

ROUGH GRAPH APPROXIMATIONS VIA SYMMETRY GROUPS: AUTOMORPHISMS AND PERMUTATIONS

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ABSTRACT. Recently, rough graphs have received considerable attention for their ability to model imprecise information in graphical data. By leveraging rough set theory, rough graphs address uncertainty through approximation and classification mechanisms. In this paper, we introduce rough graphs as mathematical objects defined through graph automorphism groups, providing a symmetry-based framework for uncertainty. Concurrently, we derive an indiscernibility relation from binary relations R via the automorphism group of its graph representation $G(R)$, where indiscernibility corresponds to orbits under the group action. This approach generalizes to define rough relations and rough graphs through an arbitrary permutation group acting on the underlying set.

Keywords: Rough set, rough relation, graphs, permutation group.

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1. INTRODUCTION

Rough set theory [18] provides a formal mathematical representation for dealing with imprecise and incomplete information in data. It defines a set of techniques and tools for analyzing and extracting knowledge from data sets that may contain uncertainties or inconsistencies. It has been used as a methodology for database mining tasks, such as knowledge discovery[21, 22], pattern recognition[25], classification, and clustering[5].

Graphs are ubiquitous in various fields, from biology (e.g., neural networks) and social science (e.g., social networks) to computer science (e.g., the internet) and transportation (e.g., airline routes). By representing complex systems by graphs, we can analyze and understand the underlying patterns and interactions within these systems.

Classical rough set theory characterizes the definability of subsets using equivalence classes as fundamental granules. However, when analyzing graph-structured data, where relationships between vertices (edges) carry critical information, we must characterize subgraphs, not merely subsets of vertices. This is essential for tasks like subgraph mining,

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graph pattern matching, and community detection. Real-world networks further introduce uncertainties from incomplete, noisy, or dynamic data. To address these challenges, extending rough set theory to graphs via rough graphs becomes necessary.

The notion of rough graphs was introduced by He and Shi [14], and since then, various approaches have been developed in the literature. These include using equivalence relations on the vertex set V [17, 2] or edge set E [16, 2, 14], independent binary relations on vertices and edges [23], graphical partitions treating graphs as incidence structures [35], graph approximations via minimum equivalent graphs and transitive closures [26], rough relations [24], and rough groups for edge rough Cayley graphs [31]. Additionally, Cao et al. [3] applied rough graphs to decision models in third-party payment platforms, while Sarwar [24] explored applications in illegal communication networks. The applications of rough graphs span numerous fields including subgraph matching [23], community detection [10], and decision modeling [3].

The motivation for this work stems from recent developments in applying rough set theory to graphs. Javaid et al. [29] applied rough set theory to graphs using the automorphism group $\text{Aut}(G)$ of a graph G . The action of $\text{Aut}(G)$ on V partitions the vertex set into orbits, making $(V, \text{Aut}(G))$ an approximation space. We note that the natural induced action of $\text{Aut}(G)$ on E similarly partitions the edge set into orbits. Thus, for a subgraph $H = (V_1, E_1)$ of G , we can approximate both V_1 in the vertex approximation space $(V, \text{Aut}(G))$ and E_1 in the edge approximation space $(E, \text{Aut}(G))$.

Furthermore, let R be a relation on a non-empty set U . It is known that R can be represented as a directed graph $D(R) = (V, A)$ where $V = U$ is the vertex set and there is a directed edge $x \rightarrow y$ iff $(x, y) \in R$. This representation allows us to use the automorphism group of $D(R)$ to define rough graphs, providing a symmetry-based framework for uncertainty that generalizes previous approaches.

This paper makes the following contributions: (1) We introduce rough graphs defined through graph automorphism groups, providing a symmetry-based framework; (2) We extend this approach to define rough relations and rough graphs through arbitrary permutation groups; (3) We establish fundamental properties of these approximations and provide illustrative examples.

The remainder of this paper is organized as follows. Section 2 provides the necessary preliminaries on rough set theory, graph theory, permutation groups, and automorphism groups. Section 3 defines rough graphs via the automorphism group of a graph and establishes their fundamental properties. Section 4 extends this concept to rough graphs derived from the automorphism group of a binary relation. Section 5 generalizes the framework further by defining rough relations and graphs via an arbitrary permutation group. Section 6 discusses potential applications, and Section 7 presents concluding remarks.

2. PRELIMINARIES

2.1. Rough Set Theory. Let R be an equivalence relation on a non-empty set U . For an element $x \in U$, the equivalence class containing x is denoted by $[x]$ and the set of equivalence classes is denoted by U/R . The pair (U, R) is called a Pawlak approximation space. Given a set $X \subseteq U$, it may not be possible to describe X precisely in (U, R) . It is possible to approximate X by lower and upper approximations.

Definition 2.1. [18] Let R be an equivalence relation on U and $[x]$ the equivalence class containing x . For any set $X \subseteq U$, the lower approximation $\underline{R}(X)$ and the upper approximation $\overline{R}(X)$ are defined as:

$$\begin{aligned}\underline{R}(X) &= \{x \in U : [x] \subseteq X\} \\ \overline{R}(X) &= \{x \in U : [x] \cap X \neq \emptyset\}\end{aligned}\tag{1}$$

For a subset $X \subseteq U$, the pair $(\underline{R}(X), \overline{R}(X))$ is called a rough set.

2.2. Graph Theory. A graph $G(V, E)$ consists of a set V of objects called vertices and a set E of 2-element subsets of V called edges. An edge, $\{u, v\}$, $u, v \in V$, is denoted by uv . Often, a graph with vertex set V and edge set E is written as the pair (V, E) . A directed graph $D(V, A)$, or a digraph, consists of vertex set V and the set A consists of ordered pairs of vertices from V . An edge from u to v is denoted by (u, v) .

The graph with $E = \emptyset$ is called an empty graph. We allow V to be empty and the graph with $V = \emptyset$ is called the null graph and is denoted by Φ . See [13] for the merits of allowing this graph. The directed graph $D(V, A)$ for which $A = V \times V$ is called a complete directed graph and is denoted by $\vec{K}_{|V|}$.

A graph $H(V_1, E_1)$ is a subgraph of a graph $G(V, E)$, denoted by $H \subseteq G$, if $V_1 \subseteq V$ and $E_1 \subseteq E$. A subgraph $H(V_1, E_1)$ of a graph G is called an induced subgraph if $V_1 \subseteq V$ and two vertices are adjacent in H whenever they are adjacent in G . A subgraph $H(V_1, E_1)$ of a graph G is called a spanning subgraph of a graph G if $V_1 = V$. A subgraph with edge set $E_1 \subseteq E$ and vertex set $V(E_1)$, which are vertices incident with edges in E_1 , is called a subgraph induced by E_1 . Graphs in this paper, unless otherwise stated, will be simple, without loops or multiple edges.

Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two subgraphs of a graph G . The union and intersection of G_1 and G_2 are the graphs $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ and $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$, respectively. It should be noted that even if they are connected in both G_1 and G_2 , they remain connected by just one edge in the union.

The complement of a graph $G(V, E)$ is defined as $G^c(V, \binom{V}{2} \setminus E)$. The relative complement of a subgraph $H(V_1, E_1)$ of $G(V, E)$ is defined as the subgraph $H^c(V, E \setminus E_1)$ of G , obtained by removing all edges of H from G (while retaining all vertices of G), see [12, 15, 9]. For a graph G and a subgraph H of G , the graph $G - H$ is obtained by removing all vertices of H from G and all edges incident to those vertices.

2.3. Permutation Groups and Automorphisms. A permutation group P on a finite set Y is a subgroup of the symmetric group S_Y , the group of all permutations of Y , and is denoted by (P, Y) . The orbit of an element $y \in Y$ under the action of P is denoted by $O(y) = \{x \in Y : \exists \alpha \in P \text{ such that } \alpha y = x\}$.

The natural induced action of P on $Y \times Y$ defined by $\alpha(u, v) = (\alpha u, \alpha v)$, for $(u, v) \in Y \times Y$ and $\alpha \in P$ partitions $Y \times Y$. Let $O(u, v) = \{(x, y) \in Y \times Y : \exists \alpha \in P \text{ such that } \alpha(u, v) = (x, y)\}$ be the orbit of (u, v) in $(P, Y \times Y)$.

Similarly, the natural induced action of P on $\binom{Y}{2}$, the set of all two element subsets of Y , defined by $\alpha\{u, v\} = \{\alpha u, \alpha v\}$, for $\{u, v\} \in \binom{Y}{2}$ and $\alpha \in P$ partitions $\binom{Y}{2}$. We denote $\{u, v\}$ by uv . Let $O(uv) = \{xy \in \binom{Y}{2} : \exists \alpha \in P \text{ such that } \alpha(uv) = (xy)\}$ be the orbit of (u, v) in $(P, \binom{Y}{2})$.

Definition 2.2 (Automorphism group of a graph [1]). An automorphism of a graph $G(V, E)$ with vertex set V and edge set E is a permutation σ of V such that $uv \in E$

if and only if $\sigma(uv) = (\sigma u)(\sigma v) \in E$. The set of automorphisms of G is a group under the composition of functions and will be denoted by $\text{Aut}(G)$.

Similarly, an automorphism of a digraph $D(V, A)$ with vertex set V and edge set A is a permutation σ of V such that $(u, v) \in A$ if and only if $\sigma(u, v) = (\sigma u, \sigma v) \in A$. The set of automorphisms of D is a group under the composition of functions and will be denoted by $\text{Aut}(D)$.

For graph theory definitions, we refer to [1]. For permutation groups, see [34, 32].

3. ROUGH GRAPH THROUGH THE AUTOMORPHISM GROUP

Let $G = (V, E)$ be a graph with vertex set V and edge set E , and let $\text{Aut}(G)$ denote its automorphism group. The action of $\text{Aut}(G)$ on V partitions V into *vertex orbits*, while its natural induced action on E partitions E into *edge orbits*. For $v \in V$, let $O(v)$ denote the orbit of v under the group action on V . For $e = uv \in E$, let $O(e)$ denote the orbit of e under the group action on E .

For a subgraph H of a graph G , we define the lower and upper approximations in $(G, \text{Aut}(G))$ as follows.

Definition 3.1. For a subgraph $H(V_1, E_1)$ of a graph $G(V, E)$, we define the lower approximation $\underline{\text{Aut}}(H)$ and the upper approximation $\overline{\text{Aut}}(H)$ as the subgraphs

$$\underline{\text{Aut}}(H) = (\underline{\text{Aut}}(V_1), \underline{\text{Aut}}(E_1)) \quad \text{and} \quad \overline{\text{Aut}}(H) = (\overline{\text{Aut}}(V_1), \overline{\text{Aut}}(E_1)), \quad (2)$$

respectively, where $\underline{\text{Aut}}(V_1)$ and $\overline{\text{Aut}}(V_1)$ are computed in $(V, \text{Aut}(G))$ and $\underline{\text{Aut}}(E_1)$ and $\overline{\text{Aut}}(E_1)$ are computed in the approximation space $(E, \text{Aut}(G))$.

Similarly, for a directed subgraph $H(V_1, A_1)$ of $D(V, A)$, the lower approximation $\underline{\text{Aut}}(H)$ and the upper approximation $\overline{\text{Aut}}(H)$ are the subgraphs

$$\underline{\text{Aut}}(H) = (\underline{\text{Aut}}(V_1), \underline{\text{Aut}}(A_1)) \quad \text{and} \quad \overline{\text{Aut}}(H) = (\overline{\text{Aut}}(V_1), \overline{\text{Aut}}(A_1)), \quad (3)$$

respectively, where $\underline{\text{Aut}}(V_1)$ and $\overline{\text{Aut}}(V_1)$ are computed in $(V, \text{Aut}(D))$ and $\underline{\text{Aut}}(A_1)$ and $\overline{\text{Aut}}(A_1)$ are computed in the approximation space $(A, \text{Aut}(D))$.

Before proceeding, we verify that these definitions yield valid subgraphs by showing that edges in the approximated edge sets have their endpoints in the corresponding approximated vertex sets.

Proposition 3.1. The structures $\underline{\text{Aut}}(H)$ and $\overline{\text{Aut}}(H)$ defined above are valid subgraphs of G .

Proof. We prove the validity for both approximations:

Lower Approximation: Let $e = uv \in \underline{\text{Aut}}(E_1)$. By definition, $O(e) \subseteq E_1$. For any vertex $u' \in O(u)$, there exists $\phi \in \text{Aut}(G)$ such that $\phi(u) = u'$. The edge $\phi(e) = \phi(u)\phi(v) = u'\phi(v)$ belongs to $O(e) \subseteq E_1$, so u' must be an endpoint of an edge in E_1 , hence $u' \in V_1$. Since u' was arbitrary, $O(u) \subseteq V_1$, implying $u \in \underline{\text{Aut}}(V_1)$. Similarly, $v \in \underline{\text{Aut}}(V_1)$. Thus, every edge in $\underline{\text{Aut}}(E_1)$ has both endpoints in $\underline{\text{Aut}}(V_1)$.

Upper Approximation: Let $e = uv \in \overline{\text{Aut}}(E_1)$. Then $O(e) \cap E_1 \neq \emptyset$, so there exists $e' \in O(e) \cap E_1$ and $\phi \in \text{Aut}(G)$ with $\phi(e) = e'$. The endpoints of e' are $\phi(u)$ and $\phi(v)$, both in V_1 since $e' \in E_1$. Thus, $\phi(u) \in O(u) \cap V_1 \neq \emptyset$, so $u \in \overline{\text{Aut}}(V_1)$. Similarly, $v \in \overline{\text{Aut}}(V_1)$. Hence, every edge in $\overline{\text{Aut}}(E_1)$ has both endpoints in $\overline{\text{Aut}}(V_1)$. $\square$

The pair $(\underline{\text{Aut}}(H), \overline{\text{Aut}}(H))$ is called an Aut-rough graph (resp. digraph).

Observation 3.1. Let $G = (V, E)$ be a graph and let $H = (V_1, E_1)$ be a subgraph of G .

- (1) If $(\text{Aut}(G), V)$ is transitive and $V_1 \subsetneq V$, then $\underline{\text{Aut}}(H) = \Phi$.

- (2) If $(\text{Aut}(G), E)$ is transitive, $E_1 \neq \emptyset$, and G has no isolated vertices, then $\overline{\text{Aut}}(H) = G$.

Proof. (1) Since $(\text{Aut}(G), V)$ is transitive, there is only one vertex orbit, namely $O(v) = V$ for all $v \in V$. For any $v \in V_1$, we have $O(v) = V \not\subseteq V_1$, because V_1 is a proper subset of V . Therefore, $\underline{\text{Aut}}(V_1) = \emptyset$.

Now consider the edge set. If $E_1 = \emptyset$, then clearly $\underline{\text{Aut}}(E_1) = \emptyset$. If $E_1 \neq \emptyset$, take any edge $e = vw \in E_1$. Since $V_1 \subsetneq V$, there exists a vertex $u \in V \setminus V_1$. By the transitivity of $(\text{Aut}(G), V)$, there exists $\phi \in \text{Aut}(G)$ such that $\phi(v) = u$. Hence $\phi(e) = \phi(v)\phi(w) = u\phi(w)$ is an edge incident with the vertex $u \notin V_1$. Since every edge of E_1 has both endpoints in V_1 , it follows that $\phi(e) \notin E_1$. Thus, $O(e) \not\subseteq E_1$. Therefore, $\underline{\text{Aut}}(E_1) = \emptyset$. Consequently, $\underline{\text{Aut}}(H) = \Phi$.

(2) Since $(\text{Aut}(G), E)$ is transitive, there is only one edge orbit, namely $O(e) = E$ for all $e \in E$. For any $e \in E$, we have $O(e) \cap E_1 = E \cap E_1 = E_1 \neq \emptyset$, since E_1 is non-empty. Therefore, $\overline{\text{Aut}}(E_1) = E$.

Now consider the vertex set. Let $u \in V$ be arbitrary. Since G has no isolated vertices, there exists an edge $e \in E$ incident with u . Since $E_1 \neq \emptyset$, take $e_1 = v_1w_1 \in E_1$, where $v_1, w_1 \in V_1$. By edge-transitivity, there exists $\phi \in \text{Aut}(G)$ such that $\phi(e_1) = e$. Thus, $\phi(e_1)$ is incident with u . The endpoints of $\phi(e_1)$ are $\phi(v_1)$ and $\phi(w_1)$, so u equals one of them. Hence $u \in O(v_1) \cup O(w_1)$. Since $v_1, w_1 \in V_1$, we have $O(v_1) \cap V_1 \neq \emptyset$ and $O(w_1) \cap V_1 \neq \emptyset$. Therefore, $u \in \overline{\text{Aut}}(V_1)$. Since $u \in V$ was arbitrary, we obtain $\overline{\text{Aut}}(V_1) = V$. Together with $\overline{\text{Aut}}(E_1) = E$, we conclude that $\overline{\text{Aut}}(H) = G$. $\square$

Note that the assumption that G has no isolated vertices is automatically satisfied when $(\text{Aut}(G), V)$ is transitive and $E \neq \emptyset$.

A subgraph H is called definable if $\underline{\text{Aut}}(H) = H = \overline{\text{Aut}}(H)$. For $i \in \{v, e\}$, the positive $\text{Pos}_i(H)$, boundary $\text{Bnd}_i(H)$, and negative $\text{Neg}_i(H)$ of a subgraph H are defined by:

$$\begin{aligned} \text{Pos}_v(H) &= \underline{\text{Aut}}(H) = \text{Pos}_e(H), \\ \text{Bnd}_v(H) &= \overline{\text{Aut}}(H) - \underline{\text{Aut}}(H), \quad \text{Bnd}_e(H) = (\underline{\text{Aut}}(H))^c \text{ of } \overline{\text{Aut}}(H), \\ \text{Neg}_v(H) &= G - \overline{\text{Aut}}(H), \quad \text{Neg}_e(H) = (\underline{\text{Aut}}(H))^c \text{ of } G. \end{aligned} \quad (4)$$

Note that $\{V(\text{Pos}_v(H)), V(\text{Bnd}_v(H)), V(\text{Neg}_v(H))\}$ partitions V , while $\{E(\text{Pos}_e(H)), E(\text{Bnd}_e(H)), E(\text{Neg}_e(H))\}$ partitions E .

The accuracy measure $\alpha(H) = (\alpha(V_1), \alpha(E_1))$ and the roughness measure $\rho(H) = (\rho(V_1), \rho(E_1))$ of $H(V_1, E_1)$ by R_v are defined by:

$$\begin{aligned} \alpha(H) &= \left(\frac{|V(\text{Pos}_v(H))| + |V(\text{Neg}_v(H))|}{|V|}, \frac{|E(\text{Pos}_e(H))| + |E(\text{Neg}_e(H))|}{|E|} \right), \\ \rho(H) &= (1 - \alpha(V_1), 1 - \alpha(E_1)) = \left(\frac{|\text{Bnd}_v(V_1)|}{|V|}, \frac{|\text{Bnd}_e(E_1)|}{|E|} \right). \end{aligned} \quad (5)$$

Here, $\alpha(H)$ quantifies the approximation quality of H under the equivalence relation R_v , while $\rho(H)$ measures the degree of incompleteness.

The vertex rough membership function and the edge rough membership function are defined as follows:

Definition 3.2. For a subgraph $H(V_1, E_1)$ of a graph $G(V, E)$, the vertex rough membership function is the function $\mu_H : V \rightarrow [0, 1]$, defined by $\mu_H(v) = \frac{|O(v) \cap V_1|}{|O(v)|}, v \in V$. The edge rough membership function is the function $\eta_H : E \rightarrow [0, 1]$, defined by $\eta_H(e) = \frac{|O(e) \cap E_1|}{|O(e)|}, e \in E$.

Observation 3.2. In $(G, \text{Aut}(G))$, we have:

$$\begin{aligned}\underline{\text{Aut}}(H) &= \left(\{v \in V : \mu_H(v) = 1\}, \{e \in E : \eta_H(e) = 1\} \right), \\ \overline{\text{Aut}}(H) &= \left(\{v \in V : \mu_H(v) > 0\}, \{e \in E : \eta_H(e) > 0\} \right).\end{aligned}\tag{6}$$

In the following examples the identity permutation is denoted by $()$.

Example 3.1. Let $G(V, E)$ be a graph with $V = \{0, 1, 2, 3, 4\}$ and $E = \{01, 12, 14, 24, 23, 34\}$. The automorphism group of G is $\text{Aut}(G) = \{(), (24)\}$. Hence,

$V/\text{Aut}(G) = \{\{0\}, \{1\}, \{3\}, \{2, 4\}\}$ and $E/\text{Aut}(G) = \{\{01\}, \{24\}, \{12, 14\}, \{23, 34\}\}$.

Let $H(V_1, E_1)$ be a subgraph of G where $V_1 = \{1, 2, 3, 4\}$ and $E_1 = \{12, 23, 34\}$. Then:

- $\underline{\text{Aut}}(H)$ has vertex set $\underline{\text{Aut}}(V_1) = \{1, 2, 3, 4\}$ and edge set $\underline{\text{Aut}}(E_1) = \{23, 34\}$.
- $\overline{\text{Aut}}(H)$ has vertex set $\overline{\text{Aut}}(V_1) = \{1, 2, 3, 4\}$ and edge set $\overline{\text{Aut}}(E_1) = \{12, 14, 23, 34\}$.
- $\alpha(H) = (1, \frac{2}{3})$ and $\rho(H) = (0, \frac{1}{3})$.

Figure (1) shows the graphs $G, H, \underline{\text{Aut}}(H)$, and $\overline{\text{Aut}}(H)$.

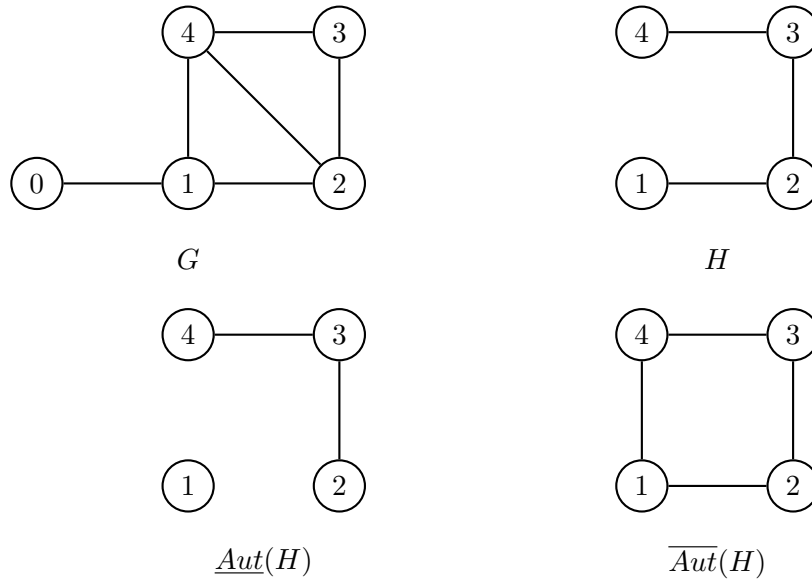


FIGURE 1. The lower and upper approximations of the subgraph H in Example 3.1.

3.1. Properties. The approximation operators in a Pawlak approximation space (U, R) are dual with respect to set complement, namely

$$\sim \underline{R}(\sim X) = \overline{R}(X) \text{ and } \underline{R}(X) = \sim \overline{R}(\sim X),\tag{7}$$

for $X \subseteq U$, where $\sim X = U \setminus X$.

For a subgraph $H(V_1, E_1)$ of a graph $G(V, E)$, there exists no subgraph $K(V_2, E_2)$ of G satisfying both $V_2 = V \setminus V_1$ and $E_2 = E \setminus E_1$, except when $H = \Phi$ (the null graph). Thus, we address duality for the vertex set $V \setminus V_1$ and edge set $E \setminus E_1$ separately, as they correspond

to distinct graph operations. We begin with the edge-set complement operation. The lower and upper approximations are dual with respect to the relative complement operation.

Lemma 3.1. (1) $(\underline{Aut}(H))^c = \overline{Aut}(H^c)$.

(2) $\underline{Aut}(H^c) = (\overline{Aut}(H))^c$.

Proof. Let $Aut(G)$ be the automorphism group of a graph $G(V, E)$ and $H(V_1, E_1)$ be a subgraph of G . It is clear that $(\underline{Aut}(H))^c, \overline{Aut}(H^c), \underline{Aut}(H^c)$ and $(\overline{Aut}(H))^c$ have the same vertex set, namely V . Moreover, $E((\underline{Aut}(H))^c) = E \setminus (\underline{Aut}(E_1)) = \overline{Aut}(E \setminus E_1)$ by (7). Also, $E((\overline{Aut}(H))^c) = E \setminus (\overline{Aut}(E_1)) = \underline{Aut}(E \setminus E_1)$ by (7). $\square$

Lemma 3.2. (1) $\overline{Aut}(G - H) \subseteq G - \underline{Aut}(H)$, and $\overline{Aut}(G - H) = G - \underline{Aut}(H)$ if H is a definable subgraph.

(2) $G - \overline{Aut}(H) = \underline{Aut}(G - H)$.

Proof. By (7), the pairs $G - \underline{Aut}(H)$ and $\overline{Aut}(G - H)$, and $G - \overline{Aut}(H)$ and $\underline{Aut}(G - H)$, share the same vertex set, respectively.

(1) If H is a definable subgraph, then $G - \underline{Aut}(H) = G - H = \overline{Aut}(G - H)$. Assume that H is not a definable subgraph and k is an edge in $\overline{Aut}(G - H)$. Note that

$$E(G - H) = E \setminus \left\{ \{e \in E : e \in E_1\} \cup \{e = vv_1 : v \in V - V_1 \text{ and } v_1 \in V_1\} \right\}.$$

Now, $O(k) \cap E(G - H) \neq \phi$ implies that:

- $O(k) \not\subseteq E_1$, that is $k \notin \underline{Aut}(E_1)$;
- $O(k)$ has an edge $e_1 = vv_1$ such that $v, v_1 \in V - V_1$. Which means that $O(v) \not\subseteq V_1$ and $O(v_1) \not\subseteq V_1$, i.e, $e_1 \in E(G - \underline{Aut}(H))$. Thus, every edge in $O(k)$ is joining a vertex in $O(v)$ to a vertex in $O(v_1)$, i.e, $k \in E(G - \underline{Aut}(H))$. Therefore $k \in E(G - \underline{Aut}(H))$. Moreover, let $e = vv_1$ be an edge in G such that $u, v \in O(x), O(x) \cap V_1 \neq \phi$ and $O(x) \cap (V - V_1) \neq \phi$. Then $e \in E(G - \underline{Aut}(H))$ but $e \notin \overline{Aut}(G - H)$.

(2) Let $k \in E(G - \overline{Aut}(H))$. Then

- $O(k) \cap E_1 = \phi$;
- if $k = vv_1$, then $O(v) \cap V_1 = \phi$ and $O(v_1) \cap V_1 = \phi$. Thus $O(k) \subseteq E(G - H)$. Thus, $k \in E(\underline{Aut}(G - H))$. Conversely, Let $k \in E(\underline{Aut}(G - H))$. Then $O(k) \subseteq E(G - H)$, which implies that:
- $O(k) \cap E_1 = \phi$;
- if $k = vv_1$, then $O(v) \cap V_1 = \phi$ and $O(v_1) \cap V_1 = \phi$.

Thus $k \in E(G - \overline{Aut}(H))$. $\square$

Remark 3.1. We note that the natural induced action of $Aut(G)$ partitions the edge set into edge orbits. Each edge orbit is either:

- (1) **Intra-orbit edges:** Edges whose both endpoints lie in the same vertex orbit, or
- (2) **Inter-orbit edges:** Edges whose endpoints lie in two distinct vertex orbits.

Furthermore, for inter-orbit edges, all edges in a given edge orbit connect the same pair of distinct vertex orbits.

Further, the lower and upper approximations satisfy the following properties.

Theorem 3.1. Let $G(V, E)$ be a graph. For subgraphs $H_1(V_1, E_1)$ and $H_2(V_2, E_2)$ of G the lower and upper approximations satisfy the following properties:

- (1) $\underline{Aut}(G) = G = \overline{Aut}(G)$, $\underline{Aut}(\Phi) = \Phi = \overline{Aut}\Phi$,

- (2) $\underline{Aut}(H_1) \subseteq H_1 \subseteq \overline{Aut}(H_1)$,
- (3) If $H_1 \subseteq H_2$, then $\underline{Aut}(H_1) \subseteq \underline{Aut}(H_2)$, $\overline{Aut}(H_1) \subseteq \overline{Aut}(H_2)$,
- (4) $\underline{Aut}(H_1 \cap H_2) = \underline{Aut}(H_1) \cap \underline{Aut}(H_2)$, $\overline{Aut}(H_1 \cap H_2) \subseteq \overline{Aut}(H_1) \cap \overline{Aut}(H_2)$,
- (5) $\underline{Aut}(H_1 \cup H_2) \supseteq \underline{Aut}(H_1) \cup \underline{Aut}(H_2)$, $\overline{Aut}(H_1 \cup H_2) = \overline{Aut}(H_1) \cup \overline{Aut}(H_2)$,
- (6) $\underline{Aut}(\underline{Aut}(H_1)) = \underline{Aut}(H_1)$, $\underline{Aut}(\overline{Aut}(H_1)) = \overline{Aut}(H_1)$,
- (7) $\underline{Aut}(\underline{Aut}(H_1)) = \underline{Aut}(H_1)$, $\overline{Aut}(\overline{Aut}(H_1)) = \overline{Aut}(H_1)$.

Proof. Note that $(V, Aut(G))$ and $(E, Aut(G))$ are Pawlak approximation spaces. Thus, for a subgraph $H(V_1, E_1)$, the approximations $\underline{Aut}(V_1)$, $\overline{Aut}(V_1)$, $\underline{Aut}(E_1)$, and $\overline{Aut}(E_1)$ satisfy these properties. Therefore the properties are satisfied directly. $\square$

4. ROUGH GRAPH THROUGH THE AUTOMORPHISM GROUP OF A RELATION

In this section, we will use the automorphism group of $D(R)$ (resp. $G(R)$) to define rough graphs. Let R be a binary relation on a non-empty set U and $D(R)$ be the graph representing R . Let $Aut(D(R))$ be the automorphism group of $D(R)$. Clearly, for $X \subseteq U$, the lower approximation $\underline{Aut}(X)$ and the upper approximation $\overline{Aut}(X)$ are:

$$\begin{aligned}\underline{Aut}(X) &= \{x \in U : O(x) \subseteq X\}, \\ \overline{Aut}(X) &= \{x \in U : O(x) \cap X \neq \emptyset\}.\end{aligned}\tag{8}$$

When R is an equivalence relation, the directed graph $D(R)$ consists of disjoint digraphs, one for each equivalence class C of R , where each digraph is the complete digraph $\vec{K}_C$ with vertex set C and edge set $C \times C$. Thus, R is partitioned into the disjoint sets $[u] \times [u]$ for $u \in U$, that is,

$$R = \bigsqcup_{u \in U} ([u] \times [u]) = \{(x, y) : x, y \in [u] \text{ for some } u \in U\},$$

where, $\bigsqcup$ denotes a disjoint union. Therefore, if there is $u, v \in U$ such that $[u] \neq [v]$ but $|[u]| = |[v]|$, then the equivalence classes $[u]$ and $[v]$ will be granulated together by $Aut(D(R))$, that is $[u]$ and $[v]$ will be replaced with a single equivalence class $[u] \cup [v]$. Therefore we have:

$$\underline{Aut}(X) \subseteq \underline{R}(X) \text{ and } \overline{R}(X) \subseteq \overline{Aut}(X),$$

where $\underline{R}(X)$ and $\overline{R}(X)$ are given by (1). Further, the equality happens when U/R has no different equivalence classes with the same size. Therefore, for $R_1 \subseteq R$, the lower approximation $\underline{R}(R_1)$ and the upper approximation $\overline{R}(R_1)$ are given by:

$$\begin{aligned}\underline{R}(R_1) &= \cup\{[u] \times [u] : [u] \times [u] \subseteq R_1\}, \\ \overline{R}(R_1) &= \cup\{[u] \times [u] : [u] \times [u] \cap R_1 \neq \emptyset\}.\end{aligned}\tag{9}$$

The lower and upper approximations of a subgraph $H = (V_1, A_1)$ of $D(R)$ are defined as follows.

Definition 4.1. Let R be an equivalence relation on U and $H = (V_1, A_1)$ be a subgraph of $D(R)$. The lower approximation $\underline{R}(H)$ and the upper approximation $\overline{R}(H)$ are:

$$\underline{R}(H) = (\underline{R}(V_1), \underline{R}(A_1)) \text{ and } \overline{R}(H) = (\overline{R}(V_1), \overline{R}(A_1)),\tag{10}$$

respectively.

If R is not an equivalence relation, then using $Aut(D(R))$, we define the lower and upper approximations of R_1 by:

$$\begin{aligned}\underline{Aut}(R_1) &= \{(x, y) \in R : O((x, y)) \subseteq R_1\} \\ \overline{Aut}(R_1) &= \{(x, y) \in R : O((x, y)) \cap R_1 \neq \emptyset\}\end{aligned}\tag{11}$$

The lower and upper approximations of $H = (V_1, R_1)$ are defined as follows.

Definition 4.2. Let R be a binary relation on a non-empty set U and $H = (V_1, A_1)$ be a subgraph of $D(R)$. We define the lower approximation $\underline{Aut}(H)$ and the upper approximation $\overline{Aut}(H)$ of H to be

$$\underline{Aut}(H) = (\underline{Aut}(V_1), \underline{Aut}(A_1)) \text{ and } \overline{Aut}(H) = (\overline{Aut}(V_1), \overline{Aut}(A_1)), \quad (12)$$

respectively.

Example 4.1. Let $U = \{1, 2, 3, 4, 5, 6\}$ and $R = \{(1, 2), (1, 5), (3, 1), (3, 6), (4, 6)\}$ be a relation on U . Then $Aut(D(R)) = \{(), (25)\}$. Then

$$V/Aut(D) = \{\{1\}, \{3\}, \{6\}, \{4\}, \{2, 5\}\}$$

and

$$A/Aut(D) = \{\{(1, 2), (1, 5)\}, \{(3, 1)\}, \{(3, 6)\}, \{(4, 6)\}\}.$$

Let $H(V_1, A_1)$ be a subgraph of D , where $V_1 = \{1, 2, 6, 3\}$ and $A_1 = \{(1, 2), (3, 1), (3, 6)\}$.

- $\underline{Aut}(H)$ has vertex set $\underline{Aut}(V_1) = \{1, 3, 6\}$ and edge set $\underline{Aut}(E_1) = \{(3, 1), (3, 6)\}$.
- $\overline{Aut}(H)$ has vertex set $\overline{Aut}(V_1) = \{1, 2, 3, 5, 6\}$ and edge set $\overline{Aut}(E_1) = \{(1, 2), (1, 5), (3, 1), (3, 6)\}$.

Figure (2) shows the lower and upper approximations.

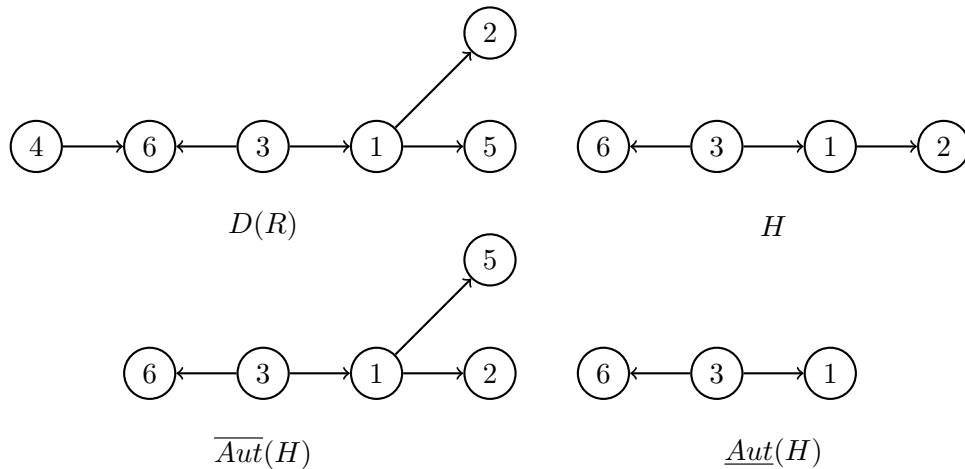


FIGURE 2. The lower and upper approximations of the subgraph H in Example (4.1)

5. ROUGH RELATIONS AND ROUGH GRAPHS VIA PERMUTATION GROUPS

Let P be a permutation group acting on a set U . Denote by $O(x)$ the orbit of an element $x \in U$. The natural induced action of P on $U \times U$, given by $g \cdot (x, y) = (g(x), g(y))$ for $g \in P$ and $(x, y) \in U \times U$, partitions $U \times U$ into disjoint orbits. Let $O((x, y))$ denote the orbit containing (x, y) . When P is transitive, the orbits of this action on $U \times U$ are called orbitals. For a relation $R \subseteq U \times U$, we define its lower and upper approximations as follows.

Definition 5.1. Let P be a permutation group on a set U . For a relation R on U , the lower approximation $\underline{P}(R)$ and the upper approximation $\overline{P}(R)$ are defined by:

$$\begin{aligned}\underline{P}(R) &= \{(x, y) \in U \times U : O((x, y)) \subseteq R\}, \\ \overline{P}(R) &= \{(x, y) \in U \times U : O((x, y)) \cap R \neq \emptyset\},\end{aligned}\tag{13}$$

respectively.

The pair $(\underline{P}(R), \overline{P}(R))$ is called a P -rough relation.

Example 5.1. Let $U = \{0, 1, 2, 3, 4\}$ and P be a permutation group on U , where

$$P = \{(), (01), (234), (243), (01)(234), (01)(243)\}.$$

Then $(U \times U)/P$ equals

$$\begin{aligned}&\left\{ \{(0, 0), (1, 1)\}, \{(0, 1), (1, 0)\}, \{(2, 2), (3, 3), (4, 4)\}, \right. \\ &\quad \{(2, 3), (2, 4), (3, 2), (3, 4), (4, 2), (4, 3)\}, \\ &\quad \{(2, 0), (3, 0), (4, 0), (2, 1), (3, 1), (4, 1)\}, \\ &\quad \left. \{(0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 4)\} \right\}.\end{aligned}$$

Let $R = \{(0, 0), (0, 1), (1, 1), (0, 3)\}$ be a relation on U . Then

- $\underline{P}(R) = \{(0, 0), (1, 1)\}$.
- $\overline{P}(R) = \{(0, 0), (1, 1), (0, 1), (1, 0), (0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 4)\}$.

In the following lemma, we present some fundamental properties of the lower and upper approximations of a relation R .

Lemma 5.1. Let P be a permutation group on a set U and R be a relation on U . Then:

- (1) If R is the identity relation (i.e., $R = \{(u, u) : u \in U\}$), then both $\underline{P}(R)$ and $\overline{P}(R)$ are identity relations.
- (2) If R is reflexive, then $\underline{P}(R)$ and $\overline{P}(R)$ are both reflexive.
- (3) If R is symmetric, then $\overline{P}(R)$ is symmetric.

Proof. (1) Since R is the identity relation, it consists of all pairs (u, u) for $u \in U$. The orbits of diagonal pairs under P are subsets of the diagonal because if $(x, y) \in O((u, u))$, then there exists $g \in P$ with $(x, y) = (g(u), g(u))$, so $x = y$. Therefore, each orbit of a diagonal pair is contained in R , so $\underline{P}(R) = R$. Moreover, any orbit intersecting R must be an orbit of a diagonal pair and hence contained in R , so $\overline{P}(R) = R$.

(2) If R is reflexive, then for any $u \in U$, $(u, u) \in R$. The orbit $O((u, u))$ consists of diagonal pairs (v, v) where v is in the orbit of u . Since R contains all diagonal pairs, we have $O((u, u)) \subseteq R$, so $(u, u) \in \underline{P}(R)$. Also, since $O((u, u)) \cap R \neq \emptyset$, we have $(u, u) \in \overline{P}(R)$. Hence, both approximations are reflexive.

(3) Suppose R is symmetric and let $(x, y) \in \overline{P}(R)$. Then there exists $(x', y') \in O((x, y)) \cap R$. Since R is symmetric, $(y', x') \in R$. Note that $(y', x') \in O((y, x))$ because for any $g \in P$, if $g(x) = x'$ and $g(y) = y'$, then $g(y) = y'$ and $g(x) = x'$. Therefore, $O((y, x)) \cap R \neq \emptyset$, so $(y, x) \in \overline{P}(R)$. Thus, $\overline{P}(R)$ is symmetric. $\square$

5.1. Rough Graphs via Permutation Groups. The concept of approximations can be extended from relations to the structure of graphs and digraphs. Let P be a permutation group on U and let $G = (U, E)$ be an ambient graph (or digraph). The action of the permutation group P induces a partition of the universal set of all possible edges—either $\binom{U}{2}$ for an undirected graph or $U \times U$ for a directed graph. This action partitions the universal set into universal orbits.

The restricted edge-orbit blocks of G are the non-empty intersections of these universal orbits with the edge set E . That is, for each universal orbit O_{univ} , the corresponding block is $O_G = O_{\text{univ}} \cap E$, provided that this intersection is non-empty. These blocks form the Pawlak partition of E used below.

We define the lower and upper approximations of subgraphs as follows.

Definition 5.2. Let P be a permutation group on a set U and let $G = (U, E)$ be a graph (resp. digraph) with edge set $E \subseteq \binom{U}{2}$ (resp. $E \subseteq U \times U$). For a subgraph $H = (V_1, E_1)$ of G , the lower approximation $\underline{P}_G(H)$ and the upper approximation $\overline{P}_G(H)$ are defined by

$$\begin{aligned}\underline{P}_G(H) &= (\underline{P}(V_1), \underline{P}(E_1) \cap E(\underline{P}(V_1))), \\ \overline{P}_G(H) &= (\overline{P}(V_1), \overline{P}(E_1)),\end{aligned}\tag{14}$$

respectively, where:

- $\underline{P}(V_1)$ and $\overline{P}(V_1)$ are computed in the approximation space (U, P) ,
- $\underline{P}(E_1)$ and $\overline{P}(E_1)$ are computed in the approximation space (E, P) ,
- $E(V')$ denotes the set of edges in G with both endpoints in V' .

The pair $(\underline{P}_G(H), \overline{P}_G(H))$ is called a P -rough graph (or P -rough digraph).

This general definition applies to any ambient graph G . The complete graph $K_{|U|}$ and the complete digraph $\vec{K}_{|U|}$ are common special cases. When G is complete, the condition $\underline{P}(E_1) \cap E(\underline{P}(V_1))$ simplifies to $\underline{P}(E_1)$ since all possible edges between vertices in $\underline{P}(V_1)$ exist in G .

Example 5.2. Let U and P be as in Example 5.1.

First, consider the complete digraph $\vec{K}_{|U|}$ with vertex set U and arc set $A = U \times U$. Let $H = (V_1, A_1)$ be the subdigraph with vertex set $V_1 = \{0, 1, 3\}$ and arc set $A_1 = R$. Then:

- $\underline{P}(H) = (\underline{P}(V_1), \underline{P}(A_1)) = (\{0, 1\}, \{(0, 0), (1, 1)\})$.
- $\overline{P}(H)$ has vertex set $\overline{P}(V_1) = U$ and arc set

$$\overline{P}(A_1) = (U, \{(0, 0), (1, 1), (0, 1), (1, 0), (0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 4)\}).$$

Next, consider the complete graph K_5 with vertex set U and edge set $E = \binom{U}{2}$. The orbits of E under P are:

$$E/P = \{\{01\}, \{23, 24, 34\}, \{02, 03, 04, 12, 13, 14\}\}.$$

Let $H = (V_1, E_1)$ be the subgraph with $V_1 = \{1, 2, 3, 4\}$ and $E_1 = \{12, 23, 34\}$. Then:

- $\underline{P}(H) = (\underline{P}(V_1), \underline{P}(E_1)) = (\{2, 3, 4\}, \emptyset)$.
- $\overline{P}(H) = (\overline{P}(V_1), \overline{P}(E_1)) = (U, \{02, 03, 04, 12, 13, 14, 23, 24, 34\})$.

Finally, consider the graph $G = (U, E)$ with $E = \{01, 12, 14, 24, 23, 34\}$. The orbits of E under P are:

$$E/P = \{\{01\}, \{23, 24, 34\}, \{12, 14\}\}.$$

Let $H = (V_1, E_1)$ be the same as above: $V_1 = \{1, 2, 3, 4\}$ and $E_1 = \{12, 23, 34\}$. Then:

- $\underline{P}_G(H) = (\underline{P}(V_1), \underline{P}(E_1) \cap E(\underline{P}(V_1))) = (\{2, 3, 4\}, \emptyset)$.
- $\overline{P}_G(H) = (\overline{P}(V_1), \overline{P}(E_1)) = (U, \{12, 14, 23, 24, 34\})$.

See Figure 3.

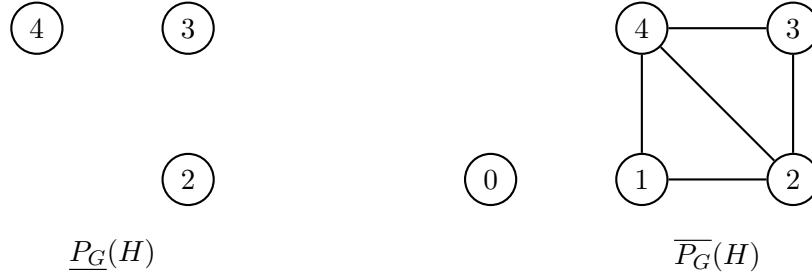


FIGURE 3. The lower and upper approximations of the subgraph H in Example (5.2)

5.2. Properties. In $(\mathcal{G}, \mathcal{P})$, where $\mathcal{G} \in \{\vec{K}_{|U|}, K_{|U|}, D, G\}$ and $\mathcal{P} \in \{P, P_D, P_G\}$, the lower and upper approximations satisfy the following relations with respect to the relative complement operation.

Lemma 5.2. (1) $\overline{\mathcal{P}}(H^c) \subseteq (\underline{\mathcal{P}}(H))^c$.
 (2) $\underline{\mathcal{P}}(H^c) = (\overline{\mathcal{P}}(H))^c$.

Proof. The proof follows similarly to the proof of Lemma 3.1. The inclusion in (1) appears because, in the general graph case, the edge set of $\underline{\mathcal{P}}(H)$ is restricted by the validity condition $\underline{\mathcal{P}}(E_1) \cap E(\underline{\mathcal{P}}(V_1))$. For (2), since H^c has vertex set V , this restriction does not remove any edge, and the usual duality of the edge approximation space gives the equality. $\square$

Lemma 5.3. (1) $\mathcal{G} - \underline{\mathcal{P}}(H) \supseteq \overline{\mathcal{P}}(\mathcal{G} - H)$, and $\mathcal{G} - \underline{\mathcal{P}}(H) = \overline{\mathcal{P}}(\mathcal{G} - H)$ if H is a definable subgraph.

(2) $\mathcal{G} - \overline{\mathcal{P}}(H) = \underline{\mathcal{P}}(\mathcal{G} - H)$.

Proof. The proof follows similarly to the proof of Lemma 3.2. $\square$

Further, the lower and upper approximation operators satisfy the following properties.

Theorem 5.1. For subgraphs $H_1(V_1, E_1)$ and $H_2(V_2, E_2)$ of $\mathcal{G}$, the lower and upper approximations satisfy the following properties:

- (1) $\underline{\mathcal{P}}(\mathcal{G}) = \mathcal{G} = \overline{\mathcal{P}}(\mathcal{G})$, $\underline{\mathcal{P}}(\Phi) = \Phi = \overline{\mathcal{P}}(\Phi)$.
- (2) $\underline{\mathcal{P}}(H_1) \subseteq H_1 \subseteq \overline{\mathcal{P}}(H_1)$.
- (3) If $H_1 \subseteq H_2$, then $\underline{\mathcal{P}}(H_1) \subseteq \underline{\mathcal{P}}(H_2)$ and $\overline{\mathcal{P}}(H_1) \subseteq \overline{\mathcal{P}}(H_2)$.
- (4) $\underline{\mathcal{P}}(H_1 \cap H_2) = \underline{\mathcal{P}}(H_1) \cap \underline{\mathcal{P}}(H_2)$, and $\overline{\mathcal{P}}(H_1 \cap H_2) \subseteq \overline{\mathcal{P}}(H_1) \cap \overline{\mathcal{P}}(H_2)$.
- (5) $\underline{\mathcal{P}}(H_1 \cup H_2) \supseteq \underline{\mathcal{P}}(H_1) \cup \underline{\mathcal{P}}(H_2)$, and $\overline{\mathcal{P}}(H_1 \cup H_2) = \overline{\mathcal{P}}(H_1) \cup \overline{\mathcal{P}}(H_2)$.
- (6) $\overline{\mathcal{P}}(\underline{\mathcal{P}}(H_1)) = \underline{\mathcal{P}}(H_1)$, and $\underline{\mathcal{P}}(\overline{\mathcal{P}}(H_1)) = \overline{\mathcal{P}}(H_1)$.
- (7) $\underline{\mathcal{P}}(\underline{\mathcal{P}}(H_1)) = \underline{\mathcal{P}}(H_1)$, and $\overline{\mathcal{P}}(\overline{\mathcal{P}}(H_1)) = \overline{\mathcal{P}}(H_1)$.

6. APPLICATIONS OF ROUGH GRAPHS DEFINED VIA AUTOMORPHISM GROUPS

The proposed notion of rough graphs leverages the symmetry of the automorphism group to approximate subgraphs. This framework has several potential applications in pattern recognition, biological network analysis, network anomaly detection, and graph compression.

Social Network Analysis. Let $G(V, E)$ be a graph representing friendship network, where $V = \{A, B, C, D, E, F\}$ and $E = \{AB, BC, CA, DE, EF, FD, AD\}$. Let $H(V_1, E_1)$ be a subgraph representing a suspected "close-knit group", where $V_1 = \{A, B, D\}$ and $E_1 = \{AB, AD\}$. Then

$$\begin{aligned} & \bullet \\ & \text{Aut}(G) = \left\{ (), (EF), (BC), (BE)(CF)(AD), (BF)(CE)(AD), \right. \\ & \quad \left. (BECF)(AD), (BC)(EF), (BFCE)(AD) \right\}, \\ & \bullet V/\text{Aut}(G) = \left\{ \{A, D\}\{B, C, E, F\} \right\}, \\ & \bullet E/\text{Aut}(G) = \left\{ \{AD\}\{BC, EF\}\{AB, AC, DE, DF\} \right\}. \end{aligned}$$

Therefore

$$\begin{aligned} & \bullet \underline{\text{Aut}}(H) = (\{A, D\}, \{AD\}), \\ & \bullet \overline{\text{Aut}}(H) = (\{A, B, C, D, E, F\}, \{AD, AB, AC, DE, DF\}). \end{aligned}$$

The lower approximation $\underline{H}$ is the set of vertices and edges contained in H under all automorphisms of G (i.e., robust against relabeling). For H with $V_1 = \{A, B, D\}$ and $E_1 = \{AB, AD\}$. The upper approximation $\overline{H}$ is the set of vertices and edges contained in some automorphic image of H (i.e., expands to structurally equivalent regions). The boundary vertices $\text{Bnd}_v(H) = \{B, C, E, F\}$ are the vertices in the upper vertex approximation but not in the lower vertex approximation. The boundary edges $\text{Bnd}_e(H) = \{AB, AC, DE, DF\}$ are the edges in the upper edge approximation but not in the lower edge approximation. Figure (4), shows the graphs $G, H, \underline{\text{Aut}}(H)$, and $\overline{\text{Aut}}(H)$.

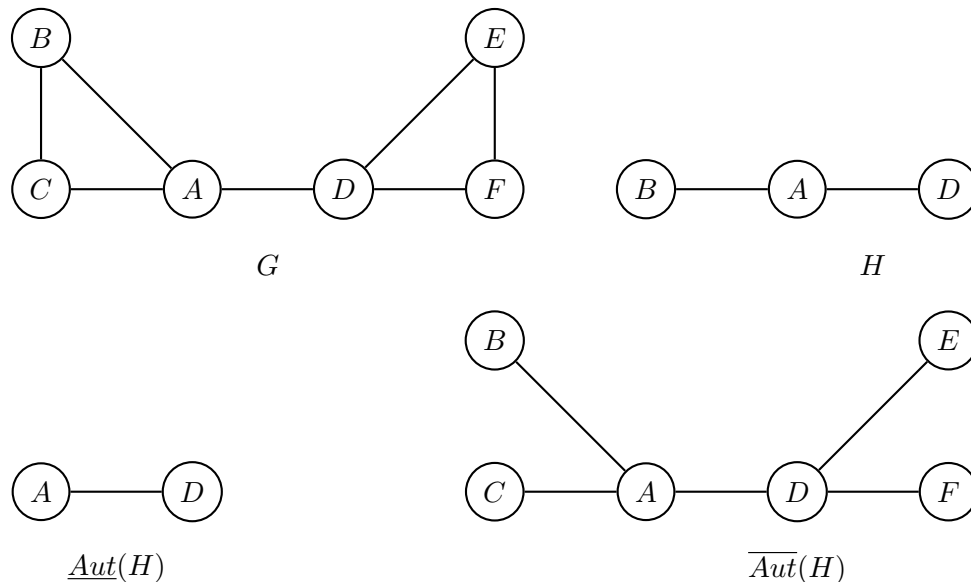


FIGURE 4. The lower and upper approximations of the subgraph H representing a suspected "close-knit group".

7. CONCLUSIONS

In this paper, we have developed a comprehensive symmetry-based framework for rough graph approximations. The main contributions are threefold.

First, we defined rough graphs using the automorphism group of a graph. By leveraging the natural action of $\text{Aut}(G)$ on both the vertex set V and the edge set E , we provided a method to approximate any subgraph H of G . We established fundamental properties of these approximations, including duality lemmas and a characterization via rough membership functions.

Second, we extended this concept by utilizing the automorphism group of a binary relation R . We showed that for an equivalence relation R , our automorphism-based approximation can coarsen the classical Pawlak approximation, providing a new granular perspective. For general relations, the automorphism group offers a natural way to define lower and upper approximations.

Third, we generalized the framework to an arbitrary permutation group P acting on a set U . This allowed us to define rough relations and rough graphs in a very general setting, subsuming the automorphism-based approach as a special case. We demonstrated that the core properties of rough approximations are preserved in this general context.

The proposed framework has significant potential for applications in areas such as social network analysis, pattern recognition in biological networks, and network anomaly detection, where symmetry and uncertainty are both present.

Future work will focus on developing efficient algorithms for computing these approximations on large graphs, exploring connections with other granular computing models, and investigating deeper algebraic properties of the approximation operators defined by group actions.

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