chi_{\mathfrak{A}_2})^{\mathfrak{G}}(\mathfrak{m}).\end{aligned}$$

This shows that

$$f(\chi_{\mathfrak{A}_1} \circ \chi_{\mathfrak{A}_2})^{\mathfrak{G}} \subseteq (f(\chi_{\mathfrak{A}_1}) \circ f(\chi_{\mathfrak{A}_2}))^{\mathfrak{G}}. \tag{40}$$

Equations (38)-(40) show that $(f(\mathfrak{A}_1) \odot f(\mathfrak{A}_2))^{\mathfrak{G}} \subseteq (f(\mathfrak{A}_1 \odot \mathfrak{A}_2))^{\mathfrak{G}}$.

(*iii*) Let $\mathfrak{m} \in \mathfrak{N}$. Then

$$\begin{aligned}(f^{-1}(\zeta_{\mathfrak{B}_1}))^{\mathfrak{G}}(\mathfrak{s}) &= \bigcap_{\mathfrak{g}\in\mathfrak{G}}(f^{-1}(\zeta_{\mathfrak{B}_1}))^{\mathfrak{g}}(\mathfrak{m}) = \bigcap_{\mathfrak{g}\in\mathfrak{G}}(f^{-1}(\zeta_{\mathfrak{B}_1}))(\mathfrak{m}^{\mathfrak{g}})\\ &= \bigcap_{\mathfrak{g}\in\mathfrak{G}}\zeta_{\mathfrak{B}_1}(f(\mathfrak{m}^{\mathfrak{g}})) = \bigcap_{\mathfrak{g}\in\mathfrak{G}}\zeta_{\mathfrak{B}_1}((f(\mathfrak{m}))^{\mathfrak{g}})\\ &\leq \bigcap_{\mathfrak{g}\in\mathfrak{G}}\zeta_{\mathfrak{B}_2}((f(\mathfrak{m}))^{\mathfrak{g}}) = \bigcap_{\mathfrak{g}\in\mathfrak{G}}\zeta^{\mathfrak{g}}_{\mathfrak{B}_2}((f(\mathfrak{m})))\\ &= \zeta^{\mathfrak{G}}_{\mathfrak{B}_2}((f(\mathfrak{m})))\\ &= f^{-1}(\zeta^{\mathfrak{G}}_{\mathfrak{B}_2})(\mathfrak{m}).\end{aligned}$$

This implies that

$$(f^{-1}(\zeta_{\mathfrak{B}_1}))^{\mathfrak{G}}(\mathfrak{m}) \leq f^{-1}(\zeta^{\mathfrak{G}}_{\mathfrak{B}_2})(\mathfrak{m}). \tag{41}$$

Similarly, we can prove that

$$(f^{-1}(\tau_{\mathfrak{B}_1}))^{\mathfrak{G}}(\mathfrak{m}) \leq f^{-1}(\tau^{\mathfrak{G}}_{\mathfrak{B}_2})(\mathfrak{m}). \tag{42}$$

Let $\mathfrak{m} \in \mathfrak{N}$. Then

$$\begin{aligned}(f^{-1}(\chi_{\mathfrak{B}_1}))^{\mathfrak{G}}(\mathfrak{m}) &= \bigcap_{\mathfrak{g}\in\mathfrak{G}}(f^{-1}(\chi_{\mathfrak{B}_1}))^{\mathfrak{g}}(\mathfrak{m}) = \bigcap_{\mathfrak{g}\in\mathfrak{G}}(f^{-1}(\chi_{\mathfrak{B}_1}))(\mathfrak{m}^{\mathfrak{g}})\\ &= \bigcap_{\mathfrak{g}\in\mathfrak{G}}\chi_{\mathfrak{B}_1}(f(\mathfrak{m}^{\mathfrak{g}})) = \bigcap_{\mathfrak{g}\in\mathfrak{G}}\chi_{\mathfrak{B}_1}((f(\mathfrak{m}))^{\mathfrak{g}})\\ &\geq \bigcap_{\mathfrak{g}\in\mathfrak{G}}\chi_{\mathfrak{B}_2}((f(\mathfrak{m}))^{\mathfrak{g}}) = \bigcap_{\mathfrak{g}\in\mathfrak{G}}\chi^{\mathfrak{g}}_{\mathfrak{B}_2}((f(\mathfrak{m})))\\ &= \chi^{\mathfrak{G}}_{\mathfrak{B}_2}((f(\mathfrak{m})))\\ &= f^{-1}(\chi^{\mathfrak{G}}_{\mathfrak{B}_2})(\mathfrak{m})\end{aligned}$$

$$(f^{-1}(\chi_{\mathfrak{B}_1}))^{\mathfrak{G}}(\mathfrak{m}) \geq f^{-1}(\chi^{\mathfrak{G}}_{\mathfrak{B}_2})(\mathfrak{m}). \tag{43}$$

This implies that $(f^{-1}(\mathfrak{B}_1))^{\mathfrak{G}} \subseteq f^{-1}({\mathfrak{B}_2}^{\mathfrak{G}})$. □

5. Conclusions

In this paper, we defined group actions on picture fuzzy (PF) ideals of near rings which are extension of both fuzzy and intuitionistic fuzzy ideals and studied properties of picture fuzzy ideals under $\mathfrak{G}$-homomorphism. We provided appropriate definitions for the operations of picture fuzzy ideals over a near ring, including product, composition and intersection as well as studied properties of images and inverse images of picture fuzzy ideals under group actions.

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