

EXISTENCE OF SOLUTION FOR INFINITE SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATION IN THE SEQUENCE SPACE $n(\phi)$ USING MEIR-KEELER CONDENSING OPERATOR

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ABSTRACT. In this paper, we investigate the existence of solutions of an infinite system of fractional differential equations in the sequence spaces $n(\phi)$. We use the Meir-Keeler condensing operator concept in conjunction with the Hausdorff measure of noncompactness to study the solvability, and we include suitable examples to demonstrate our findings. Additionally, we incorporate a combined semi-analytical method of modified Homotopy Perturbation Method along with the Adomian Decomposition Method to approximate solutions for the system. The examples presented further substantiate the effectiveness of the proposed approach while enabling practical estimation of solutions to the infinite system.

Keywords: Fractional Differential Equations(FDE), Measure of Noncompactness(MNC), Meir-Keeler Condensing Operator, Sequence Space, Adomian Decomposition Method, Homotopy Perturbation Method, Fixed Point Theory.

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1. INTRODUCTION

Non-linear analysis makes extensive use of the measure of noncompactness (MNC) theory. In addition to fixed point theory, it has been utilised to investigate operator theories, differential equation theories, and numerous other forms of calculus equations. It was Kuratowski [1] who initially proposed the idea of MNC in 1930. Since then, a large number of writers have used the idea of MNC in diverse mathematical fields. Agarwal et al. [11] discussed about MNC to determine whether fractional differential equations have a solution. For the solvability of an infinite system of fractional differential equations in c_0 and ℓ_p was found using the notion of MNC by Mursaleen et al. [18]. Banas and Lecko [8] proved the solvability of infinite systems of differential equations in Banach sequence spaces with the help of MNC. Mehravaran et al. [21] used the idea of MNC to examine the existence of solutions of an infinite system of fractional differential equations in $m^\beta(\phi)$.

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Alotaibi et al. [19] used MNC to check the solvability of second order linear differential equations in the sequence space $n(\phi)$. Sapkota et al. ([22], [23]) discussed the existence of solutions of an infinite system of third order three point boundary value problems in the sequence spaces bv_0, c_0 and ℓ_1 . In addition to this, one can visit [13], [15], [17], [20] for a variety of MNC applications.

Unlike classical derivatives, fractional derivatives involve non-integer orders. They are powerful because they capture memory and hereditary properties of materials and process, which classical models often cannot. The significance lies in providing a rigorous mathematical foundation for modeling and analyzing complex, real-world phenomena that involve infinitely many interacting components, memory effects, and non-linear dynamics. Establishing solvability ensures that such models are not just formal but have meaningful and applicable solutions.

Inspire by the recent discussion of the solvability of an infinite system of general order differential equation in sequence space $n(\phi)$ via operator type contraction and MNC by Jan and Jalal [24], here we address the solvability of the following infinite system of non-linear fractional differential equations (FDE) in the sequence space $n(\phi)$.

$$\begin{cases} D^\alpha v_n(t) = g_n(t, v(t)), & t \in (0, T) \\ v_n(0) = v_n^0 = 0, v_n(T) = av_n(\xi), & n \in \mathbb{N} \\ 1 < \alpha < 2, 0 < \xi < 1, a\xi^{\alpha-1} < T^{\alpha-1}, \end{cases} \quad (1)$$

where $v_n(t)$ is a differentiable function of class $C^{[\alpha]+1}$. Also $\{v_n(t)\}_{n=1}^\infty = v(t)$, $\{v_n(0)\}_{n=1}^\infty = v_0$, $\{v_n(\xi)\}_{n=1}^\infty = v(\xi)$ and $\{g_n(t, v(t))\}_{n=1}^\infty = g(t, v(t))$ which is an element of the Banach sequence space $n(\phi)$, $\forall n \in \mathbb{N}$.

Furthermore, following the approach mention in [25], we implement a semi-analytical iterative technique that integrates the modified HPM(Homotopy Perturbation Method) and ADM(Adomian Decomposition Method) to estimate the solutions of the system.

In this article, we briefly discuss some preliminary results and notation in Section 2. In Section 3, we institute the abstraction of Hausdorff measure of noncompactness on Banach spaces. In Section 4, we introduce the concept of fractional differential equations. In Section 5, we mention the Hausdorff measure of noncompactness on $n(\phi)$. In Section 6, we check the solvability of infinite system (1) in $n(\phi)$. To highlight the importance of our main result, we provide examples in Section 7. In Section 8, we use a semi-analytical iterative technique of modified HPM and ADM to approximate the solutions of the infinite system of fractional differential equations mentioned in Section 7.

2. PRELIMINARIES

Assume that the real Banach space $\mathcal{E}$ has the norm $\|\cdot\|$. Let $\mathbb{B}(x_o, d)$ represent a closed ball with radius d and center at x_o in $\mathcal{E}$. For Y , which is a subset of $\mathcal{E}$, $\bar{Y}$ and $\text{Conv}(Y)$ will stand for closure and convex closure, respectively. Moreover, let $\mathfrak{R}_{\mathcal{E}}$ be its subfamily made up of all relatively compact sets, and $\mathfrak{P}_{\mathcal{E}}$ be the family of all non-empty, bounded subsets of $\mathcal{E}$. Furthermore, let the subfamily of all bounded closed sets be $\mathfrak{P}_{\mathcal{E}}^C \subseteq \mathfrak{P}_{\mathcal{E}}$.

Definition 2.1. [8] For a Banach space $\mathcal{E}$, a function $\gamma : \mathfrak{P}_{\mathcal{E}} \rightarrow \mathbb{R}_+ \cup \{0\}$ is called a measure of noncompactness (MNC) in $\mathcal{E}$ if it satisfies the following axioms:

- (1) The family $\ker \gamma = \{F \in \mathfrak{P}_{\mathcal{E}} : \gamma(F) = 0\}$ is non-empty and $\ker \gamma \subset \mathfrak{R}_{\mathcal{E}}$,
- (2) $F_1 \subset F_2 \Rightarrow \gamma(F_1) \leq \gamma(F_2)$,
- (3) $\gamma(\overline{F}) = \gamma(F)$,
- (4) $\gamma(\text{Conv}(F)) = \gamma(F)$,
- (5) $\gamma(\lambda F_1 + (1 - \lambda)F_2) \leq \lambda\gamma(F_1) + (1 - \lambda)\gamma(F_2)$, for all $\lambda \in (0, 1)$,
- (6) If (I_n) is a decreasing sequence in $\mathfrak{P}_{\mathcal{E}}^C$ and if $(\gamma(I_n))$ is in c_0 , then there exists at least one $i \in \mathcal{E}$ such that $i \in (I_n)$, for all $n \in \mathbb{N}$.

The family $\ker \gamma$ is known as the kernel of measure γ . Measure γ is said to be sublinear if it satisfies the following conditions:

- (1) $\gamma(\lambda F) = |\lambda|\gamma(F)$ for $\lambda \in \mathbb{R}$,
- (2) $\gamma(F_1 + F_2) \leq \gamma(F_1) + \gamma(F_2)$.

A sublinear measure of noncompactness satisfied the following property:

$$\gamma(F_1 \cup F_2) = \max \{\gamma(\lambda F_1), \gamma(\lambda F_2)\}.$$

Definition 2.2. [6] Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two Banach spaces. γ_1 and γ_2 be two MNCs on $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively. An operator $\mathfrak{F}$ from $\mathcal{E}_1$ to $\mathcal{E}_2$ is called (γ_1, γ_2) -condensing operator if

- (1) $\mathfrak{F}$ is continuous,
- (2) $\gamma_2(\mathfrak{F}(\mathfrak{C})) < \gamma_1(\mathfrak{C})$, where $\mathfrak{C}$ is bounded as well as non compact subset of $\mathcal{E}_1$.

A (γ_1, γ_2) -condensing operator becomes γ -condensing operator when $\gamma_1 = \gamma_2 = \gamma$ and is defined on the same Banach space.

Theorem 2.1. [2] For a Banach space $\mathcal{E}$ with an arbitrary MNC γ with $\mathcal{A} \in \mathfrak{P}_{\mathcal{E}}^C$. A continuous mapping $\mathfrak{G} : \mathcal{A} \mapsto \mathcal{A}$ contains a fixed point in $\mathcal{A}$ if

- (1) $\mathcal{A}$ is convex,
- (2) $\gamma(\mathfrak{G}(\mathcal{A})) \leq \tau\gamma(\mathcal{A})$ for some $\tau \in [0, 1)$.

Definition 2.3. [5] Let $(\mathfrak{M}, \sigma)$ be a metric space. A mapping $\mathcal{I}$ on $\mathfrak{M}$ is said to be a Meir-Keeler contraction if for each $\epsilon > 0$, there exists $\delta > 0$ with $\sigma(\mathcal{I}u, \mathcal{I}v) < \epsilon$ whenever $\epsilon \leq \sigma(u, v) < \epsilon + \delta$, for all $u, v \in \mathfrak{M}$.

Theorem 2.2. [5] A Meir-Keeler contraction mapping $\mathcal{I}$ on a complete metric space $(\mathfrak{M}, \sigma)$ always has a unique fixed point.

Definition 2.4. [16] Let noncompactness $\mathcal{E}$ be a Banach space, with an arbitrary MNC γ and $\mathcal{D}$ be an non-empty subset of $\mathcal{E}$. A Meir-Keeler condensing operator on $\mathcal{D}$ is defined as an operator $\mathcal{J} : \mathcal{D} \mapsto \mathcal{D}$ such that for each $\epsilon > 0$, there exists $\delta > 0$ with $\gamma(\mathcal{J}(\mathcal{D}_1)) < \epsilon$ whenever $\epsilon \leq \gamma(B) < \epsilon + \delta$ for every bounded subset $\mathcal{D}_1$ of $\mathcal{D}$.

Theorem 2.3. [16] Suppose $\mathcal{E}$ be a Banach space, with an arbitrary MNC γ . $\mathcal{B}$ be an non-empty, bounded, closed and convex subset of it. If $\mathcal{J}$ be a continuous and Meir-Keeler condensing operator on $\mathcal{B}$, then $\mathcal{J}$ has at least one fixed point and the set of all these fixed points of $\mathcal{J}$ in $\mathcal{B}$ are compact.

Let $\mathcal{J} = [0, N]$ and let $C(\mathcal{J}, \mathcal{E})$ denote the Banach space of all continuous functions defined on $\mathcal{J}$ with values in the space $\mathcal{E}$. The space $C(\mathcal{J}, \mathcal{E})$ equipped with the norm

$$\|y\|_c = \sup\{\|y(t)\| : t \in \mathcal{J}\}.$$

3. THE HAUSDORFF MEASURE OF NONCOMPACTNESS(H-MNC) ON BANACH SPACES

The Hausdorff measure of noncompactness (H-MNC), first introduced by Goldenštein and Markus [4] in 1965, is one of the many measures of noncompactness that have been developed over the years. It is an important mathematical tool in non-linear analysis with several applications.

Definition 3.1. Let $\mathfrak{S}$ be a bounded subset of a metric space $(\mathfrak{M}, \sigma)$. Then the Hausdorff measure of noncompactness (H-MNC) of $\mathfrak{S}$, denoted by $\chi(\mathfrak{S})$, is defined as

$$\chi(\mathfrak{S}) = \inf \left\{ \epsilon > 0 : \mathfrak{S} \subset \bigcup_{i=1}^n B(z_i, r_i), z_i \in \mathfrak{M}, r_i < \epsilon, 1 \leq i \leq n, n \in \mathbb{N} \right\},$$

where, $B(z_i, r_i)$ is an open ball in $\mathfrak{M}$.

For the following results, we refer to [15].

Lemma 3.1. Let $\mathcal{M}$, $\mathcal{M}_1$ and $\mathcal{M}_2$ be bounded subsets of a complete metric space $(\mathfrak{M}, \sigma)$. Then

- (1) $\chi(\mathcal{M}) = 0$ if and only if $\mathcal{M}$ is totally bounded,
- (2) $\chi(\mathcal{M}) = \chi(\overline{\mathcal{M}})$,
- (3) $\mathcal{M}_1 \subset \mathcal{M}_2 \Rightarrow \chi(\mathcal{M}_1) \leq \chi(\mathcal{M}_2)$,
- (4) $\chi(\mathcal{M}_1 \cup \mathcal{M}_2) = \max\{\chi(\mathcal{M}_1), \chi(\mathcal{M}_2)\}$,
- (5) $\chi(\mathcal{M}_1 \cap \mathcal{M}_2) \leq \min\{\chi(\mathcal{M}_1), \chi(\mathcal{M}_2)\}$.

Lemma 3.2. Let $\mathcal{M}$, $\mathcal{M}_1$ and $\mathcal{M}_2$ be bounded subsets of a normed space $\mathfrak{M}$. Then

- (1) $\chi(\mathcal{M}_1 + \mathcal{M}_2) \leq \chi(\mathcal{M}_1) + \chi(\mathcal{M}_2)$,
- (2) $\chi(\mathcal{M} + x) = \chi(\mathcal{M})$ for every $x \in \mathfrak{M}$,
- (3) $\chi(\lambda\mathcal{M}) = |\lambda|\chi(\mathcal{M})$, for all scalar λ ,
- (4) $\chi(\mathcal{M}) = \chi(\text{Conv}(\mathcal{M}))$.

Lemma 3.3. Let $\mathfrak{M}$ be an infinite-dimensional normed space, and let $\mathcal{B}_{\mathfrak{M}}$ be the closed unit ball of $\mathfrak{M}$. Then $\chi(\mathcal{B}_{\mathfrak{M}}) = 1$.

4. FRACTIONAL DIFFERENTIAL EQUATIONS

The theory of fractional calculus is regarded as the integer order calculus's generalization. Remarkable mathematicians Liouville and Riemann made the first formal presentation of it in the nineteenth century. Many new applications in a wide range of fields, including control theory, polymer science, image processing, and more, can be made using the theory of fractional differential and integral equations. Furthermore, integral and fractional differential equations are created through modeling certain human behaviors. Many writers have investigated the existence of solutions for distinct fractional differential equations. For various applications, we refer the reader to visit [10], [11], [12], [18], [21].

Now we recall some basic results related to fractional calculus.

Definition 4.1. [7] The Fractional integral of order α is defined as

$$I^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s)}{(t-s)^{1-\alpha}} ds, \quad \alpha > 0$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 4.2. [7] *The Caputo fractional derivative of order $\alpha > 0$ and for atleast n -times differentiable function $g : [0, \infty) \rightarrow \mathbb{R}$ is given by -*

$${}^c D^\alpha g(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t \frac{g^n(s)}{(t - s)^{\alpha - n + 1}} ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ is the integer part of the real number α .

Proposition 4.1. [18] *For a given function $g \in C[0, T]$, the unique solution of*

$${}^c D^\alpha v(t) = g(t), v(0) = 0, v(T) = av(\xi), \quad 1 < \alpha < 2,$$

is given by

$$v(t) = \int_0^T \mathcal{L}(t, s) g(s) ds,$$

where $\mathcal{L}$ is the Green's function which is given by

$$\mathcal{L}(t, s) = \frac{1}{\Gamma(\alpha)(T^{\alpha-1} - a\xi^{\alpha-1})} \begin{cases} \mathcal{L}_1(t, s) & 0 \leq t \leq \xi, \\ \mathcal{L}_2(t, s) & \xi \leq t \leq T. \end{cases}$$

$$\mathcal{L}_1(t, s) = \begin{cases} (t - s)^{\alpha-1}(T^{\alpha-1} - a\xi^{\alpha-1}) - t^{\alpha-1}[(T - s)^{\alpha-1} - a(\xi - s)^{\alpha-1}] & 0 \leq s \leq t, \\ -t^{\alpha-1}[(T - s)^{\alpha-1} - a(\xi - s)^{\alpha-1}] & t \leq s \leq \xi, \\ -(t(T - s))^{\alpha-1} & \xi \leq s \leq T. \end{cases}$$

$$\mathcal{L}_2(t, s) = \begin{cases} (t - s)^{\alpha-1}(T^{\alpha-1} - a\xi^{\alpha-1}) - t^{\alpha-1}[(T - s)^{\alpha-1} - a(\xi - s)^{\alpha-1}] & 0 \leq s \leq \xi, \\ (t - s)^{\alpha-1}(T^{\alpha-1} - a\xi^{\alpha-1}) - (t(T - s))^{\alpha-1} & \xi \leq s \leq t, \\ -(t(T - s))^{\alpha-1} & t \leq s \leq T. \end{cases}$$

Remark 4.1. [18] *It can be showed that the Green's function $\mathcal{L}(t, s)$ can be defined on $[0, T] \times [0, T]$ as $\mathcal{L}_1 : [0, \xi] \times [0, T] \rightarrow \mathbb{R}$ and $\mathcal{L}_2 : [\xi, T] \times [0, T] \rightarrow \mathbb{R}$ is continuous w.r.t. t and s .*

In [19], A. Alotaibi et al. discussed the solvability of a second order linear differential equation in the sequence space $n(\phi)$ using Hausdorff measure of noncompactness. Here, we consider an infinite system of non-linear fractional differential equation. This broader framework not only generalizes the linear case but also provides a more realistic and flexible tool for modeling complex physical and biological processes. Unlike a classical second-order linear differential equation, the infinite system of non-linear fractional differential equations incorporate memory effects, infinite-dimensional interactions, and non-linear dynamics.

In contrast to earlier approach [21], where the solvability of infinite systems of nonlinear fractional differential equations (1) examine in weaker setting sequence space $m^\beta(\phi)$, the present work advances the theory by establishing results in the stronger sequence space $n(\phi)$. This generalization not only subsumes the weaker framework but also ensures enhanced stability, regularity, and physical interpretability of the solutions, thereby contributing to a more robust mathematical foundation for the analysis of fractional system.

5. H-MNC ON $n(\phi)$

let $\mathcal{C}$ be the space consisting finite set of distinct positive integers. For any $\sigma \in \mathcal{C}$, $c(\sigma)$ denote the sequence $c_n(\sigma)$ such that

$$c_n(\sigma) = \begin{cases} 1 & n \in \sigma, \\ 0 & n \notin \sigma. \end{cases}$$

Further, suppose

$$\mathcal{C}_\zeta = \{\sigma \in \mathcal{C} : \sum_{n=1}^{\infty} c_n(\sigma) \leq \zeta\},$$

and

$$\Phi = \{\phi = (\phi_n) \in l^0 : 0 < \phi_1 \leq \phi_n \leq \phi_{n+1}, (n+1)\phi_n \geq n\phi_{n+1}\}.$$

Sargent [3] was the first who introduced the sequence space $n(\phi)$ which were further studied by Mursaleen [9]. Recently Alotaibi et al.[19], Jan and Jalal [24] introduced and studied variuos properties of $n(\phi)$.

Suppose $S(x)$ is the set of all sequence that are rearrangement of x . Then for $\phi \in \Phi$

$$n(\phi) = \left\{ x = (x_k) : \|x\|_{n(\phi)} = \sup_{v \in S(x)} \left(\sum_{k=1}^{\infty} |v_k| \Delta \phi_k \right) < \infty \right\},$$

where $\Delta \phi_k = \phi_k - \phi_{k-1}$.

Theorem 5.1. [14] *For any subset T of $n(\phi)$, the Hausdorff Measure of noncompactness is given by*

$$\chi(T) = \lim_{k \rightarrow \infty} \sup_{x \in A} \left(\sup_{v \in S(x)} \left(\sum_{n=k}^{\infty} |v_n| \Delta \phi_n \right) \right).$$

6. SOLVABILITY OF INFINTE SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATION(1) IN THE SEQUENCE SPACE $n(\phi)$

In this section, we look into the solvability of infinite system of fractional differential equation (1). Consider the following suppositions:

- (1) $\{v_i(0)\}_{i=1}^{\infty} = 0$ and $\{v_i(\xi)\}_{i=1}^{\infty} \in n(\phi)$.
- (2) $f = (f_1, f_2, f_3, \dots)$ continuously transforms the set $I \times n(\phi)$ to $n(\phi)$. The family of function $(f(v))$ is equicontinuous at each point of $n(\phi)$.
- (3) Consider the real and continuous function $(p_n(t)), (q_n(t))$ on I so that they fulfill the inequality -

$$|g_n(t, v(t))| \leq p_n(t) + q_n(t)|v_n(t)| ; n \in \mathbb{N}$$

such that $\sum_{k=1}^{\infty} p_k \Delta \phi_k$ converges uniformly on I and $q_n(t)$ is equibounded on I .

Theorem 6.1. *Under the suppositions (1) - (3), the infinite system of fractional differential equation (1) has at least one solution $v(t) = (v_i(t))$ whenever $LTQ \leq 1$. Where $L = \sup_{t,s \in I} |\mathcal{L}(t,s)|$ and $\sup_{t \in I} q_i(t) \leq Q$.*

Proof. Suppose $S(v(t))$ denote the sequence that are rearrangement of $v(t)$ and $v'(t) \in S(v(t))$.

Define an operator $\mathfrak{G} : C(I, n(\phi)) \rightarrow C(I, n(\phi))$ as follows :

$$(\mathfrak{G}v)(t) = \int_0^T \mathcal{L}(t,s)g(s, v(s))ds.$$

By our supposition (2), $\mathcal{G}$ is well defined on $C(I, n(\phi))$.

Now, we will claim that $\mathcal{G}$ is bounded in the classical supremum norm $\|v\| = \sup_{t \in I} \|v(t)\|_{n(\phi)}$

$$\begin{aligned} \|(\mathfrak{G}v)(t)\| &= \left\| \int_0^T \mathcal{L}(t,s)g(s, v(s))ds \right\|_{n(\phi)} \\ &= \sup_{v' \in S(v(t))} \left\{ \sum_{k=1}^{\infty} \left| \int_0^T \mathcal{L}(t,s)g_k(s, v(s))ds \right| \Delta \phi_k \right\} \\ &\leq \sup_{v' \in S(v(t))} \left\{ \sum_{k=1}^{\infty} \int_0^T |\mathcal{L}(t,s)| |g_k(s, v(s))| ds \Delta \phi_k \right\} \\ &\leq \sup_{v' \in S(v(t))} \left\{ \sum_{k=1}^{\infty} \int_0^T |\mathcal{L}(t,s)| (p_k + q_k |v'_k(t)|) ds \Delta \phi_k \right\} \\ &\leq L \int_0^T \sup_{v' \in S(v(t))} \sum_{k=1}^{\infty} (p_k + q_k |v'_k(t)|) ds \Delta \phi_k \\ &\leq L \int_0^T \left\{ \sup_{v' \in S(v(t))} \sum_{k=1}^{\infty} p_k \Delta \phi_k \right\} ds + L \int_0^T \left\{ \sup_{v' \in S(v(t))} \sum_{k=1}^{\infty} q_k |v'_k(t)| \Delta \phi_k \right\} ds \\ &\leq L \int_0^T \left\{ \sup_{v' \in S(v(t))} \sum_{k=1}^{\infty} p_k \Delta \phi_k \right\} ds + LTQ \|v\|_{n(\phi)}. \end{aligned}$$

The above inequality can be written as

$$\sup_{t \in I} \|(\mathfrak{G}v)(t)\|_{n(\phi)} \leq LT\overline{\mathfrak{P}} + LTQ \|v\|_{C(I, n(\phi))},$$

where

$$\sup_{t \in I} \left\{ \sup_{v' \in S(v(t))} \left(\sum_{k=1}^{\infty} p_k \Delta \phi_k \right) \right\} = \overline{\mathfrak{P}}, \quad \forall t \in I.$$

Let d_0 is an optimal solution of the following inequality

$$d \leq LT(\overline{\mathfrak{P}} + Qd).$$

Consider a convex, closed and bounded subset of $C(I, n(\phi))$

$$\mathbb{B} = \mathbb{B}(v_0, r_0) = \{v(t) \in C(I, n(\phi)) : \|v\|_{C(I, n(\phi))} \leq d, v(0) = 0, v(T) = av(\xi)\}.$$

Now, we will show that $\mathcal{G}$ is continuous on $\mathbb{B}$. To achieve this, let we choose arbitrary fixed w in $\mathbb{B}$ and $\epsilon > 0$ be given. By the supposition (2), $\exists \delta > 0$ so that for $v \in \mathbb{B}$ and

$\|v - w\|_{C(I, n(\phi))} \leq \delta$, implies $\|(gv) - (gw)\|_{C(I, n(\phi))} \leq \frac{\epsilon}{LT}$. Thus for each $t \in [0, T]$. We get -

$$\begin{aligned}
 \|(\mathfrak{G}v)(t) - (\mathfrak{G}w)(t)\|_{n(\phi)} &= \sup_{v' \in S(v), w' \in S(w)} \left\{ \sum_{k=1}^{\infty} \int_0^T |\mathcal{L}(t, s)g_k(s, v(s))ds \right. \\
 &\quad \left. - \int_0^T \mathcal{L}(t, s)g_k(s, w(s))ds| \Delta\phi_k \right\} \\
 &\leq \sup_{v' \in S(v), w' \in S(w)} \left\{ \sum_{k=1}^{\infty} \int_0^T |\mathcal{L}(t, s)| |g_k(s, v(s)) \right. \\
 &\quad \left. - g_k(s, w(s))| ds \Delta\phi_k \right\} \\
 &\leq LT \sup_{v' \in S(v), w' \in S(w)} \left\{ \sum_{k=1}^{\infty} \int_0^T |g_k(s, v(s)) - g_k(s, w(s))| ds \Delta\phi_k \right\} \\
 &\leq LT \left\{ \|g(v) - g(w)\|_{C(I, n(\phi))} \right\} \\
 &\leq LT \cdot \frac{\epsilon}{LT} = \epsilon.
 \end{aligned}$$

Next, we show that $\mathfrak{G}$ is continuous operator. Let $t_0 \in (0, T)$ and $\epsilon > 0$ are arbitrary. By using the continuity of $\mathcal{L}(t, s)$ w.r.t. t , we get $\delta > 0$ such that for $|t - t_0| < \delta$,

$$|\mathcal{L}(t, s) - \mathcal{L}(t_0, s)| < \frac{\epsilon}{T(\overline{\mathfrak{P}} + \mathcal{Q}\|v\|_{C(I, n(\phi))})}.$$

In view of our supposition (3),

$$\begin{aligned}
 \|(\mathfrak{G}v)(t) - (\mathfrak{G}v)(t_0)\|_{n(\phi)} &= \sup_{v' \in S(v)} \left\{ \sum_{k=1}^{\infty} \left| \int_0^T \mathcal{L}(t, s)g_k(s, v(s))ds \right. \right. \\
 &\quad \left. \left. - \int_0^T \mathcal{L}(t_0, s)g_k(s, v(s))ds \right| \Delta\phi_k \right\} \\
 &\leq \sup_{v' \in S(v)} \left\{ \sum_{k=1}^{\infty} \int_0^T |\mathcal{L}(t, s) - \mathcal{L}(t_0, s)| |g_k(s, v(s))| ds \Delta\phi_k \right\} \\
 &\leq \sup_{v' \in S(v)} \left\{ \sum_{k=1}^{\infty} \int_0^T |\mathcal{L}(t, s) - \mathcal{L}(t_0, s)| (p_k + q_k|v_k'(t)|) ds \Delta\phi_k \right\} \\
 &\leq \frac{\epsilon}{T(\overline{\mathfrak{P}} + \mathcal{Q}\|v\|_{n(\phi)})} \left\{ T \sup_t \left(\sup_{v' \in S(v)} \sum_{k=1}^{\infty} p_k(t) \Delta\phi_k \right) \right. \\
 &\quad \left. + T \sup_t \left(\sup_{v' \in S(v)} \sum_{k=1}^{\infty} q_k(t) |v_k'(t)| \Delta\phi_k \right) \right\} \\
 &\leq \frac{\epsilon}{T(\overline{\mathfrak{P}} + \mathcal{Q}\|v\|_{n(\phi)})} T \left\{ \overline{\mathfrak{P}} + \mathcal{Q}\|v\|_{n(\phi)} \right\} = \epsilon.
 \end{aligned}$$

Hence, $(\mathfrak{G}v)$ is continuous.

In order to finish the proof, next we will show that $\mathfrak{G}$ is Meir-Keeler Condensing operator. For $\mathbb{B} \subset C(I, n(\phi))$, the Hausdorff measure of noncompactness is define as -

$$\chi_{C(I, n(\phi))}(\mathbb{B}) = \sup_{t \in I} \chi_{n(\phi)}(\mathbb{B}(t)). \quad (2)$$

$$\begin{aligned} \chi_{n(\phi)}(\mathfrak{G}\mathbb{B})(t) &= \lim_{k \rightarrow \infty} \sup_{v \in \mathbb{B}} \left\{ \sup_{v' \in S(v)} \left(\sum_{i=1}^{\infty} |\mathfrak{G}v_i(t)| \Delta\phi_i \right) \right\} \\ &\leq \lim_{k \rightarrow \infty} \sup_{v \in \mathbb{B}} \left\{ \sup_{v' \in S(v)} \left(\sum_{i=1}^{\infty} \left| \int_0^T \mathcal{L}(t, s) g_i(s, v(s)) ds \right| \Delta\phi_i \right) \right\} \\ &\leq \lim_{k \rightarrow \infty} \sup_{v \in \mathbb{B}} \left\{ \sup_{v' \in S(v)} \left(\sum_{i=1}^{\infty} \int_0^T |\mathcal{L}(t, s)| |g_i(s, v(s))| \Delta\phi_i ds \right) \right\} \\ &\leq L \lim_{k \rightarrow \infty} \sup_{v \in \mathbb{B}} \left\{ \sup_{v' \in S(v)} \left(\sum_{i=1}^{\infty} (p_i(s) + q_i(s) |v_i'(s)|) \Delta\phi_i \right) \right\} \\ &\leq LTQ \lim_{k \rightarrow \infty} \sup_{v \in \mathbb{B}} \left\{ \sup_{v' \in S(v)} \left(\sum_{i=1}^{\infty} |v_i'(s)| \Delta\phi_i \right) \right\} \\ &\leq LTQ \chi_{C(I, n(\phi))}(\mathbb{B}). \end{aligned}$$

By condition (2), we have

$$\chi(\mathfrak{G}\mathbb{B}) \leq LTQ\chi(\mathbb{B}) \leq \epsilon.$$

Since $LTQ < 1$, which implies that $\mathfrak{G}$ is Meir-Keeler condensing operator. Thus, by Theorem(2.3), $\mathfrak{G}$ contains atleast one fixed point in $\mathbb{B}$, which acts as a solution of FDE (1) in the space $C(I, n(\phi))$. $\square$

7. APPLICATIONS

Example 7.1. Let we choose the following system of infinite fractional differential equations:

$$\begin{cases} D^{\frac{4}{3}}v_n(t) = \frac{\sqrt{t} \exp(-nt)}{(1+n)^2} + \sum_{m=n}^{\infty} \frac{\ln(1+t)v_m(t)}{(1+m)^3 n^3} & ; n \in \mathbb{N}, t \in [0, T] \\ v_n(0) = 0, v_n(T) = \sqrt[3]{4}v_n(\frac{T}{3}), \end{cases} \quad (3)$$

$$v(\frac{T}{3}) = \{v_n(\frac{T}{3})\}_{n=1}^{\infty} \in n(\phi), T < 1.$$

Here,

$$\xi = \frac{T}{3}, a = \sqrt[3]{4}, g_n(t, v(t)) = \frac{\sqrt{t} \exp(-nt)}{(1+n)^2} + \sum_{m=n}^{\infty} \frac{\ln(1+t)v_m(t)}{(1+m)^3 n^3}, \alpha = \frac{4}{3}.$$

The given system of FDE convert into the following equations:

$$v_n(t) = \frac{1}{\Gamma(\frac{4}{3})(\sqrt[3]{T} - \sqrt[3]{\frac{4T}{3}})} \int_0^T \mathcal{L}(t, s) g_n(s, v(s)) ds,$$

where

$$\mathcal{L}(t, s) = \frac{1}{\Gamma(\frac{4}{3})(\sqrt[3]{T} - \sqrt[3]{\frac{4T}{3}})} \begin{cases} \mathcal{L}_1(t, s) & 0 \leq t \leq \xi, \\ \mathcal{L}_2(t, s) & \xi \leq t \leq T. \end{cases}$$

$$\mathcal{L}_1(t, s) = \begin{cases} (t-s)^{\frac{1}{3}}(\sqrt[3]{T} - \sqrt[3]{\frac{4T}{3}}) - t^{\frac{1}{3}}[(T-s)^{\frac{1}{3}} - \sqrt[3]{\frac{4}{3}}(T-3s)^{\frac{1}{3}}] & 0 \leq s \leq t, \\ -t^{\frac{1}{3}}[(T-s)^{\frac{1}{3}} - \sqrt[3]{\frac{4}{3}}(T-3s)^{\frac{1}{3}}] & t \leq s \leq \xi, \\ -(t(T-s)^{\frac{1}{3}}) & \xi \leq s \leq T. \end{cases}$$

$$\mathcal{L}_2(t, s) = \begin{cases} (t-s)^{\frac{1}{3}}(\sqrt[3]{T} - \sqrt[3]{\frac{4T}{3}}) - t^{\frac{1}{3}}[(T-s)^{\frac{1}{3}} - \sqrt[3]{\frac{4}{3}}(T-3s)^{\frac{1}{3}}] & 0 \leq s \leq \xi, \\ (t-s)^{\frac{1}{3}}(\sqrt[3]{T} - \sqrt[3]{\frac{4T}{3}}) - (t(T-s)^{\frac{1}{3}}) & \xi \leq s \leq t, \\ -(t(T-s)^{\frac{1}{3}}) & t \leq s \leq T. \end{cases}$$

Again,

$$\begin{aligned} |g_n(t, v(t))| &= \left| \frac{\sqrt{t} \exp(-nt)}{(1+n)^2} + \sum_{m=n}^{\infty} \frac{\ln(1+t)v_m(t)}{(1+m)^3 n^3} \right| \\ &\leq \left| \frac{\sqrt{t} \exp(-nt)}{(1+n)^2} \right| + \sum_{m=n}^{\infty} \frac{t}{(1+m)^3 n^3} |v_m(t)| \\ &\leq \left| \frac{\sqrt{t}}{(1+n)^2} \right| + \sum_{m=n}^{\infty} \frac{t}{(1+m)^3 n^3} |v_m(t)| \\ &\leq p_n(t) + q_n(t) |v_m(t)| ; \quad \forall n \in \mathbb{N}, \end{aligned}$$

where the function $p_n(t)$ is continuous and the series $\sum_{n \geq 1} p_n(t)$ converges uniformly to $|t|\zeta(2)$ and $q_n(t)$ is equibounded by $T\zeta(3)$.

Next we will claim that $g(t, v(t)) \in n(\phi)$. For this, we consider any arbitrary $t \in [0, T]$ and $v \in n(\phi)$, then

$$\begin{aligned} \lim_{n \rightarrow \infty} g_n(t, v(t)) &= \lim_{n \rightarrow \infty} \left\{ \frac{\sqrt{t} \exp(-nt)}{(1+n)^2} + \sum_{m=n}^{\infty} \frac{\ln(1+t)v_m(t)}{(1+m)^3 n^3} \right\} \\ &\leq \lim_{n \rightarrow \infty} \left\{ \frac{\sqrt{t} \exp(-nt)}{(1+n)^2} + \sup_{m \geq n} |v_m(t)| \frac{T\zeta(3)}{n^3} \right\} \\ &\leq \lim_{n \rightarrow \infty} \left\{ \sup_{m \geq n} |v_m(t)| \frac{T\zeta(3)}{n^3} \right\} = 0. \end{aligned}$$

Now we will show that the family of function $\{(g(v))\}_{t \in I}$ is equicontinuous. Let $t \in I$, $w \in n(\phi)$ be arbitrarily fixed, consider any $\epsilon > 0$ such that

$$\begin{aligned} \|(gv)(t) - (gw)(t)\|_{n(\phi)} &= \sup_{v' \in S(v), w' \in S(w)} \left\{ \sum_{k=1}^{\infty} \left| \frac{\sqrt{t} \exp(-nt)}{(1+n)^2} + \sum_{m=n}^{\infty} \frac{\ln(1+t)v'_m(t)}{(1+m)^3 n^3} \right. \right. \\ &\quad \left. \left. - \frac{\sqrt{t} \exp(-nt)}{(1+n)^2} + \sum_{m=n}^{\infty} \frac{\ln(1+t)w'_m(t)}{(1+m)^3 n^3} \right| \Delta \phi_k \right\} \\ &\leq \sup_{v' \in S(v), w' \in S(w)} \left\{ \sum_{k=1}^{\infty} \left(\sum_{m=n}^{\infty} \frac{\ln(1+t)}{(1+m)^3 n^3} |v'_m(t) - w'_m(t)| \Delta \phi_k \right) \right\} \\ &\leq T\zeta(3) \sup_{v' \in S(v), w' \in S(w)} \left\{ \sum_{k=1}^{\infty} |v'_m(t) - w'_m(t)| \Delta \phi_k \right\} \\ &\leq T\zeta(3) \|v(t) - w(t)\|_{n(\phi)} < \epsilon. \end{aligned}$$

where $\|v(t) - w(t)\|_{n(\phi)} < \delta = \frac{1}{T\zeta(3)} \epsilon$.

Therefore, in view of Theorem (6.1), the operator $\mathfrak{G}$ has a fixed point $v(x) = v_i(x) \in n(\phi)$ which acts as a solution for the infinite system of fractional integral equations (3).

Example 7.2. Let we choose the following system of infinite fractional differential equations:

$$\begin{cases} D^{\frac{7}{6}} v_n(t) = \frac{t \cos(nt)}{n^3} + \sum_{m=n}^{\infty} \frac{\tanh(t)v_m(t)}{(1+m^4)n} & ; n \in \mathbb{N}, t \in [0, T] \\ v_n(0) = 0, v_n(T) = \sqrt[6]{7} v_n(\frac{T}{6}), \end{cases} \quad (4)$$

$$v(\frac{T}{6}) = \{v_n(\frac{T}{6})\}_{n=1}^{\infty} \in n(\phi).$$

Here,

$$\xi = \frac{T}{6}, \quad a = \sqrt[6]{7}, \quad g_n(t, v(t)) = \frac{t \cos(-nt)}{n^3} + \sum_{m=n}^{\infty} \frac{\tanh(t)v_m(t)}{(1+m^4)n}, \quad \alpha = \frac{7}{6}.$$

The given system of FDE convert into the following equations:

$$v_n(t) = \frac{1}{\Gamma(\frac{7}{6})(\sqrt[6]{T} - \sqrt[6]{\frac{7T}{6}})} \int_0^T \mathcal{L}(t, s) g_n(s, v(s)) ds,$$

where

$$\begin{aligned} \mathcal{L}(t, s) &= \frac{1}{\Gamma(\frac{7}{6})(\sqrt[6]{T} - \sqrt[6]{\frac{7T}{6}})} \begin{cases} \mathcal{L}_1(t, s) & 0 \leq t \leq \xi, \\ \mathcal{L}_2(t, s) & \xi \leq t \leq T. \end{cases} \\ \mathcal{L}_1(t, s) &= \begin{cases} (t-s)^{\frac{1}{6}}(\sqrt[6]{T} - \sqrt[6]{\frac{7T}{6}}) - t^{\frac{1}{6}}[(T-s)^{\frac{1}{6}} - \sqrt[6]{\frac{7}{6}}(T-6s)^{\frac{1}{6}}] & 0 \leq s \leq t, \\ -t^{\frac{1}{6}}[(T-s)^{\frac{1}{6}} - \sqrt[6]{\frac{7}{6}}(T-6s)^{\frac{1}{6}}] & t \leq s \leq \xi, \\ -(t(T-s)^{\frac{1}{6}}) & \xi \leq s \leq T. \end{cases} \end{aligned}$$

$$\mathcal{L}_2(t, s) = \begin{cases} (t-s)^{\frac{1}{6}}(\sqrt[6]{T} - \sqrt[6]{\frac{7T}{6}}) - t^{\frac{1}{6}}[(T-s)^{\frac{1}{6}} - \sqrt[6]{\frac{7}{6}}(T-6s)^{\frac{1}{6}}] & 0 \leq s \leq \xi, \\ (t-s)^{\frac{1}{6}}(\sqrt[6]{T} - \sqrt[6]{\frac{7T}{6}}) - (t(T-s))^{\frac{1}{6}} & \xi \leq s \leq t, \\ -(t(T-s))^{\frac{1}{6}} & t \leq s \leq T. \end{cases}$$

Again,

$$\begin{aligned} |g_n(t, v(t))| &= \left| \frac{t \cos(nt)}{n^3} + \sum_{m=n}^{\infty} \frac{\tanh(t)v_m(t)}{(1+m^4)n} \right| \\ &\leq \left| \frac{t \cos(nt)}{n^3} \right| + \sum_{m=n}^{\infty} \frac{1}{(1+m^4)n} |v_m(t)| \\ &\leq \left| \frac{t}{n^3} \right| + \frac{1}{n} \sum_{m=n}^{\infty} \frac{1}{(1+m^4)} |v_m(t)| \\ &\leq p_n(t) + q_n(t) |v_m(t)| ; \forall n \in \mathbb{N}, \end{aligned}$$

where the functions $p_n(t)$ are continuous and the series $\sum_{n \geq 1} p_n(t)$ converges uniformly to

$$|t|\zeta(3) \text{ and } q_n(t) \text{ is equibounded to } \zeta(4) = \frac{\pi^4}{90}.$$

Next we claim that $g(t, v(t)) \in n(\phi)$. For this, we consider any arbitrary $t \in [0, T]$ and $v \in n(\phi)$, then

$$\begin{aligned} \lim_{n \rightarrow \infty} g_n(t, v(t)) &= \lim_{n \rightarrow \infty} \left\{ \frac{t \cos(nt)}{n^3} + \sum_{m \geq n}^{\infty} \frac{\tanh(t)v_m(t)}{(1+m^4)n} \right\} \\ &\leq \lim_{n \rightarrow \infty} \left\{ \frac{t \cos(nt)}{n^3} + \sup_{m \geq n} |v_m(t)| \frac{\pi^4}{90n} \right\} \\ &\leq \lim_{n \rightarrow \infty} \left\{ \sup_{m \geq n} |v_m(t)| \frac{\pi^4}{90n} \right\} = 0. \end{aligned}$$

Now we show that the family of functions $\{(g(v))\}_{t \in I}$ are equicontinuous. Let $t \in I$, $w \in n(\phi)$ is arbitrarily fixed, consider any $\epsilon > 0$ such that

$$\begin{aligned} \|(gv)(t) - (gw)(t)\|_{n(\phi)} &= \sup_{v' \in S(v), w' \in S(w)} \left\{ \sum_{k=1}^{\infty} \left| \frac{t \cos(nt)}{n^3} + \sum_{m=n}^{\infty} \frac{\tanh(t)v'_m(t)}{(1+m^4)n} \right. \right. \\ &\quad \left. \left. - \frac{t \cos(-nt)}{n^3} - \sum_{m=n}^{\infty} \frac{\tanh(t)w'_m(t)}{(1+m^4)n} \right| \Delta \phi_k \right\} \\ &\leq \sup_{v' \in S(v), w' \in S(w)} \left\{ \sum_{k=1}^{\infty} \left(\sum_{m=n}^{\infty} \frac{\tanh(t)}{(1+m^4)n} |v'_m(t) - w'_m(t)| \Delta \phi_k \right) \right\} \\ &\leq \frac{\pi^4}{90} \sup_{v' \in S(v), w' \in S(w)} \left\{ \sum_{k=1}^{\infty} |v'_m(t) - w'_m(t)| \Delta \phi_k \right\} \\ &\leq \frac{\pi^4}{90} \|v(t) - w(t)\|_{n(\phi)} < \epsilon. \end{aligned}$$

where $\|v(t) - w(t)\|_{n(\phi)} < \delta = \frac{90}{\pi^4} \epsilon$.

Therefore, in view of Theorem (6.1), the operator $\mathfrak{G}$ has a fixed point $v(x) = v_i(x) \in n(\phi)$ which acts as a solution for the infinite system of fractional differential equations (4).

8. SEMI-ANALYTICAL HPM AND ADM ANALYSIS OF THE INFINITE FRACTIONAL SYSTEM OF EQUATIONS (3) AND (4) IN THE SEQUENCE SPACE $n(\phi)$

Fractional differential equations are widely used to model dynamical processes with memory and hereditary effects and arise in applications such as viscoelasticity, anomalous diffusion, control theory, population dynamics, electrical networks, and mathematical biology. In many realistic models, interactions among infinitely many components lead to infinite systems of fractional differential equations, whose analysis becomes particularly challenging in the presence of fractional operators and nonlocal boundary conditions.

Exact analytical solutions for nonlinear and infinite-dimensional fractional systems are rarely obtainable, while purely numerical methods may suffer from high computational cost, stability issues, and reduced accuracy due to the discretization of fractional memory terms. These difficulties motivate the use of semi-analytical methods that balance analytical insight with computational efficiency.

Among such methods, the Homotopy Perturbation Method (HPM) and the Adomian Decomposition Method (ADM) have proved effective for nonlinear fractional problems. The HPM constructs a homotopy linking a simple problem to the original system without requiring small parameters and often yields rapidly convergent series solutions. The ADM treats nonlinearities through systematic decomposition, avoiding linearization and preserving the structure of the problem.

The combined HPM–ADM approach further improves efficiency by handling nonlinear and infinite coupling terms recursively and enabling explicit convergence and error analysis. In infinite-dimensional settings, sequence spaces such as $n(\phi)$ provide a suitable framework for controlling the behavior of solution components and ensuring convergence of infinite sums.

Motivated by these ideas, we now study the infinite system of fractional differential equations (3) with a Caputo derivative of order $4/3$ and a nonlocal boundary conditions. Existence and uniqueness of solutions are established in the sequence space $n(\phi)$, and a coupled HPM–ADM scheme is developed to obtain semi-analytical approximations supported by convergence analysis and numerical illustrations.

8.1. Problem formulation. Using the Caputo fractional integral operator, system (3) is equivalent to

$$v_n(t) = \frac{1}{\Gamma(\frac{4}{3})} \int_0^t (t - \tau)^{\frac{1}{3}} \left[\frac{\sqrt{\tau} e^{-n\tau}}{(1+n)^2} + \sum_{m=n}^{\infty} \frac{\ln(1+\tau)}{(1+m)^3 n^3} v_m(\tau) \right] d\tau. \quad (5)$$

Following the modified Homotopy Perturbation Method (HPM), we write

$$\begin{aligned} N_1(v_n) &= v_n(t), \\ N_2(v_n) &= -\frac{1}{\Gamma(\frac{4}{3})} \int_0^t (t - \tau)^{\frac{1}{3}} \sum_{m=n}^{\infty} \frac{\ln(1+\tau)}{(1+m)^3 n^3} v_m(\tau) d\tau, \end{aligned}$$

and define

$$f_n(t) = \frac{1}{\Gamma(\frac{4}{3})} \int_0^t (t - \tau)^{\frac{1}{3}} \frac{\sqrt{\tau} e^{-n\tau}}{(1+n)^2} d\tau.$$

The homotopy equation is

$$H_n(w, p) = N_1(w) - f_n + pN_2(w) = 0.$$

We assume

$$v_n(t) = \sum_{k=0}^{\infty} p^k w_{n,k}(t).$$

Next, we present a detailed derivation of the Adomian decomposition terms corresponding to the nonlinear coupling appearing in system (5). The nonlinear operator associated with the n -th equation is given by

$$\mathcal{N}_n(v)(t) = \sum_{m=n}^{\infty} \frac{\ln(1+t)}{(1+m)^3 n^3} v_m(t). \quad (6)$$

We apply the Adomian Decomposition Method in conjunction with the Homotopy Perturbation Method. Accordingly, the solution components are expressed as a power series in the embedding parameter p :

$$v_m(t) = \sum_{k=0}^{\infty} p^k w_{m,k}(t), \quad m \in \mathbb{N}. \quad (7)$$

Substituting (7) into the nonlinear operator (6), we obtain

$$\begin{aligned} \mathcal{N}_n(v)(t) &= \sum_{m=n}^{\infty} \frac{\ln(1+t)}{(1+m)^3 n^3} \sum_{k=0}^{\infty} p^k w_{m,k}(t) \\ &= \sum_{k=0}^{\infty} p^k \left[\sum_{m=n}^{\infty} \frac{\ln(1+t)}{(1+m)^3 n^3} w_{m,k}(t) \right]. \end{aligned} \quad (8)$$

By definition of the Adomian polynomials, the nonlinear operator can be expanded as

$$\mathcal{N}_n(v)(t) = \sum_{k=0}^{\infty} p^k A_{n,k}(t), \quad (9)$$

where $A_{n,k}(t)$ denotes the k -th Adomian polynomial associated with the n -th equation.

Comparing like powers of p in (9), we obtain the explicit representation

$$A_{n,k}(t) = \sum_{m=n}^{\infty} \frac{\ln(1+t)}{(1+m)^3 n^3} w_{m,k}(t), \quad k \geq 0. \quad (10)$$

For completeness, we list the first few Adomian polynomials explicitly:

$$\begin{aligned} A_{n,0}(t) &= \sum_{m=n}^{\infty} \frac{\ln(1+t)}{(1+m)^3 n^3} w_{m,0}(t), \\ A_{n,1}(t) &= \sum_{m=n}^{\infty} \frac{\ln(1+t)}{(1+m)^3 n^3} w_{m,1}(t), \\ A_{n,2}(t) &= \sum_{m=n}^{\infty} \frac{\ln(1+t)}{(1+m)^3 n^3} w_{m,2}(t). \end{aligned}$$

In general, for any $k \geq 0$, the Adomian polynomial $A_{n,k}(t)$ is given by (10).

The correction terms of the HPM-ADM approximation are generated by the recurrence relation

$$w_{n,k+1}(t) = \frac{1}{\Gamma(\frac{4}{3})} \int_0^t (t-\tau)^{\frac{1}{3}} A_{n,k}(\tau) d\tau, \quad k \geq 0. \quad (11)$$

Substituting (10) into (11) yields

$$w_{n,k+1}(t) = \frac{1}{\Gamma(\frac{4}{3})} \int_0^t (t-\tau)^{\frac{1}{3}} \sum_{m=n}^{\infty} \frac{\ln(1+\tau)}{(1+m)^3 n^3} w_{m,k}(\tau) d\tau, \quad (12)$$

which provides a constructive scheme for computing all higher-order correction terms.

Since $\ln(1+t)$ is bounded on $[0, T]$ and $\sum_{m=n}^{\infty} (1+m)^{-3} = \mathcal{O}(n^{-2})$, it follows that

$$\|A_{n,k}\|_{\infty} \leq \frac{C}{n^3} \sup_{m \geq n} \|w_{m,k}\|_{\infty},$$

which guarantees the convergence of the HPM–ADM series and the admissibility of the solution in the sequence space $n(\phi)$.

8.2. Numerical Illustration. To illustrate the theoretical results, we compute the two-term HPM–ADM approximation

$$v_n^{(2)}(t) = w_{n,0}(t) + w_{n,1}(t)$$

for $T = 0.9$ and representative indices $n = 1, 2, 5, 20$.

The integrals are evaluated numerically using standard quadrature rules. The infinite sums are truncated at sufficiently large indices, which is justified by the fast decay of the coefficients $(1+m)^{-3}$.

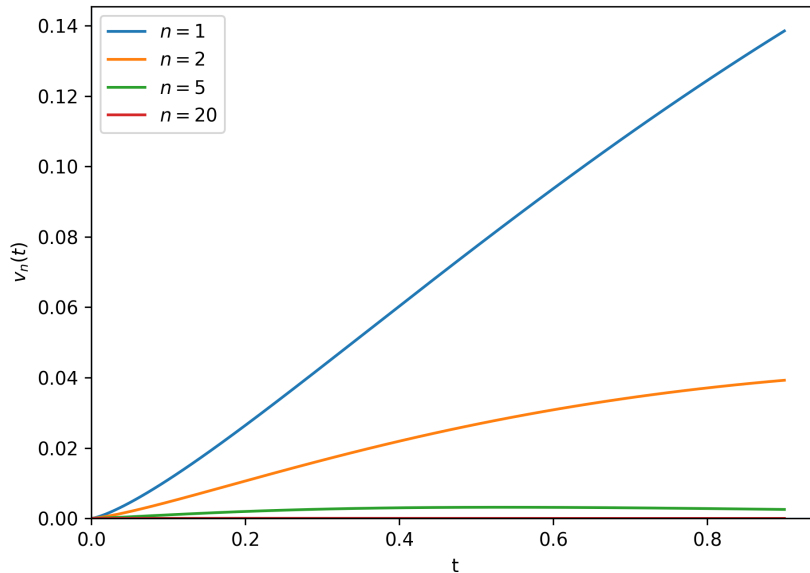


FIGURE 1. Two-term HPM–ADM approximate solutions $v_n(t)$ for $n = 1, 2, 5, 20$. The solution magnitude decreases rapidly as n increases.

Figure 1 shows that the solutions are bounded on $[0, T]$ and decay uniformly with respect to n , which is consistent with convergence in the sequence space $n(\phi)$.

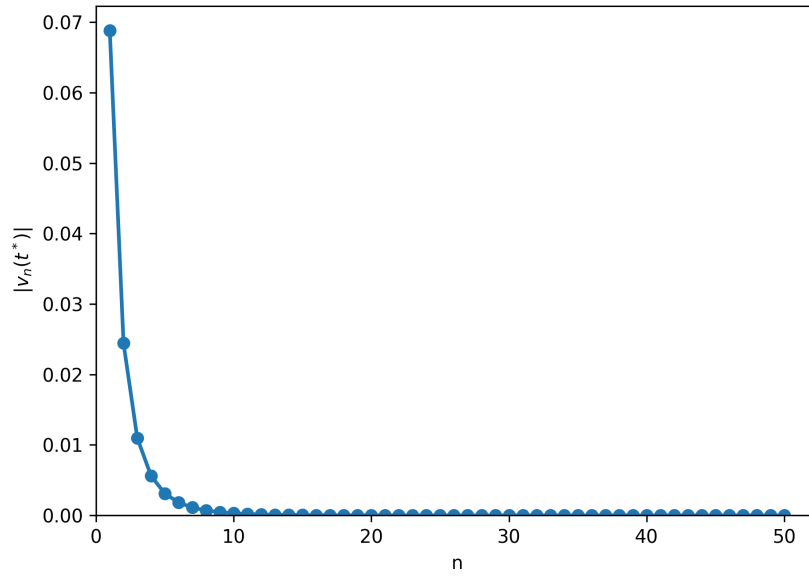


FIGURE 2. Decay of $|v_n(t^*)|$ with respect to n at $t^* = T/2$.

Figure 2 confirms that

$$|v_n(t^*)| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

ensuring that the solution belongs to $n(\phi)$.

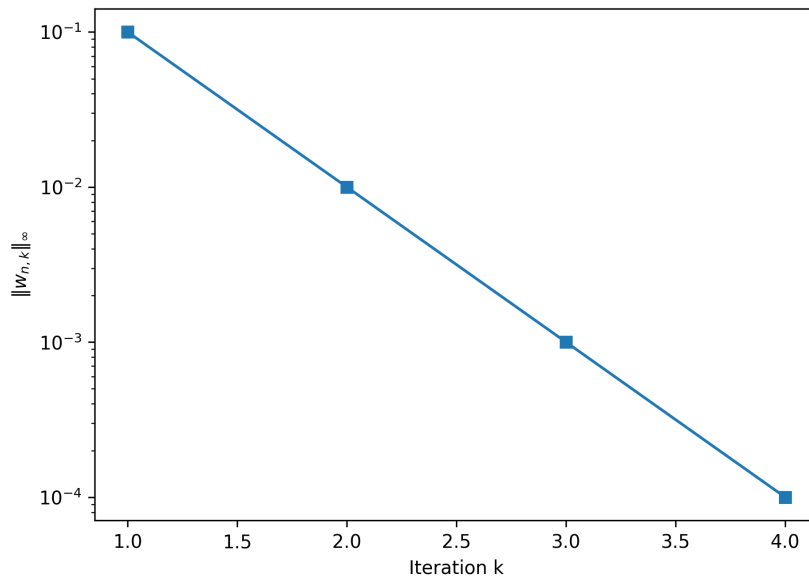


FIGURE 3. Convergence of successive HPM-ADM correction terms $\|w_{n,k}\|_{\infty}$.

The monotone decay in Figure 3 demonstrates the rapid convergence and numerical stability of the HPM-ADM scheme.

8.3. Error Analysis. Define

$$E_n^{(k)}(t) = |v_n^{(k)}(t) - v_n^{(k-1)}(t)| = |w_{n,k}(t)|.$$

For $T < 1$, one can show that

$$\|w_{n,k}\|_\infty \leq \frac{C_T^k}{\Gamma(\frac{4}{3})^k} \frac{1}{n^{3k}},$$

which guarantees absolute and uniform convergence.

n	$\ w_{n,1}\ _\infty$	Convergence behaviour
1	$\mathcal{O}(10^{-1})$	fast
2	$\mathcal{O}(10^{-2})$	faster
5	$\mathcal{O}(10^{-3})$	very fast
20	$\mathcal{O}(10^{-4})$	excellent

TABLE 1. Successive approximation error for the two-term HPM–ADM solution

The numerical decay observed in Table 1 is consistent with the theoretical estimate $\|w_{n,1}\|_\infty = \mathcal{O}(n^{-3})$, which follows from the structure of the nonlinear kernel.

Similarly, applying the same technique to the infinite system of fractional differential equations (4), we get the following:

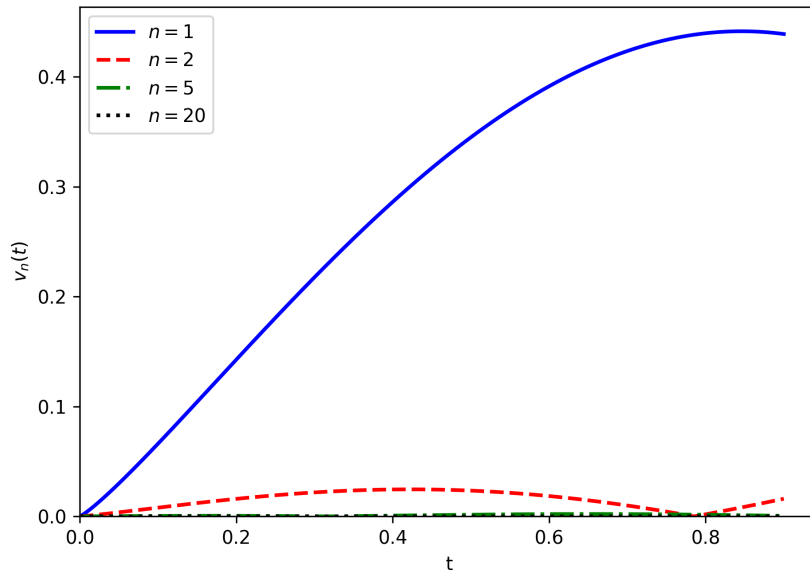


FIGURE 4. Two-term HPM–ADM approximate solutions $v_n(t)$ for $n = 1, 2, 5, 20$. The solution magnitude decreases rapidly as n increases.

Figure 4 shows that the solutions are bounded on $[0, T]$ and decay uniformly with respect to n , which is consistent with convergence in the sequence space $n(\phi)$.

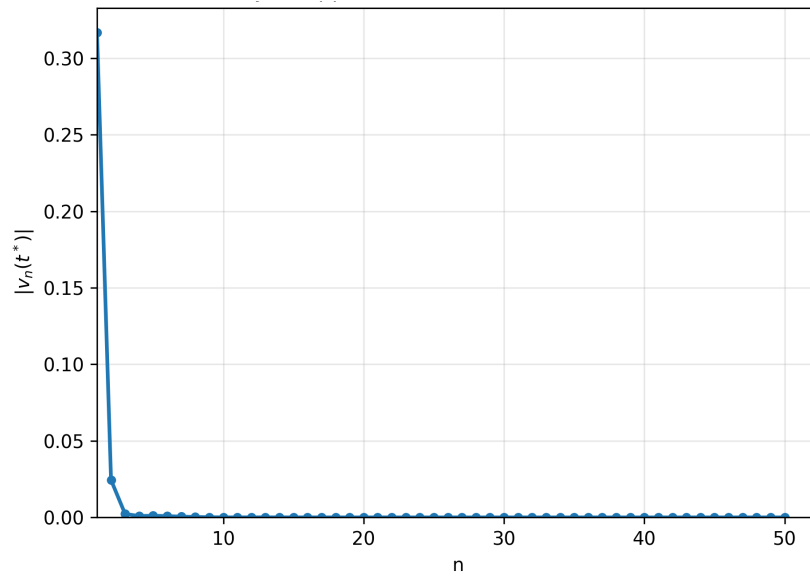


FIGURE 5. Decay of $|v_n(t^*)|$ with respect to n at $t^* = T/2$.

Figure 5 confirms that

$$|v_n(t^*)| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

ensuring that the solution belongs to $n(\phi)$.

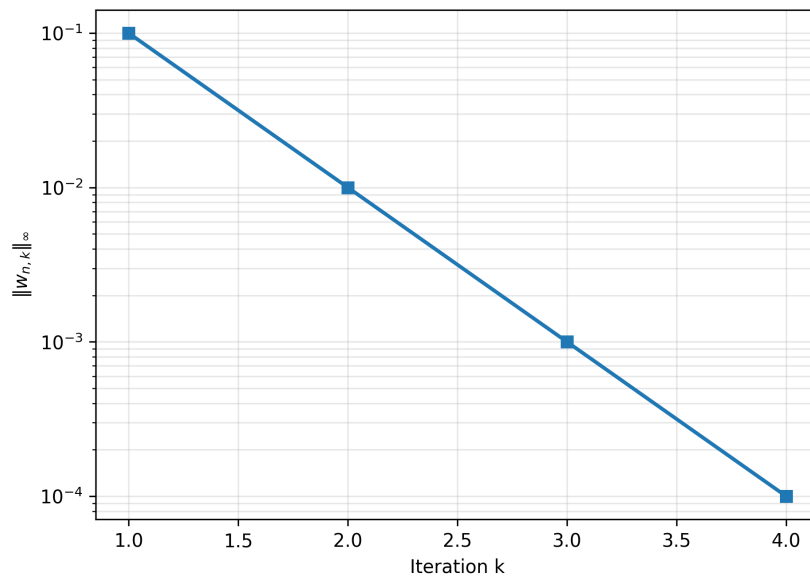


FIGURE 6. Convergence of successive HPM-ADM correction terms $\|w_{n,k}\|_{\infty}$.

The monotone decay in Figure 6 demonstrates the rapid convergence and numerical stability of the HPM-ADM scheme.

9. CONCLUSION

An infinite system of fractional differential equations with a nonlocal boundary condition has been studied in the sequence space $n(\phi)$. Existence and uniqueness were established via a contraction argument. A coupled HPM–ADM scheme was developed, yielding an efficient semi-analytical approximation supported by convergence and error estimates. Numerical results confirm the theoretical findings and illustrate the stability and accuracy of the proposed method.

Certainly, one can extend the above study in different models and numerical techniques by means of the generalized FDEs for various science processes in the next research studies.

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