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# INTRODUCING A NOVEL SUBCLASS OF HARMONIC FUNCTIONS WITH CLOSE-TO-CONVEX PROPERTIES

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Abstract. In this paper, we introduce a new subclass of close-to-convex harmonic functions. We present a sufficient coefficient condition for a function to be a member of this class. Furthermore, we establish a distortion theorem. These results lay the groundwork for extending the findings to function classes involving higher-order derivatives.

Keywords: Close-to-convex functions, Harmonic functions, Starlike functions, Distortion

AMS Subject Classification: 30C45, 30C50

## 1. Introduction

In the realm of harmonic functions, every function $\mathfrak{f}$ belonging to the class $\mathcal{SH}^0$ can be expressed as $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$, where

$$\mathfrak{u}(z) = z + \sum_{m=2}^{\infty} u_m z^m, \quad \mathfrak{v}(z) = \sum_{m=2}^{\infty} v_m z^m \tag{1}$$

with both functions being analytic in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. Provided that $|\mathfrak{u}'(z)| > |\mathfrak{v}'(z)|$ in $\mathbb{E}$, $\mathfrak{f}$ is locally univalent and sense-preserving in $\mathbb{E}$. It's noteworthy that when $\mathfrak{v}(z)$ is identically zero, $\mathcal{SH}^0$ contracts to class $\mathcal{S}$.

The subclasses of $\mathcal{S}$ that map $\mathbb{E}$ onto starlike and close-to-convex domains, respectively, are denoted by $\mathcal{S}^*$ and $\mathcal{K}$. Similarly, $\mathcal{SH}^{0,*}$ and $\mathcal{KH}^0$ represent subclasses of $\mathcal{SH}^0$ that map $\mathbb{E}$ onto their respective domains. (For further details, refer to [1, 3])

In 2005, Gao and Zhou [4] introduced the class

$$\mathcal{K}_s = \left\{ \mathfrak{u} \in \mathcal{S} : \operatorname{Re}\left\{ \frac{z^2 \mathfrak{u}'(z)}{-\phi(z)\phi(-z)} \right\} > 0 \text{ for } z \in \mathbb{E} \right\}.$$

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In 2011, Şeker [5] introduced the class

$$\mathcal{K}(k,\gamma)=\left\{\mathfrak{u}\in\mathcal{S}:\operatorname{Re}\left\{\frac{z^k\mathfrak{u}'(z)}{\phi_k(z)}\right\}>\gamma,\quad 0\le\gamma<1\text{ and }z\in\mathbb{E}\right\}$$

where $k\in\mathbb{Z}^+$, $\phi(z)$ is an analytic function of the form

$$\phi(z)=z+\sum_{m=2}^{\infty}c_m z^m\in S^*\left(\frac{k-1}{k}\right),$$

and $\phi_k(z)$, which was previously introduced by Wang et al. [24], is defined as

$$\phi_k(z)=\prod_{v=0}^{k-1}\mu^{-v}\phi(\mu^v z),\quad\text{where}\quad\mu^k=1 \tag{2}$$

with $\mu=e^{2\pi i/k}$.

The classes $\mathcal{K}_s$ and $\mathcal{K}(k,\gamma)$, introduced by Gao and Zhou [4] and Şeker [5], respectively, focus solely on analytic functions and are independent of the variable $\overline{z}$. This restriction prevents the study of properties of harmonic functions that depend on the variable $\overline{z}$. This leads us to define a new function class that includes the co-analytic part of harmonic functions.

Observe that if $k=2$ in the class $\mathcal{K}(k,\gamma)$, the class $\mathcal{K}(\gamma)$ studied by Kowalczyk et al. [2] is obtained. For $k=2$ and $\gamma=0$, the class $\mathcal{K}_s$ studied by Gao and Zhou [4] is obtained. It is clear that the class $\mathcal{K}(k,\gamma)$ encompasses both classes. The class $\mathcal{KH}^0(k,\gamma)$, which we will define shortly, covers the class $\mathcal{K}(k,\gamma)$ since it is defined using harmonic conjugates of analytic functions belonging to the classes $\mathcal{K}(k,\gamma)$. Therefore, the class $\mathcal{KH}^0(k,\gamma)$ is a generalization of the $\mathcal{K}(k,\gamma)$, $\mathcal{K}_s$, and $\mathcal{K}(\gamma)$ classes.

This generalization allows for a broader investigation of geometric properties, such as distortion bounds and close-to-convexity, for harmonic functions that depend on $\overline{z}$.

**Definition 1.1.** *The class $\mathcal{KH}^0(k,\gamma)$ is defined as the collection of functions $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}\in\mathcal{SH}^0$ that adhere to the following inequality:*

$$\operatorname{Re}\left\{\frac{z^k\mathfrak{u}'(z)}{\phi_k(z)}-\gamma\right\}>\left|\frac{z^k\mathfrak{v}'(z)}{\phi_k(z)}\right|, \tag{3}$$

*where $0\le\gamma<1$ and $\phi_k(z)$ is given by (2).*

Specifically, when $\mathfrak{v}(z)\equiv 0$, the class $\mathcal{KH}^0(k,\gamma)$ reduces to the class $\mathcal{K}(k,\gamma)$. Also, by setting $\mathfrak{v}(z)\equiv 0$, $k=2$ and $\gamma=0$, we obtain $\mathcal{KH}^0(2,0)=\mathcal{K}_s$. The inclusion of the co-analytic term $\overline{\mathfrak{v}(z)}$ extends these classes, providing a more general framework that accommodates both analytic and co-analytic components. Additionally, when $\phi(z)=z$, and by appropriately selecting the parameters, the class $\mathcal{KH}^0(k,\gamma)$ can be reduced to several well-known subclasses of harmonic functions, as outlined below.

i $\mathcal{KH}^0(k,0)=\mathcal{PH}^0$ [6].
ii $\mathcal{KH}^0(k,\gamma)=\mathcal{PH}^0(\gamma)$ [7, 8].
ii $\mathcal{KH}^0(k,0)=\mathcal{WH}^0(0)$ ([9]).
iv $\mathcal{KH}^0(k,\gamma)=\mathcal{WH}^0(0,\gamma)$ ([10]).
v $\mathcal{KH}^0(k,\gamma)=\mathcal{AH}^0(1,0,\gamma)$ ([11]).
vi $\mathcal{KH}^0(k,0)=\mathcal{RH}^0(0,0)$ ([12]).

For further details on harmonic function classes defined by differential inequality, see [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23].

In this work, we investigate the distortion theorems and coefficient bounds for functions in the class $\mathcal{KH}^0(k,\gamma)$ and demonstrate that functions within this class exhibit close-to-convex behavior.

## 2. Examples of Functions in the Class $\mathcal{KH}^0(k,\gamma)$

**Example 2.1.** *Let $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} = z + \frac{1-\gamma}{m}\overline{z}^m$ and $\phi(z) = z$. For $0 \leq \gamma < 1$ and $|z| < 1$, we have*

$$\operatorname{Re}\left\{\frac{z^k\mathfrak{u}'(z)}{\phi_k(z)} - \gamma\right\} = 1 - \gamma > (1-\gamma)\,|z|^{m-1} = \left|\frac{z^k\mathfrak{v}'(z)}{\phi_k(z)}\right|.$$

*Hence, $\mathfrak{f} \in \mathcal{KH}^0(k,\gamma)$.*

The following examples can be given for the special case of the parameters in Example 2.1.

**Example 2.2.** *Let $\mathfrak{f} = z + \frac{99}{200}\overline{z}^2$, $\gamma = \frac{1}{100}$ and $\phi(z) = z$. Then $\mathfrak{f} \in \mathcal{KH}^0(k, \frac{33}{100})$. The unit disk is mapped to a starlike region by the function $f$. The depiction in Figure 1 showcases the image of the set $\mathbb{E}$ under the transformation defined by $\mathfrak{f}(z) = z + \frac{99}{200}\overline{z}^2$.*

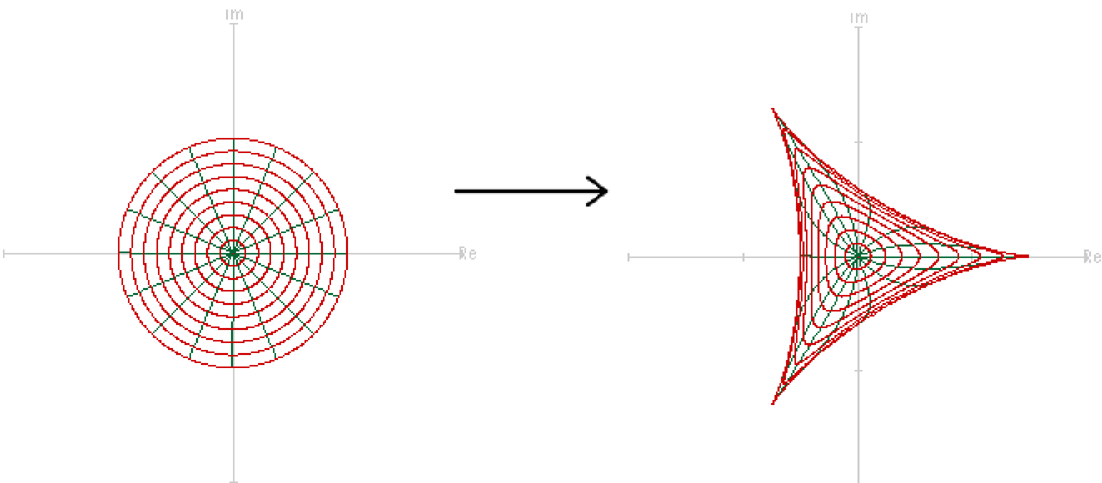


Figure 1. Under the map $\mathfrak{f} = z + \frac{99}{200}\overline{z}^2$, the image of the unit disk.

**Example 2.3.** *Let $\mathfrak{f} = z + \frac{1}{10}\overline{z}^2$, $\gamma = \frac{4}{5}$, and $\phi(z) = z$. Then $\mathfrak{f} \in \mathcal{KH}^0(k, \frac{4}{5})$. The unit disk is mapped to a convex region by the function $f$. The depiction in Figure 2 showcases the image of the set $\mathbb{E}$ under the transformation defined by $\mathfrak{f} = z + \frac{1}{10}\overline{z}^2$.*

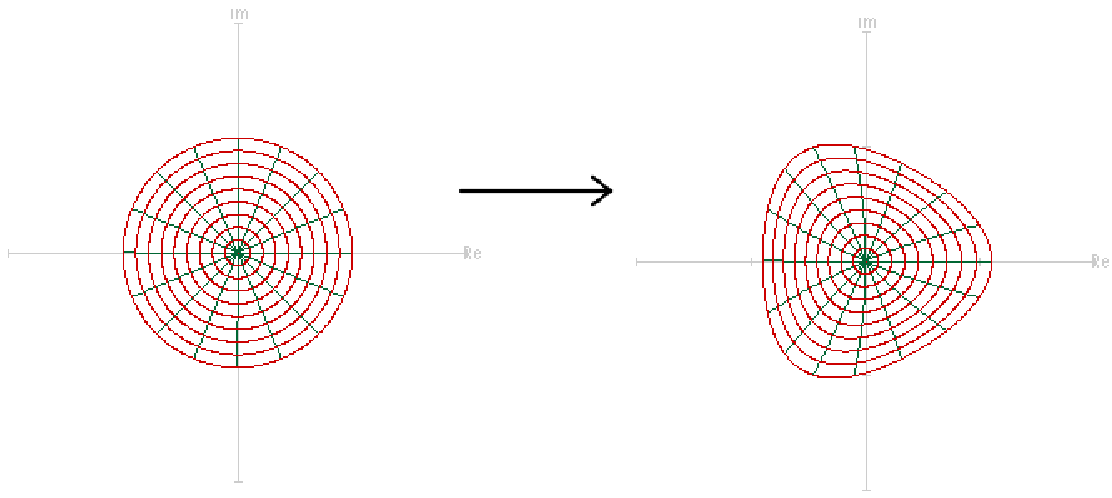


FIGURE 2. Under the map $\mathfrak{f} = z + \frac{1}{10}\overline{z}^2$, the image of the unit disk.

**Example 2.4.** *Let* $\mathfrak{f} = z + \frac{33}{100}\overline{z}^3$, $\gamma = \frac{1}{100}$ *and* $\phi(z) = z$*. Then* $\mathfrak{f} \in \mathcal{KH}^0(k, \frac{33}{100})$*. The unit disk is mapped to a starlike region by the function* $f$*. The depiction in Figure 3 showcases the image of the set* $\mathbb{E}$ *under the transformation defined by* $\mathfrak{f} = z + \frac{33}{100}\overline{z}^3$.

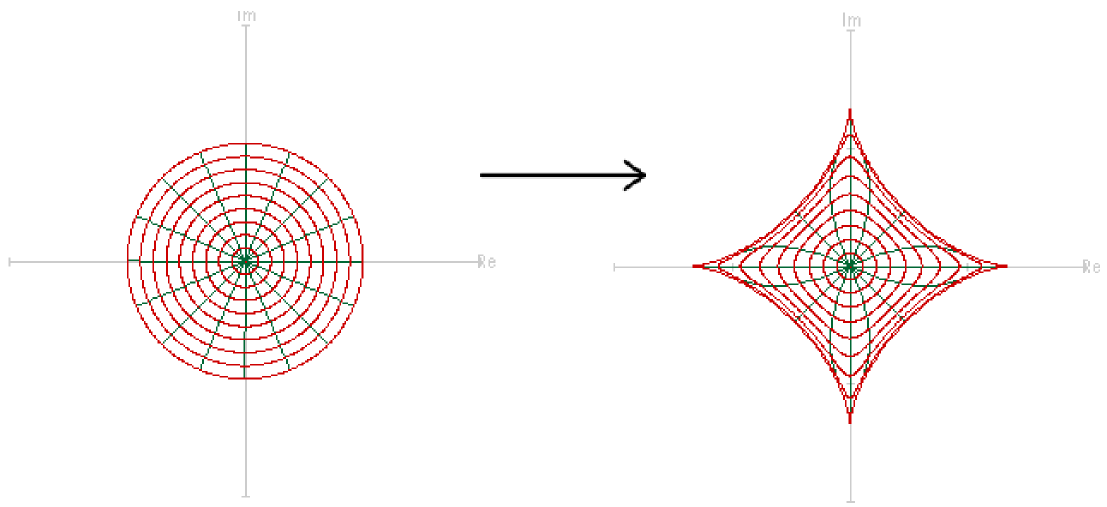


FIGURE 3. Under the map $\mathfrak{f} = z + \frac{33}{100}\overline{z}^3$, the image of the unit disk.

**Example 2.5.** *Let* $\mathfrak{f} = z + \frac{1}{15}\overline{z}^3$, $\gamma = \frac{4}{5}$ *and* $\phi(z) = z$*. Then* $\mathfrak{f} \in \mathcal{KH}^0(k, \frac{4}{5})$*. The unit disk is mapped to a convex region by the function* $f$*. The depiction in Figure 4 showcases the image of the set* $\mathbb{E}$ *under the transformation defined by* $\mathfrak{f} = z + \frac{1}{15}\overline{z}^3$.

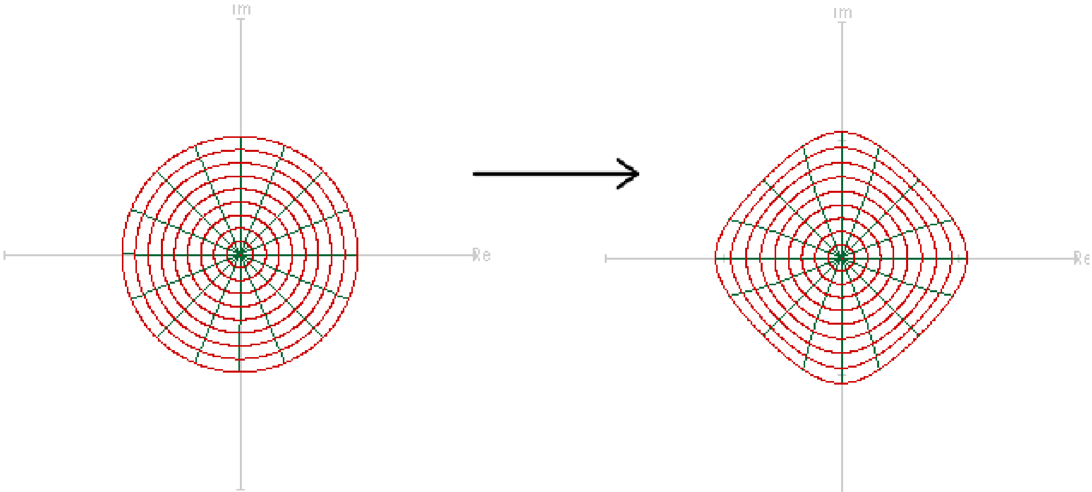


FIGURE 4. Under the map $\mathfrak{f} = z + \frac{1}{15}\overline{z}^3$, the image of the unit disk.

**Example 2.6.** *Let* $\mathfrak{f} = z + \frac{99}{500}\overline{z}^5$, $\gamma = \frac{1}{100}$ *and* $\phi(z) = z$. *Then* $\mathfrak{f} \in \mathcal{KH}^0(k, \frac{99}{500})$. *The unit disk is mapped to a starlike region by the function* $f$. *The depiction in Figure 5 showcases the image of the set* $\mathbb{E}$ *under the transformation defined by* $\mathfrak{f} = z + \frac{99}{500}\overline{z}^5$.

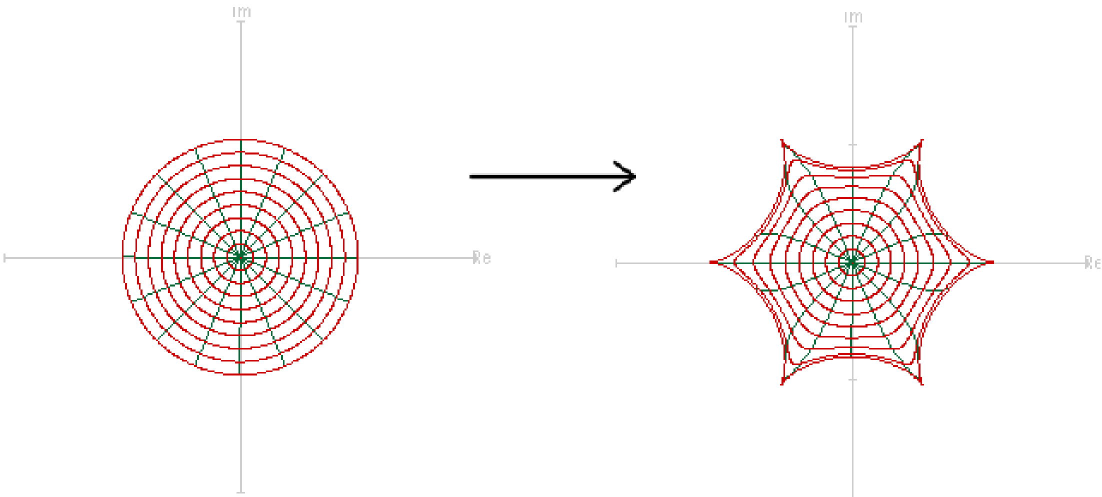


FIGURE 5. Under the map $\mathfrak{f} = z + \frac{99}{500}\overline{z}^5$, the image of the unit disk.

**Example 2.7.** *Let* $\mathfrak{f} = z + \frac{1}{25}\overline{z}^5$, $\gamma = \frac{4}{5}$ *and* $\phi(z) = z$. *Then* $\mathfrak{f} \in \mathcal{KH}^0(k, \frac{4}{5})$. *The unit disk is mapped to a convex region by the function* $f$. *The depiction in Figure 6 showcases the image of the set* $\mathbb{E}$ *under the transformation defined by* $\mathfrak{f} = z + \frac{1}{25}\overline{z}^5$.

## 3. Geometric Properties of the class $\mathcal{KH}^0(k, \gamma)$

First, we give a result that establishes a sufficient condition for $\mathfrak{f} \in \mathcal{SH}^0$ to be close-to-convex, which comes from Clunie and Sheil-Small [1].

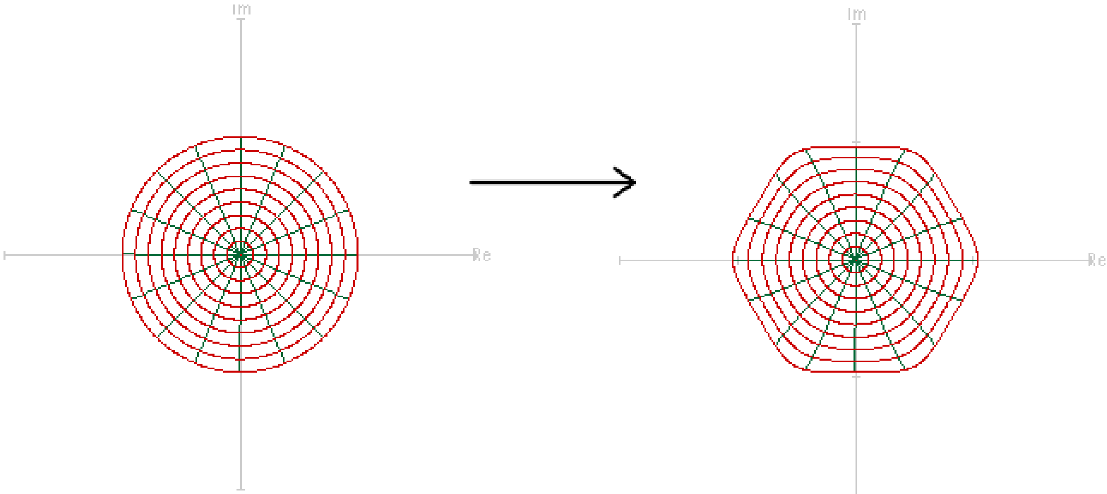


FIGURE 6. Under the map $\mathfrak{f} = z + \frac{1}{25}\overline{z}^5$, the image of the unit disk.

**Lemma 3.1.** *Let $\mathfrak{u}$ and $\mathfrak{v}$ be analytic functions in $\mathbb{E}$, such that $|\mathfrak{v}'(0)| < |\mathfrak{u}'(0)|$, and for each $\varepsilon$ $(|\varepsilon| = 1)$, $F_\varepsilon = \mathfrak{u} + \varepsilon\mathfrak{v}$ is close-to-convex. Then, $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$ is close-to-convex in $\mathbb{E}$.*

The result we will present now establishes a connection between the $\mathcal{KH}^0(k,\gamma)$ harmonic function class and the $\mathcal{K}(k,\gamma)$ analytic function class.

**Theorem 3.1.** *$\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{KH}^0(k,\gamma)$ if and only if $F_\varepsilon = \mathfrak{u} + \varepsilon\mathfrak{v} \in \mathcal{K}(k,\gamma)$ for each $\varepsilon$ $(|\varepsilon| = 1)$.*

*Proof.* Assume $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{KH}^0(k,\gamma)$. For each $\varepsilon$ $(|\varepsilon| = 1)$, we have

$$\begin{aligned}
&\mathrm{Re}\left\{\frac{z^k F_\varepsilon{}'(z)}{\phi_k(z)}\right\} \\
&= \mathrm{Re}\left\{\frac{z^k \mathfrak{u}'(z)}{\phi_k(z)}\right\} + \varepsilon \mathrm{Re}\left\{\frac{z^k \mathfrak{v}'(z)}{\phi_k(z)}\right\} \\
&> \mathrm{Re}\left\{\frac{z^k \mathfrak{u}'(z)}{\phi_k(z)}\right\} - \left|\frac{z^k \mathfrak{v}'(z)}{\phi_k(z)}\right| > \gamma \ \ (z \in \mathbb{E})\,.
\end{aligned}$$

Hence, $F_\varepsilon \in \mathcal{K}(k,\gamma)$ for each $\varepsilon$ $(|\varepsilon| = 1)$.

Conversely, suppose $F_\varepsilon = \mathfrak{u} + \varepsilon\mathfrak{v} \in \mathcal{K}(k,\gamma)$. Then,

$$\mathrm{Re}\left\{\frac{z^k \mathfrak{u}'(z)}{\phi_k(z)}\right\} > \mathrm{Re}\left[-\varepsilon\frac{z^k \mathfrak{v}'(z)}{\phi_k(z)}\right] + \gamma \ \ (z \in \mathbb{E})\,.$$

Choosing an appropriate $\varepsilon$ $(|\varepsilon| = 1)$ yields

$$\mathrm{Re}\left\{\frac{z^k \mathfrak{u}'(z)}{\phi_k(z)} - \gamma\right\} > \left|\frac{z^k \mathfrak{v}'(z)}{\phi_k(z)}\right| \ \ (z \in \mathbb{E})\,,$$

and thus $\mathfrak{f} \in \mathcal{KH}^0(k,\gamma)$. □

We will now demonstrate that harmonic functions belonging to the class $\mathcal{KH}^0(k,\gamma)$ map the open unit disk onto a close-to-convex region. To this end, we first state the following lemma and subsequently establish that functions in the class $\mathcal{K}(k,\gamma)$ are close-to-convex within the open unit disk.

**Lemma 3.2.** [24] *Let* $\phi(z) = z + \sum_{m=2}^{\infty} c_m z^m \in \mathcal{S}^*\left(\frac{k-1}{k}\right)$ *for* $k \geq 1$. *Then,*

$$\Phi_k(z) = \frac{\phi_k(z)}{z^{k-1}} = z + \sum_{m=2}^{\infty} C_m z^m \in S^* \tag{4}$$

*where* $\phi_k(z)$ *is given by* (2).

**Theorem 3.2.** *If* $F$ *is a function in the class* $\mathcal{K}(k,\gamma)$, *then* $F$ *is close-to-convex of order* $\gamma$ *in the region* $\mathbb{E}$.

*Proof.* Let $F \in \mathcal{K}(k,\gamma)$. We have

$$\operatorname{Re}\left\{\frac{z^k F'(z)}{\phi_k(z)}\right\} = \operatorname{Re}\left\{\frac{zF'(z)}{\Phi_k(z)}\right\} > \gamma,$$

where $\Phi_k(z)$ is given by (4). Therefore, the function $F$ is close-to-convex of order $\gamma$ since $\Phi_k(z) \in \mathcal{S}^*$. □

**Theorem 3.3.** *Every function in the class* $\mathcal{KH}^0(k,\gamma)$ *is close-to-convex within the region* $\mathbb{E}$.

*Proof.* Let $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$ belong to class $\mathcal{KH}^0(k,\gamma)$. Then by Theorem 3.1 the function $F_\varepsilon = \mathfrak{u} + \varepsilon\mathfrak{v}$ belongs to class $\mathcal{K}(k,\gamma)$ and by Theorem 3.2 also close to convex in $\mathbb{E}$. Therefore, by Lemma 3.1, $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{KH}^0(k,\gamma)$ is also close to convex in $\mathbb{E}$. □

In the following result, we derive a coefficient bound for functions belonging to the class $\mathcal{KH}^0(k,\gamma)$.

**Theorem 3.4.** *Let* $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{KH}^0(k,\gamma)$. *For* $m \geq 2$, *the following inequalities hold:*

$$|u_m| + |v_m| \leq \gamma + m(1-\gamma).$$

*For the function* $\mathfrak{f}(z) = z + [\gamma + m(1-\gamma)]z^m$, *every outcome is sharp and every equality is holds.*

*Proof.* Suppose that $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{KH}^0(k,\gamma)$. $F_\varepsilon = \mathfrak{u} + \varepsilon\mathfrak{v} \in \mathcal{K}(k,\gamma)$ for $\varepsilon$ $(|\varepsilon| = 1)$, according to Theorem 3.1. With respect to every $\varepsilon$ $(|\varepsilon| = 1)$, we possess

$$\operatorname{Re}\left\{\frac{z^k F_\varepsilon{}'(z)}{\phi_k(z)}\right\} = \operatorname{Re}\left\{\frac{zF_\varepsilon{}'(z)}{\Phi_k(z)}\right\} = \operatorname{Re}\left\{\frac{z\left(\mathfrak{u}'(z) + \varepsilon\mathfrak{v}'(z)\right)}{\Phi_k(z)}\right\} > \gamma.$$

for $z \in \mathbb{E}$. On the other hand, there is an analytic function $\mathfrak{P}(z) = 1 + \sum\limits_{m=1}^{\infty} \mathfrak{p}_m z^m$ in $\mathbb{E}$ whose real part is positive, satisfying

$$\frac{z\left(\mathfrak{u}'(z) + \varepsilon\mathfrak{v}'(z)\right)}{\Phi_k(z)} = \gamma + (1-\gamma)\,\mathfrak{P}(z). \tag{5}$$

or

$$z\left(\mathfrak{u}'(z) + \varepsilon\mathfrak{v}'(z)\right) = [\gamma + (1-\gamma)\,\mathfrak{P}(z)]\,\Phi_k(z). \tag{6}$$

Upon comparing the coefficients in (6), it can be observed that

$$m(u_m + \varepsilon v_m) = C_m + (1-\gamma)\mathfrak{p}_{m-1} + (1-\gamma)\mathfrak{p}_1 C_{m-1} + \cdots + (1-\gamma)\mathfrak{p}_{m-2}C_2. \tag{7}$$

Since $\Phi_k(z)$ is starlike, we have $|C_m| \leq m$, and since $Re\{\mathfrak{P}(z)\} > 0$, we have $|\mathfrak{p}_m| \leq 2$ for $m \geq 1$. Hence, by equation 7, we have

$$m\,|u_m + \varepsilon v_m| \leq m\,[\gamma + m(1-\gamma)]\,. \tag{8}$$

Since $\varepsilon$ $(|\varepsilon| = 1)$ is arbitrary, it follows that the proof is concluded. The function $\mathfrak{f}(z) = z + [\gamma + m(1-\gamma)]z^m$, demonstrates the sharpness of inequality. □

Now, we provide the necessary coefficient condition for a harmonic function to belong to the class $\mathcal{KH}^0(k,\gamma)$.

**Theorem 3.5.** *Let $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}\in\mathcal{SH}^0$ with the series expansions given by (1). If the following inequality holds:*

$$\sum_{m=2}^{\infty} 2m\left(|u_m|+|v_m|\right)+\sum_{m=2}^{\infty}\left(|1-2\gamma|+1\right)|C_m|\le 2(1-\gamma), \tag{9}$$

*then $\mathfrak{f}\in\mathcal{KH}^0(k,\gamma)$.*

*Proof.* Consider $\mathfrak{u}$ and $\mathfrak{v}$ as functions with series expansions given by (1). Let $F_\varepsilon=\mathfrak{u}+\varepsilon\mathfrak{v}$ for each $\varepsilon$ ($|\varepsilon|=1$). Define the functions $\phi_k(z)$ and $\Phi_k(z)$ as given by (2) and (4), respectively. Then we can express $A$ as

$$\begin{aligned}
A &= \left|zF_\varepsilon'-\frac{\phi_k(z)}{z^{k-1}}\right|-\left|zF_\varepsilon'+\frac{(1-2\gamma)\phi_k(z)}{z^{k-1}}\right| \\
&= \left|z(\mathfrak{u}+\varepsilon\mathfrak{v})'-\Phi_k(z)\right|-\left|z(\mathfrak{u}+\varepsilon\mathfrak{v})'+(1-2\gamma)\Phi_k(z)\right| \\
&= \left|\sum_{m=2}^{\infty} m(u_m+\varepsilon v_m)z^m-\sum_{m=2}^{\infty} C_m z^m\right| \\
&- \left|(2-2\gamma)z+\sum_{m=2}^{\infty} m(u_m+\varepsilon v_m)z^m+(1-2\gamma)\sum_{m=2}^{\infty} C_m z^m\right| \\
&\le \sum_{m=2}^{\infty} m\left|u_m+\varepsilon v_m\right||z|^m+\sum_{m=2}^{\infty}|C_m||z|^m \\
&- \left((2-2\gamma)|z|-\sum_{m=2}^{\infty} m\left|u_m+\varepsilon v_m\right||z|^m-|1-2\gamma|\sum_{m=2}^{\infty}|C_m||z|^m\right) \\
&< \left\{-2(1-\gamma)+\sum_{m=2}^{\infty} 2m\left|u_m+\varepsilon v_m\right|+\left(|1-2\gamma|+1\right)\sum_{m=2}^{\infty}|C_m|\right\}|z|
\end{aligned}$$

Since $\varepsilon$ ($|\varepsilon|=1$) is arbitrary, and from the inequality (9), we obtain that $A<0$. This implies that $F_\varepsilon=\mathfrak{u}+\varepsilon\mathfrak{v}$ belongs to the class $\mathcal{K}(k,\gamma)$, and consequently, according to Theorem 2, it shows that $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}$ belongs to the class $\mathcal{KH}^0(k,\gamma)$. □

The result we present now provides the distortion bounds for functions in the class $\mathcal{KH}^0(k,\gamma)$.

**Theorem 3.6.** *Assuming $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}\in\mathcal{KH}^0(k,\gamma)$, the following inequalities hold for all $z$:*

$$|z|+\sum_{m=2}^{\infty}(-1)^{m-1}\left[m(1-\gamma)+\gamma\right]|z|^m\le|\mathfrak{f}(z)|\le|z|+\sum_{m=2}^{\infty}\left[m(1-\gamma)+\gamma\right]|z|^m.$$

*These are sharp inequality for the function $\mathfrak{f}(z)=z+\sum_{m=2}^{\infty}[m(1-\gamma)+\gamma]z^m$.*

*Proof.* Let $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}\in\mathcal{KH}^0(k,\gamma)$. Then using Theorem 3.1, $F_\varepsilon=\mathfrak{u}+\varepsilon\mathfrak{v}\in\mathcal{K}(k,\gamma)$ for each $\varepsilon$ ($|\varepsilon|=1$). Additionally, from Theorem 3.4 in [5], we obtain

$$\frac{1-(1-2\gamma)|z|}{(1+|z|)^3}\le\left|F_\varepsilon'(z)\right|\le\frac{1+(1-2\gamma)|z|}{(1-|z|)^3} \tag{10}$$

Since

$$\begin{aligned}\left|F_{\varepsilon}'(z)\right| &= \left|\mathfrak{u}'(z)+\varepsilon\mathfrak{v}'(z)\right| \\ &\leq 1+\sum_{m=2}^{\infty} m[m(1-\gamma)+\gamma]\,|z|^{m-1}\end{aligned}$$

and

$$\begin{aligned}\left|F_{\varepsilon}'(z)\right| &= \left|\mathfrak{u}'(z)+\varepsilon\mathfrak{v}'(z)\right| \\ &\geq 1+\sum_{m=2}^{\infty}(-1)^{m-1} m[m(1-\gamma)+\gamma]\,|z|^{m-1},\end{aligned}$$

in particular, we get

$$\left|\mathfrak{u}'(z)\right|+\left|\mathfrak{v}'(z)\right| \leq 1+\sum_{m=2}^{\infty} m[m(1-\gamma)+\gamma]\,|z|^{m-1}$$

and

$$\left|\mathfrak{u}'(z)\right|-\left|\mathfrak{v}'(z)\right| \geq 1+\sum_{m=2}^{\infty}(-1)^{m-1} m[m(1-\gamma)+\gamma]\,|z|^{m-1}.$$

Assume $\Gamma$ is the radial segment extending from 0 to $z$, so

$$\begin{aligned}|\mathfrak{f}(z)| &= \left|\int_{\Gamma}\frac{\partial\mathfrak{f}}{\partial\mathfrak{z}}d\mathfrak{z}+\frac{\partial\mathfrak{f}}{\partial\bar{\mathfrak{z}}}d\bar{\mathfrak{z}}\right| \leq \int_{\Gamma}\left(\left|\mathfrak{u}'(\mathfrak{z})\right|+\left|\mathfrak{v}'(\mathfrak{z})\right|\right)|d\mathfrak{z}| \\ &\leq \int_{0}^{|z|}\left(1+\sum_{m=2}^{\infty} m[m(1-\gamma)+\gamma]\,|\rho|^{m-1}\right)d\rho \\ &= |z|+\sum_{m=2}^{\infty}[m(1-\gamma)+\gamma]\,|z|^{m},\end{aligned}$$

and

$$\begin{aligned}|\mathfrak{f}(z)| &\geq \int_{\Gamma}\left(\left|\mathfrak{u}'(\mathfrak{z})\right|-\left|\mathfrak{v}'(\mathfrak{z})\right|\right)|d\mathfrak{z}| \\ &\geq \int_{0}^{|z|}\left(1+\sum_{m=2}^{\infty}(-1)^{m-1} m[m(1-\gamma)+\gamma]\,|\rho|^{m-1}\right)d\rho \\ &= |z|+\sum_{m=2}^{\infty}(-1)^{m-1}[m(1-\gamma)+\gamma]\,|z|^{m}.\end{aligned}$$

□

**Theorem 3.7.** *The class $\mathcal{KH}^0(k,\gamma)$ is closed under convex combinations.*

*Proof.* Suppose $\mathfrak{f}_\alpha=\mathfrak{u}_\alpha+\overline{\mathfrak{v}_\alpha}\in\mathcal{KH}^0(k,\gamma)$ for $\alpha=1,2,...,p$ and $\sum_{\alpha=1}^{p} s_\alpha=1$ $(0\leq s_\alpha\leq 1)$. The convex combination of functions $\mathfrak{f}_\alpha$ $(\alpha=1,2,...,p)$ can be expressed as:

$$\mathfrak{f}(z)=\sum_{\alpha=1}^{p} s_\alpha\mathfrak{f}_\alpha(z)=\mathfrak{u}(z)+\overline{\mathfrak{v}(z)},$$

where

$$\mathfrak{u}(z)=\sum_{\alpha=1}^{p}s_\alpha\mathfrak{u}_\alpha(z)\quad\text{and}\quad\mathfrak{v}(z)=\sum_{\alpha=1}^{p}s_\alpha\mathfrak{v}_\alpha(z).$$

Both $\mathfrak{u}$ and $\mathfrak{v}$ are analytic within the open unit disk $\mathbb{E}$, satisfying initial conditions $\mathfrak{u}(0)=\mathfrak{v}(0)=\mathfrak{u}'(0)-1=\mathfrak{v}'(0)=0$ and

$$\operatorname{Re}\left[\frac{z^k\mathfrak{u}'(z)}{\phi_k(z)}-\gamma\right]=\operatorname{Re}\left[\sum_{\alpha=1}^{p}s_\alpha\left(\frac{z^k\mathfrak{u}'_\alpha(z)}{\phi_k(z)}-\gamma\right)\right]>\sum_{\alpha=1}^{p}s_\alpha\left|\frac{z^k\mathfrak{v}'_\alpha(z)}{\phi_k(z)}\right|=\left|\frac{z^k\mathfrak{v}'(z)}{\phi_k(z)}\right|$$

showing that $\mathfrak{f}\in\mathcal{KH}^0(k,\gamma)$. □

## 4. Conclusions

In this paper, we introduced a new class of harmonic functions denoted by $\mathcal{KH}^0(k,\gamma)$. We established a relationship between $\mathcal{KH}^0(k,\gamma)$ and $\mathcal{K}(k,\gamma)$. We demonstrated that $\mathcal{KH}^0(k,\gamma)$ is close-to-convex. For functions in the $\mathcal{KH}^0(k,\gamma)$ class, we derived coefficient bounds and distortion theorems. Finally, we proved that $\mathcal{KH}^0(k,\gamma)$ is closed under convolution.

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