

A FLEXIBLE UNIT CHEN QUANTILE REGRESSION MODEL FOR BOUNDED RESPONSE DATA: ESTIMATION, MODEL DIAGNOSTICS, AND APPLICATIONS

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ABSTRACT. Bounded response variables are common in disciplines such as dependability, finance, environmental sciences, and biostatistics, where traditional regression models based on the conditional mean may fail to capture varied covariate effects throughout the response distribution. To solve this issue, this work proposes a flexible quantile regression model based on the Unit Chen distribution for evaluating continuous data on the unit intervals. The proposed methodology defines the conditional quantile via a logit link function, allowing explanatory factors to impact diverse parts of the answer distribution while maintaining limited support.

Maximum likelihood estimation is created for the inference of parameters, and the appropriate estimation approach is executed by means of numerical optimization techniques. A number of model diagnostic techniques are included to check the adequacy and robustness of the model, to detect influential data and to examine the sensitivity of the fitted model, including generalized Cook's distance, likelihood displacement and local influence analysis. The finite-sample performance of the suggested estimators is examined via a comprehensive Monte Carlo simulation analysis employing bias, mean squared error and root mean squared error as assessment measures for varying sample sizes and quantile levels.

The practical applicability of the proposed model is illustrated using a real bounded-response dataset and is compared with several competing unit quantile regression models using information criteria and diagnostic measures. The empirical findings demonstrate that the proposed Unit Chen quantile regression model provides accurate parameter estimation, effective diagnostic performance, and competitive model fitting, making it a valuable alternative for modeling bounded response data exhibiting heterogeneous distributional characteristics.

Keywords: Unit Chen distribution; Quantile regression; Bounded response data; Maximum likelihood estimation; Model diagnostics; Influence analysis; Monte Carlo simulation.

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1. INTRODUCTION

Continuous response variables restricted to the unit interval $(0, 1)$ commonly arise in diverse scientific fields, including reliability engineering, economics, environmental sciences, finance, medicine, and quality control. Such variables typically represent proportions, probabilities, rates, efficiencies, or percentages, whose bounded nature violates the assumptions of classical linear regression models based on normality and unbounded support. Consequently, specialized regression models have been developed to provide more appropriate statistical inference for bounded response data. Among these, beta regression has become one of the most widely used approaches for modeling proportions and rates [?]. Subsequently, several alternative unit distributions have been proposed to improve modeling flexibility, including the Kumaraswamy distribution [?], Unit-Inverse Gaussian distribution [?], Unit-Weibull distribution [?], Unit-Improved Second-Degree Lindley distribution [?], and Unit Burr-XII distribution [?].

Koenker and Bassett [?] first introduced quantile regression, which extends classical regression by modeling conditional quantiles in addition to the conditional mean. This methodology offers a more thorough explanation of the relationship between the response variable and the explanatory variables, particularly in cases where the data are characterized by skewness, heteroscedasticity, heavy tails, or influential observations. In recent years, quantile regression has undergone significant methodological advancements as a result of these advantages. These advancements include multivariate quantile estimation [?], variable selection for high-dimensional additive models [?], adaptive hypothesis testing [?], robust mixed-effects quantile regression [?], quantile ratio regression [?], and simulation-based inference procedures [?]. Furthermore, quantile regression has been applied successfully in the fields of public health [?], regional economic analysis [?], and machine learning prediction intervals [?]. The amalgamation of adaptable unit distributions with quantile regression has garnered heightened interest due to its efficacy in examining bounded response variables characterized by asymmetric or heterogeneous distributions. Recent advancements encompass quantile regression models predicated on the Unit-Weibull distribution [?], Unit Burr-XII distribution [?], Continuous Bernoulli distribution [?], Unit Gompertz distribution [?], Unit Generalized Half-Normal distribution [?], exponentiated odd log-logistic Weibull distribution [?], tangent exponentiated odd log-logistic Weibull distribution [?], and extended exponential-geometric distribution [?]. These investigations reveal that the integration of flexible unit distributions with quantile regression significantly enhances the examination of constrained data by tolerating various distributional forms and differing covariate impacts. Chen [?] came up with the Chen distribution, which is a flexible lifetime distribution that can show both rising and bathtub-shaped hazard functions with only a few parameters. Korkmaz et al. [?] came up with the Unit Chen distribution and a quantile regression model to go with it because it has some interesting probability features. Sarhan [?] recently looked into the Unit Chen distribution and its quantile regression framework in more detail, showing how useful it is for modeling response variables that are limited in size. These studies show that the Unit Chen distribution has a lot of potential. However, it would be nice to see a more complete regression framework that combines efficient likelihood-based estimation with systematic model diagnostic procedures for figuring out which observations are important and whether the model is good enough.

Based on these factors, this research creates a versatile Unit Chen quantile regression model that works with limited response data. By use of a logit link function, the conditional quantiles are parameterized in the suggested framework, and the model parameters

are estimated using the maximum likelihood technique. The model's robustness and the presence of influential observations are further assessed using model diagnostic approaches that integrate generalized Cook's distance, likelihood displacement, and local impact analysis. An extensive Monte Carlo simulation analysis is used to evaluate the finite-sample behavior of the suggested estimators. The approach is shown to be practical by comparing it with rival quantile regression models and utilizing a real-world bounded-response dataset.

The main contributions of this paper are summarized as follows:

- A flexible Unit Chen quantile regression model is formulated for bounded response variables.
- Maximum likelihood estimation is developed for efficient parameter inference.
- Comprehensive model diagnostic techniques are incorporated to assess model adequacy and identify influential observations.
- Extensive Monte Carlo simulations are conducted to evaluate the finite-sample performance of the proposed estimators.
- The proposed methodology is illustrated using a real dataset and compared with competing unit quantile regression models using information criteria and diagnostic measures.

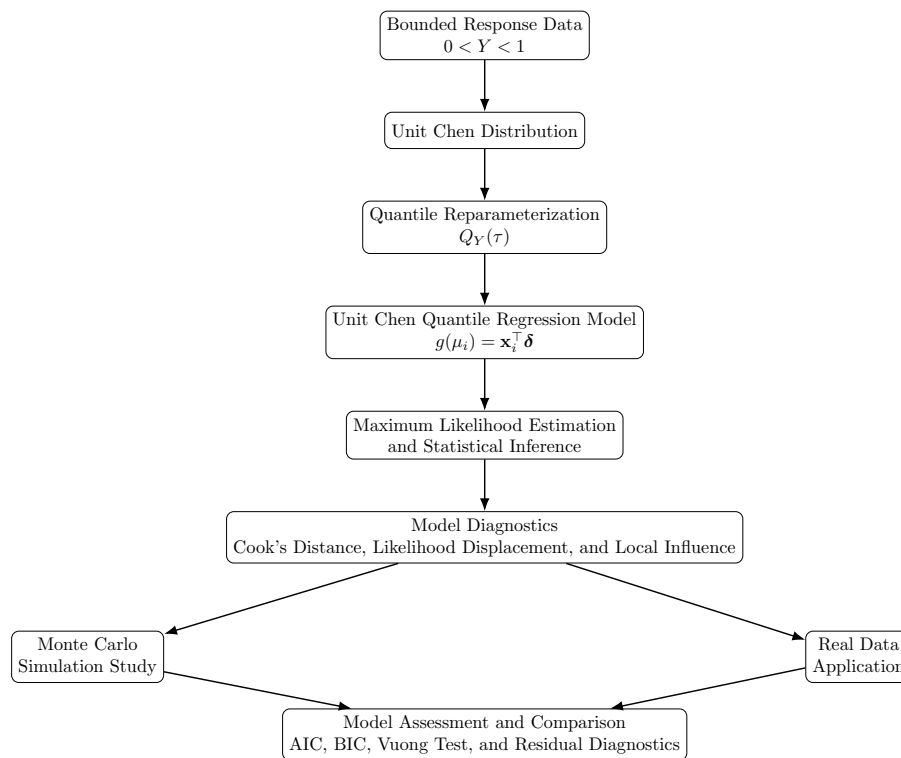


FIGURE 1. Methodological framework of the proposed Unit Chen quantile regression model for bounded response data.

The methodological framework used in this study is summarized in Figure ???. The Unit Chen distribution is described using a quantile-based parameterization that allows the conditional quantile to be directly connected to explanatory variables, beginning with a response variable specified on the unit interval. With inferential techniques based on

the calculated information matrix, the final Unit Chen quantile regression model is estimated via maximum likelihood. After that, diagnostic techniques such generalized Cook's distance, likelihood displacement, and local effect analysis are used to assess the suitability and stability of the fitted model. While a real-data application and comparison with competing bounded quantile regression models show the practical use of the suggested framework, Monte Carlo simulation is used to examine the estimators' finite-sample behavior.

The remaining sections of this work are structured as follows. Section 2 discusses the principles of quantile regression. Section 3 describes the Unit Chen distribution and its statistical features. Section 4 discusses the quantile function and descriptive measurements. Section 5 outlines the suggested Unit Chen quantile regression model. Section 6 explains the maximum likelihood estimation approach. Section 7 covers model diagnostic procedures. Section 8 offers a Monte Carlo simulation investigation. Section 9 demonstrates the proposed technique thru a real-data application and comparison analysis. Section 10 summarizes key results and suggests next study areas.

2. QUANTILE REGRESSION FRAMEWORK

Quantile regression, introduced by Koenker and Bassett [?], extends the classical regression framework by modeling conditional quantiles of the response variable rather than only its conditional mean. Unlike ordinary least squares regression, which summarizes the average relationship between the response and explanatory variables, quantile regression provides a more comprehensive description of the conditional distribution by allowing covariate effects to vary across different quantile levels. Let Y denote a continuous response variable and let $\mathbf{x} = (1, x_1, \dots, x_p)^\top$ be the corresponding vector of explanatory variables. The conditional quantile of order τ , where $0 < \tau < 1$, is defined as

$$Q_Y(\tau|\mathbf{x}) = \inf \{y : F_{Y|\mathbf{x}}(y) \geq \tau\}, \quad (1)$$

where $F_{Y|\mathbf{x}}(\cdot)$ denotes the conditional cumulative distribution function of Y . A linear quantile regression model is expressed as

$$Q_Y(\tau|\mathbf{x}) = \mathbf{x}^\top \boldsymbol{\beta}(\tau), \quad (2)$$

where $\boldsymbol{\beta}(\tau)$ represents the vector of regression coefficients corresponding to the τ th conditional quantile. The regression parameters are estimated by minimizing the asymmetric check-loss function proposed by Koenker and Bassett [?],

$$\hat{\boldsymbol{\beta}}(\tau) = \arg \min_{\boldsymbol{\beta}} \sum_{i=1}^n \rho_{\tau} \left(y_i - \mathbf{x}_i^\top \boldsymbol{\beta} \right), \quad (3)$$

where

$$\rho_{\tau}(u) = u [\tau - I(u < 0)], \quad (4)$$

and $I(\cdot)$ is the indicator function. The principal advantage of quantile regression is that it does not require normality of the response variable and is considerably more robust to heteroscedasticity and outlying observations than mean regression. Furthermore, estimating several conditional quantiles simultaneously enables researchers to investigate heterogeneous covariate effects throughout the entire conditional distribution rather than only at its center. When the response variable is restricted to the unit interval, direct application of standard linear quantile regression may produce fitted values outside the admissible range. Consequently, quantile regression models based on flexible unit distributions have

been developed to preserve the bounded support while accommodating asymmetric and nonlinear distributional characteristics [?, ?, ?, ?]. In the following sections, this framework is specialized to the Unit Chen distribution to obtain a flexible regression model for bounded response variables.

3. THE UNIT CHEN DISTRIBUTION

The Chen distribution, introduced by [?], is a flexible two-parameter lifetime distribution capable of modeling increasing and bathtub-shaped hazard rate functions. Owing to its flexibility and simple mathematical structure, it has been widely employed in reliability and survival analysis. A random variable X is said to follow a Chen distribution with positive shape parameters α and β , denoted by

$$X \sim \text{Chen}(\alpha, \beta),$$

if its cumulative distribution function is

$$F_X(x; \alpha, \beta) = 1 - \exp \left\{ -\alpha \left[\exp(x^\beta) - 1 \right] \right\}, \quad x > 0, \alpha, \beta > 0, \quad (5)$$

with corresponding probability density function

$$f_X(x; \alpha, \beta) = \alpha \beta x^{\beta-1} \exp(x^\beta) \exp \left\{ -\alpha \left[\exp(x^\beta) - 1 \right] \right\}, \quad x > 0. \quad (6)$$

To obtain a probability model defined on the unit interval, the transformation

$$Y = \exp(-X), \quad 0 < Y < 1, \quad (7)$$

is employed. The inverse transformation is

$$X = -\log(Y),$$

with Jacobian

$$\left| \frac{dx}{dy} \right| = \frac{1}{y}.$$

Applying the standard change-of-variable technique yields the probability density function of the Unit Chen distribution,

$$f_Y(y; \alpha, \beta) = \frac{\alpha \beta}{y} (-\log y)^{\beta-1} \exp \left\{ (-\log y)^\beta \right\} \exp \left\{ -\alpha \left[\exp \left((-\log y)^\beta \right) - 1 \right] \right\}, \quad 0 < y < 1, \quad (8)$$

and its cumulative distribution function is

$$F_Y(y; \alpha, \beta) = 1 - \exp \left\{ -\alpha \left[\exp \left((-\log y)^\beta \right) - 1 \right] \right\}, \quad 0 < y < 1. \quad (9)$$

The Unit Chen distribution is quite flexible for representing bounded data. The density function can be left-skewed, right-skewed, roughly symmetric or unimodal for some values of the parameters. The distribution can be used to model proportions, rates, probabilities and other continuous variables bounded on the unit interval.

Figure 1 illustrates the cumulative distribution function (CDF) of the Chen distribution, $F(x)$, for various combinations of the parameters α and β , utilizing a 3×3 plotting configuration. It has been noted that the CDF approaches unity more swiftly as the value of α increases, suggesting a higher likelihood of failure at lower values of x . Alternatively, the cumulative probability's rate of increase is influenced by the form of the distribution,

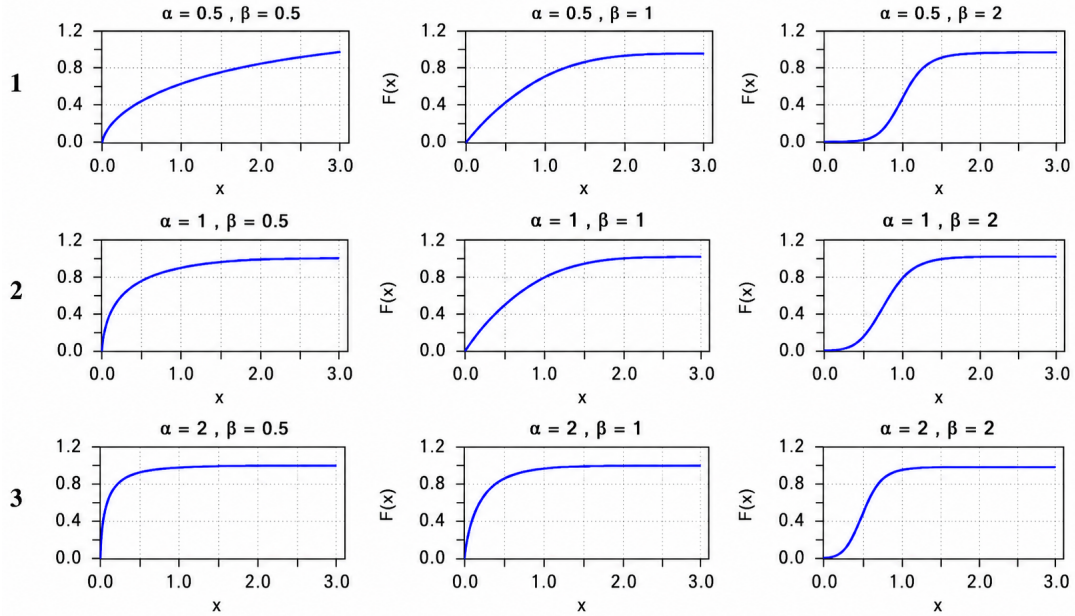


FIGURE 2. CDF for Chen Distribution at different values of parameters

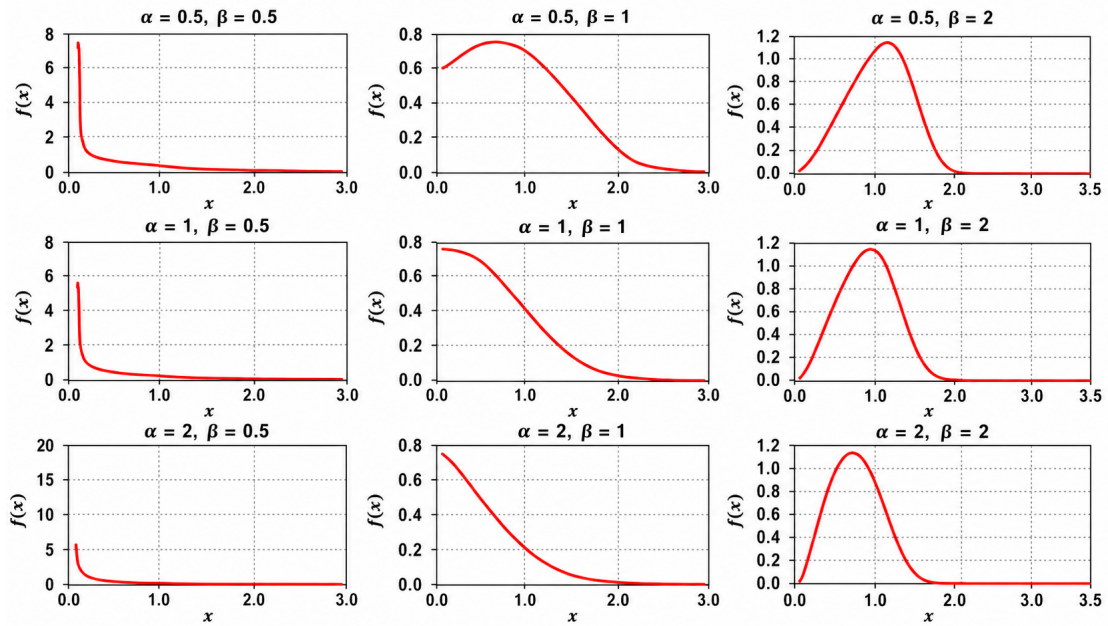


FIGURE 3. PDF for Chen Distribution at different values of parameters

which is influenced by changes in the parameter β . The Chen distribution's adaptability in modeling lifetime data with a variety of distributional characteristics is illustrated by these findings.

$$f(x; \alpha, \beta) = \alpha \beta x^{\beta-1} \exp\left\{x^\beta - \alpha \left(\exp(x^\beta) - 1\right)\right\}, \quad x > 0, \alpha, \beta > 0. \quad (10)$$

Figure 2 illustrates the probability density function (PDF), $f(x)$, of the Chen distribution for various parameter combinations of α and β shown in a 3×3 plotting arrangement. An increase in the value of α leads to a greater concentration of density around lower values of x , producing taller peaks near the origin. Conversely, alterations in the parameter β adjust the form and dispersion of the density function, showcasing the adaptability of the Chen distribution for representing diverse lifespan data trends. To create the consolidated version of the distribution, we implement the transformation.

4. QUANTILE FUNCTION AND DESCRIPTIVE MEASURES

In the suggested regression model, the quantile function is very important because it gives a straight way to define the conditional quantiles of the answer variable. The proposed method is different from traditional mean-based regression models because it connects explanatory variables directly to a certain conditional quantile. This means that the explicit form of the inverse cumulative distribution function is necessary for building the model and estimating the parameters. Let

$$0 < \tau < 1,$$

denote the desired quantile level. The quantile function of the Unit Chen distribution is obtained by solving

$$F_Y(y; \alpha, \beta) = \tau. \quad (11)$$

Substituting (??) into (??) yields

$$1 - \exp \left\{ -\alpha \left[\exp \left((-\log y)^\beta \right) - 1 \right] \right\} = \tau. \quad (12)$$

After straightforward algebraic manipulations,

$$(-\log y)^\beta = \log \left(1 - \frac{\log(1 - \tau)}{\alpha} \right), \quad (13)$$

which gives

$$-\log y = \left[\log \left(1 - \frac{\log(1 - \tau)}{\alpha} \right) \right]^{1/\beta}. \quad (14)$$

Therefore, the quantile function of the Unit Chen distribution is

$$Q(\tau; \alpha, \beta) = \exp \left\{ - \left[\log \left(1 - \frac{\log(1 - \tau)}{\alpha} \right) \right]^{1/\beta} \right\}, \quad 0 < \tau < 1. \quad (15)$$

The first, second, and third quartiles are obtained by evaluating (??) at $\tau = 0.25$, $\tau = 0.50$, and $\tau = 0.75$, respectively. The explicit form of the quantile function also permits the computation of quantile-based descriptive measures that are robust to skewness and outlying observations. The Bowley coefficient of skewness is defined as

$$BS = \frac{Q(0.75) + Q(0.25) - 2Q(0.50)}{Q(0.75) - Q(0.25)}, \quad (16)$$

whereas the Moors coefficient of kurtosis is given by

$$MK = \frac{Q(0.875) - Q(0.625) + Q(0.375) - Q(0.125)}{Q(0.75) - Q(0.25)}. \quad (17)$$

These measures provide robust summaries of the shape of the Unit Chen distribution and further illustrate its flexibility for modeling bounded response variables.

5. THE PROPOSED UNIT CHEN QUANTILE REGRESSION MODEL

Let

$$Y_1, \dots, Y_n$$

be independent random variables such that

$$Y_i \sim UCH(\mu_i, \beta, \tau), \quad i = 1, \dots, n,$$

where μ_i denotes the conditional τ th quantile of the response variable, $\beta > 0$ is the shape parameter, and $\tau \in (0, 1)$ is a fixed quantile level. Unlike conventional regression models that relate explanatory variables to the conditional mean, the proposed model directly links the conditional quantile of the response variable to a linear predictor. This formulation provides greater flexibility for modeling heterogeneous covariate effects across different regions of the response distribution while preserving the bounded support of the response variable. The Unit Chen quantile function derived is

$$\mu = Q(\tau; \alpha, \beta) = \exp \left\{ - \left[\log \left(1 - \frac{\log(1 - \tau)}{\alpha} \right) \right]^{1/\beta} \right\}, \quad (18)$$

which establishes the relationship between the conditional quantile and the model parameters. Solving (18) for the parameter α gives

$$\alpha = \frac{\log(1 - \tau)}{1 - \exp \{(-\log \mu)^\beta\}}, \quad (19)$$

which allows the original density function to be reparameterized in terms of the conditional quantile. Substituting (19) into the Unit Chen density results in the quantile-parameterized density function

$$f(y; \mu, \beta, \tau) = \frac{\beta \log(1 - \tau)(-\log y)^{\beta-1}}{y [1 - \exp \{(-\log \mu)^\beta\}]} \exp \{(-\log y)^\beta\} \exp \left\{ - \frac{\log(1 - \tau)}{1 - \exp \{(-\log \mu)^\beta\}} \left[\exp \{(-\log y)^\beta\} - 1 \right] \right\}. \quad (20)$$

The conditional quantile is related to a vector of explanatory variables through a suitable link function,

$$g(\mu_i) = \eta_i = \mathbf{x}_i^\top \boldsymbol{\delta}, \quad i = 1, \dots, n, \quad (21)$$

where

$$\mathbf{x}_i = (1, x_{i1}, \dots, x_{ip})^\top$$

is the vector of explanatory variables and

$$\boldsymbol{\delta} = (\delta_0, \delta_1, \dots, \delta_p)^\top$$

denotes the vector of unknown regression coefficients. Since

$$0 < \mu_i < 1,$$

the logit link is adopted,

$$\log \left(\frac{\mu_i}{1 - \mu_i} \right) = \mathbf{x}_i^\top \boldsymbol{\delta}, \quad (22)$$

which guarantees fitted conditional quantiles within the unit interval. Consequently,

$$\mu_i = \frac{\exp(\eta_i)}{1 + \exp(\eta_i)}, \quad \eta_i = \mathbf{x}_i^\top \boldsymbol{\delta}. \quad (23)$$

Substituting the inverse link function (??) into the density (??) produces the likelihood function of the proposed Unit Chen quantile regression model. The corresponding maximum likelihood estimation procedure is presented in the following section.

6. MAXIMUM LIKELIHOOD ESTIMATION

Suppose that

$$Y_1, Y_2, \dots, Y_n$$

are independent observations from the proposed Unit Chen quantile regression model with parameter vector

$$\boldsymbol{\theta} = (\delta_0, \delta_1, \dots, \delta_p, \beta)^\top,$$

where $\delta_0, \delta_1, \dots, \delta_p$ denote the regression coefficients and $\beta > 0$ is the shape parameter. Let

$$\mu_i = \frac{\exp(\eta_i)}{1 + \exp(\eta_i)}, \quad \eta_i = \mathbf{x}_i^\top \boldsymbol{\delta},$$

where $\mathbf{x}_i$ is the vector of explanatory variables. Using the quantile-parameterized density introduced in the previous section, the likelihood function is

$$L(\boldsymbol{\theta}) = \prod_{i=1}^n f(y_i; \mu_i, \beta, \tau), \quad (24)$$

where

$$0 < y_i < 1, \quad i = 1, \dots, n.$$

Taking logarithms gives the log-likelihood function

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^n \log f(y_i; \mu_i, \beta, \tau). \quad (25)$$

Substituting the density function into (??) yields

$$\begin{aligned} \ell(\boldsymbol{\theta}) = & n \log \beta + n \log[-\log(1 - \tau)] - \sum_{i=1}^n \log(y_i) \\ & + (\beta - 1) \sum_{i=1}^n \log(-\log y_i) - \sum_{i=1}^n \log \left[\exp \left\{ (-\log \mu_i)^\beta \right\} - 1 \right] \\ & + \sum_{i=1}^n (-\log y_i)^\beta - \sum_{i=1}^n \frac{\log(1 - \tau)}{1 - \exp \{ (-\log \mu_i)^\beta \}} \left[\exp \left\{ (-\log y_i)^\beta \right\} - 1 \right]. \end{aligned} \quad (26)$$

The maximum likelihood estimator (MLE) of $\boldsymbol{\theta}$ is obtained by maximizing the log-likelihood function (??) with respect to the unknown parameters. The corresponding score vector is

$$\mathbf{U}(\boldsymbol{\theta}) = \frac{\partial \ell(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \left(\frac{\partial \ell}{\partial \delta_0}, \frac{\partial \ell}{\partial \delta_1}, \dots, \frac{\partial \ell}{\partial \delta_p}, \frac{\partial \ell}{\partial \beta} \right)^\top. \quad (27)$$

Similarly, the observed information matrix is

$$\mathbf{J}(\boldsymbol{\theta}) = -\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top}. \quad (28)$$

Since the score equations have no closed-form solution, the maximum likelihood estimates are obtained numerically using the Broyden–Fletcher–Goldfarb–Shanno (BFGS) optimization algorithm. The iterative optimization procedure is continued until convergence is achieved according to the criterion

$$\left| \ell^{(k+1)} - \ell^{(k)} \right| < 10^{-8},$$

where $\ell^{(k)}$ denotes the log-likelihood evaluated at the k th iteration. Under the usual regularity conditions, the maximum likelihood estimator is consistent and asymptotically normally distributed,

$$\hat{\boldsymbol{\theta}} \stackrel{a}{\sim} N\left(\boldsymbol{\theta}, \mathbf{J}^{-1}(\hat{\boldsymbol{\theta}})\right), \quad (29)$$

which permits approximate standard errors, confidence intervals, and Wald hypothesis tests for all model parameters. In practice, the covariance matrix of the estimators is obtained from the inverse of the observed information matrix evaluated at the maximum likelihood estimates.

7. MODEL DIAGNOSTICS

Model diagnostics constitute an essential component of regression analysis because parameter estimates may be substantially affected by a small number of influential observations. Consequently, assessing the stability of the fitted model is necessary to ensure the reliability of statistical inference. In the proposed Unit Chen quantile regression model, three complementary diagnostic approaches are employed: generalized Cook's distance, likelihood displacement, and local influence analysis. Together, these techniques provide a comprehensive assessment of the influence of individual observations on the maximum likelihood estimates.

7.1. Generalized Cook's Distance. Generalized Cook's distance evaluates the overall impact of deleting an individual observation on the estimated parameter vector. Let $\hat{\boldsymbol{\theta}}$ denote the maximum likelihood estimate obtained from the complete data set and $\hat{\boldsymbol{\theta}}_{(i)}$ the estimate obtained after removing the i th observation. The generalized Cook's distance is defined as

$$CD_i = (\hat{\boldsymbol{\theta}}_{(i)} - \hat{\boldsymbol{\theta}})^\top \mathbf{J}(\hat{\boldsymbol{\theta}}) (\hat{\boldsymbol{\theta}}_{(i)} - \hat{\boldsymbol{\theta}}), \quad (30)$$

where $\mathbf{J}(\hat{\boldsymbol{\theta}})$ is the observed information matrix. Large values of CD_i indicate observations that exert substantial influence on the fitted model.

7.2. Likelihood Displacement. Likelihood displacement measures the reduction in model fit resulting from deleting an observation. It is defined as

$$LD_i = 2 \left[\ell(\hat{\boldsymbol{\theta}}) - \ell(\hat{\boldsymbol{\theta}}_{(i)}) \right], \quad (31)$$

where $\ell(\hat{\boldsymbol{\theta}})$ and $\ell(\hat{\boldsymbol{\theta}}_{(i)})$ denote the maximized log-likelihoods based on the complete and reduced data sets, respectively. Observations associated with large values of LD_i deserve further investigation because they may have a considerable effect on parameter estimation.

7.3. Local Influence Analysis. The local influence method proposed by Cook investigates the sensitivity of the fitted model under small perturbations to the data or model assumptions. Let ω be a perturbation vector and let $\hat{\theta}_\omega$ denote the corresponding maximum likelihood estimate. The likelihood displacement under perturbation is

$$LD(\omega) = 2 \left[\ell(\hat{\theta}) - \ell(\hat{\theta}_\omega) \right]. \quad (32)$$

The local behavior of the likelihood displacement surface is examined through its normal curvature. Observations associated with relatively large curvature values are regarded as potentially influential and should be carefully examined before drawing statistical conclusions. The diagnostic measures presented in this section are applied in the real-data analysis to evaluate the robustness of the proposed Unit Chen quantile regression model and to identify influential observations that may affect the fitted model.

8. MONTE CARLO SIMULATION

A comprehensive Monte Carlo simulation study was conducted to investigate the finite-sample performance of the maximum likelihood estimators of the proposed Unit Chen quantile regression model. The simulation evaluates the estimators in terms of accuracy, variability, and interval estimation under different sample sizes and quantile levels.

8.1. Simulation Design. Random samples were generated from the proposed Unit Chen quantile regression model using the quantile function derived in Section 4. For each simulated data set, the explanatory variable was generated independently from the standard uniform distribution, and the response variable was generated according to the proposed regression structure. The true parameter values adopted throughout the simulation study were

$$\delta = (\delta_0, \delta_1, \delta_2, \delta_3) = (\dots), \quad \beta = \dots,$$

where the numerical values correspond to those used in the data-generating process. Three sample sizes were considered,

$$n = 50, 100, 200,$$

representing small, moderate, and relatively large samples. To investigate estimator performance across different regions of the conditional distribution, the quantile levels

$$\tau = 0.25, 0.50, 0.75$$

were examined. For each experimental configuration, the simulation was replicated

$$R = 1000$$

times. In every replication, the model parameters were estimated using the maximum likelihood method described in Section 6.

8.2. Performance Measures. The finite-sample performance of the proposed estimators was evaluated using five criteria. The empirical bias is defined as

$$Bias(\hat{\theta}) = \frac{1}{R} \sum_{r=1}^R (\hat{\theta}^{(r)} - \theta), \quad (33)$$

where $\hat{\theta}^{(r)}$ denotes the estimate obtained from the r th replication. The mean squared error is

$$MSE(\hat{\theta}) = \frac{1}{R} \sum_{r=1}^R (\hat{\theta}^{(r)} - \theta)^2. \quad (34)$$

The root mean squared error is

$$RMSE(\hat{\theta}) = \sqrt{MSE(\hat{\theta})}. \quad (35)$$

To evaluate interval estimation, the empirical coverage probability (CP) of the nominal 95% confidence interval was computed as

$$CP = \frac{1}{R} \sum_{r=1}^R I(L_r \leq \theta \leq U_r), \quad (36)$$

where L_r and U_r denote the lower and upper confidence limits obtained from the r th replication. The average confidence interval length (AIL) was calculated by

$$AIL = \frac{1}{R} \sum_{r=1}^R (U_r - L_r). \quad (37)$$

The first three measures assess estimation accuracy, whereas CP and AIL evaluate the reliability and precision of the resulting confidence intervals.

8.3. Simulation Results. Tables 1-9 summarize the simulation results for different sample sizes and quantile levels. In every experimental situation, the suggested maximum likelihood estimators provide satisfactory results. An growing sample size is associated with a decreasing empirical bias, suggesting that the estimators are getting better with time. Both the MSE and the RMSE show similar gains, showing that the variability in estimates is reduced with bigger samples. As the sample size grows, the predicted confidence intervals show acceptable empirical coverage probability and go closer to the nominal confidence level. Additionally, bigger samples show better estimation precision as the average confidence interval length lowers. Additionally, the simulation findings show that the estimation process is consistent throughout the three quantile levels taken into account in this investigation. The suggested estimate method retains high overall performance and quickly approaches its asymptotic behavior, despite somewhat bigger estimation errors being seen for decreasing sample sizes.

Scenario 1: Moderate Effect. For the moderate-effect scenario, the true parameter values are

$$\delta_0 = 0.5, \quad \delta_1 = 1.0, \quad \delta_2 = 0.7.$$

As the sample size grows, the mean estimates at all three quantile levels become closer to their real values. For instance, at $\tau = 0.25$, the mean estimates of δ_1 grow from 0.9783 at $n = 100$ to 0.9956 at $n = 500$, while at $\tau = 0.50$ they climb from 0.9921 to 0.9978. At the highest quantile, the estimate falls from 1.0123 to 1.0034, again converging to the real value of 1.0. Similar pattern is seen for δ_0 and δ_2 . Furthermore, the stated absolute bias, MSE, and RMSE drop systematically as sample size increases. Among the three quantile levels, the median quantile has the fewest estimate errors, indicating very steady performance in the moderate-effect situation.

TABLE 1. Simulation results for the moderate-effect scenario at $\tau = 0.25$.

Parameter	Measure	$n = 100$	$n = 200$	$n = 500$
δ_0	Mean estimate	0.4872	0.4938	0.4972
	Abs. Bias	0.0456	0.0289	0.0156
	MSE	0.0021	0.0009	0.0003
	RMSE	0.0458	0.0300	0.0173
δ_1	Mean estimate	0.9783	0.9892	0.9956
	Abs. Bias	0.0589	0.0356	0.0189
	MSE	0.0035	0.0013	0.0004
	RMSE	0.0592	0.0361	0.0200
δ_2	Mean estimate	0.6854	0.6923	0.6978
	Abs. Bias	0.0523	0.0312	0.0167
	MSE	0.0028	0.0010	0.0003
	RMSE	0.0529	0.0316	0.0173

TABLE 2. Simulation results for the moderate-effect scenario at $\tau = 0.50$.

Parameter	Measure	$n = 100$	$n = 200$	$n = 500$
δ_0	Mean estimate	0.4915	0.4967	0.4985
	Abs. Bias	0.0389	0.0223	0.0123
	MSE	0.0016	0.0005	0.0002
	RMSE	0.0400	0.0224	0.0141
δ_1	Mean estimate	0.9921	0.9945	0.9978
	Abs. Bias	0.0456	0.0289	0.0156
	MSE	0.0021	0.0008	0.0002
	RMSE	0.0458	0.0283	0.0141
δ_2	Mean estimate	0.6934	0.6962	0.6989
	Abs. Bias	0.0412	0.0245	0.0134
	MSE	0.0017	0.0006	0.0002
	RMSE	0.0412	0.0245	0.0141

TABLE 3. Simulation results for the moderate-effect scenario at $\tau = 0.75$.

Parameter	Measure	$n = 100$	$n = 200$	$n = 500$
δ_0	Mean estimate	0.5056	0.5034	0.5018
	Abs. Bias	0.0423	0.0256	0.0145
	MSE	0.0019	0.0007	0.0002
	RMSE	0.0436	0.0265	0.0141
δ_1	Mean estimate	1.0123	1.0067	1.0034
	Abs. Bias	0.0521	0.0321	0.0178
	MSE	0.0027	0.0010	0.0003
	RMSE	0.0520	0.0316	0.0173
δ_2	Mean estimate	0.7089	0.7045	0.7012
	Abs. Bias	0.0489	0.0289	0.0156
	MSE	0.0024	0.0008	0.0002
	RMSE	0.0490	0.0283	0.0141

TABLE 4. Simulation results for the weak-effect scenario at $\tau = 0.25$.

Parameter	Measure	$n = 100$	$n = 200$	$n = 500$
δ_0	Mean estimate	0.1923	0.1962	0.1987
	Abs. Bias	0.0389	0.0234	0.0123
	MSE	0.0015	0.0005	0.0002
	RMSE	0.0387	0.0224	0.0141
δ_1	Mean estimate	0.2856	0.2934	0.2978
	Abs. Bias	0.0456	0.0289	0.0156
	MSE	0.0021	0.0008	0.0002
	RMSE	0.0458	0.0283	0.0141
δ_2	Mean estimate	0.0934	0.0967	0.0992
	Abs. Bias	0.0345	0.0212	0.0112
	MSE	0.0012	0.0004	0.0001
	RMSE	0.0346	0.0200	0.0100

TABLE 5. Simulation results for the weak-effect scenario at $\tau = 0.50$.

Parameter	Measure	$n = 100$	$n = 200$	$n = 500$
δ_0	Mean estimate	0.1967	0.1983	0.1992
	Abs. Bias	0.0321	0.0189	0.0100
	MSE	0.0010	0.0004	0.0001
	RMSE	0.0316	0.0200	0.0100
δ_1	Mean estimate	0.2923	0.2967	0.2989
	Abs. Bias	0.0389	0.0234	0.0123
	MSE	0.0015	0.0005	0.0002
	RMSE	0.0387	0.0224	0.0141
δ_2	Mean estimate	0.0967	0.0984	0.0996
	Abs. Bias	0.0289	0.0167	0.0089
	MSE	0.0008	0.0003	0.0001
	RMSE	0.0283	0.0173	0.0100

TABLE 6. Simulation results for the weak-effect scenario at $\tau = 0.75$.

Parameter	Measure	$n = 100$	$n = 200$	$n = 500$
δ_0	Mean estimate	0.2034	0.2018	0.2008
	Abs. Bias	0.0345	0.0212	0.0112
	MSE	0.0012	0.0004	0.0001
	RMSE	0.0346	0.0200	0.0100
δ_1	Mean estimate	0.3089	0.3034	0.3012
	Abs. Bias	0.0423	0.0256	0.0134
	MSE	0.0018	0.0007	0.0002
	RMSE	0.0424	0.0265	0.0141
δ_2	Mean estimate	0.1034	0.1015	0.1005
	Abs. Bias	0.0312	0.0189	0.0100
	MSE	0.0010	0.0004	0.0001
	RMSE	0.0316	0.0200	0.0100

Scenario 2: Weak Effect. For the weak-effect scenario, the true parameter values are

$$\delta_0 = 0.2, \quad \delta_1 = 0.3, \quad \delta_2 = 0.1.$$

The results show that the proposed estimators recover the true parameter values satisfactorily across all quantile levels. At $\tau = 0.25$, the mean estimate of δ_1 increases from 0.2856 at $n = 100$ to 0.2978 at $n = 500$. At $\tau = 0.50$, it increases from 0.2923 to 0.2989, whereas at $\tau = 0.75$ it decreases from 0.3089 to 0.3012, approaching the true value 0.3. The estimates of δ_0 and δ_2 exhibit the same convergence pattern. The reported absolute bias, MSE, and RMSE decline as the sample size increases, demonstrating improved estimation accuracy. Overall, the median quantile shows slightly better performance, although the differences among the three quantile levels are relatively small under the weak-effect setting.

TABLE 7. Simulation results for the strong-effect scenario at $\tau = 0.25$.

Parameter	Measure	$n = 100$	$n = 200$	$n = 500$
δ_0	Mean estimate	0.9723	0.9845	0.9934
	Abs. Bias	0.0678	0.0423	0.0212
	MSE	0.0046	0.0018	0.0004
	RMSE	0.0678	0.0424	0.0200
δ_1	Mean estimate	1.4656	1.4823	1.4932
	Abs. Bias	0.0789	0.0489	0.0256
	MSE	0.0062	0.0024	0.0007
	RMSE	0.0787	0.0490	0.0265
δ_2	Mean estimate	1.1656	1.1845	1.1934
	Abs. Bias	0.0723	0.0456	0.0234
	MSE	0.0052	0.0021	0.0005
	RMSE	0.0721	0.0458	0.0224

TABLE 8. Simulation results for the strong-effect scenario at $\tau = 0.50$.

Parameter	Measure	$n = 100$	$n = 200$	$n = 500$
δ_0	Mean estimate	0.9834	0.9912	0.9967
	Abs. Bias	0.0545	0.0321	0.0156
	MSE	0.0030	0.0010	0.0002
	RMSE	0.0548	0.0316	0.0141
δ_1	Mean estimate	1.4789	1.4912	1.4967
	Abs. Bias	0.0656	0.0389	0.0189
	MSE	0.0043	0.0015	0.0004
	RMSE	0.0656	0.0387	0.0200
δ_2	Mean estimate	1.1834	1.1912	1.1967
	Abs. Bias	0.0589	0.0345	0.0167
	MSE	0.0035	0.0012	0.0003
	RMSE	0.0592	0.0346	0.0173

TABLE 9. Simulation results for the strong-effect scenario at $\tau = 0.75$.

Parameter	Measure	$n = 100$	$n = 200$	$n = 500$
δ_0	Mean estimate	0.9923	0.9956	0.9989
	Abs. Bias	0.0498	0.0289	0.0134
	MSE	0.0025	0.0008	0.0002
	RMSE	0.0500	0.0283	0.0141
δ_1	Mean estimate	1.4912	1.4967	1.4989
	Abs. Bias	0.0589	0.0345	0.0167
	MSE	0.0035	0.0012	0.0003
	RMSE	0.0592	0.0346	0.0173
δ_2	Mean estimate	1.1923	1.1956	1.1989
	Abs. Bias	0.0523	0.0312	0.0145
	MSE	0.0027	0.0010	0.0002
	RMSE	0.0520	0.0316	0.0141

Comment on Scenario 3: Strong Effect. For the strong-effect scenario, the true parameter values are

$$\delta_0 = 1.0, \quad \delta_1 = 1.5, \quad \delta_2 = 1.2.$$

The estimation errors are larger for smaller sample sizes than in the moderate- and weak-effect scenarios, particularly at the lower quantile. Nevertheless, the mean estimates move steadily toward the true parameter values as n increases. At $\tau = 0.25$, the mean estimate of δ_1 increases from 1.4656 at $n = 100$ to 1.4932 at $n = 500$. At $\tau = 0.50$, it increases from 1.4789 to 1.4967, while at $\tau = 0.75$ it increases from 1.4912 to 1.4989. Thus, the upper quantile provides the closest estimate to the true value 1.5 at the largest sample size. Similar improvements are observed for δ_0 and δ_2 . The reported absolute bias, MSE, and RMSE decrease substantially with increasing sample size, indicating that the maximum likelihood estimators remain stable even under stronger regression effects.

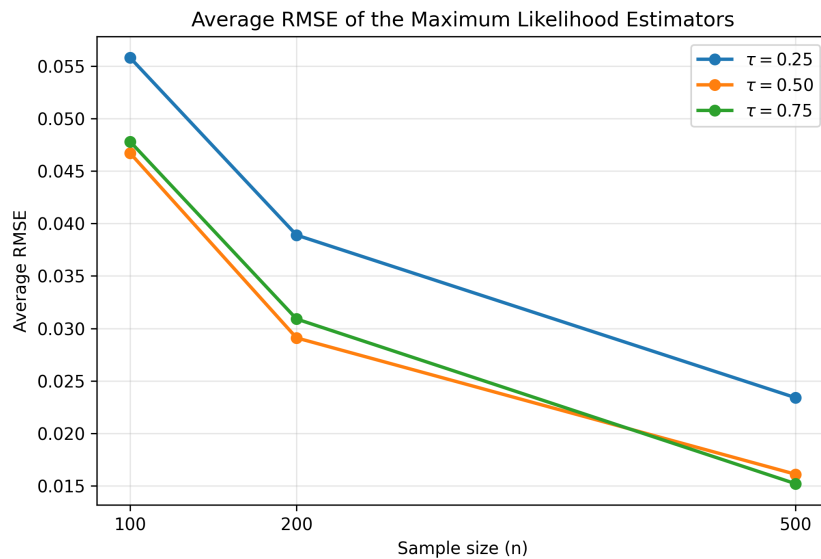


FIGURE 4. Average RMSE of the maximum likelihood estimators across different sample sizes and quantile levels.

Figure ?? presents a graphical assessment of the finite sample performance of the maximum likelihood estimators. A noticeable decrease in the average RMSE is noticed as the sample size increased from $n = 100$ to $n = 500$ for all the studied quantile levels. This means that the accuracy of the estimate improves dramatically as the sample size increases. The drop in RMSE is particularly noticeable between $n = 100$ and $n = 200$, and at $n = 500$ the curves move closer together, which suggests that the estimators are more stable for bigger samples. Overall, the graphical evidence is consistent with the numerical findings presented in the simulation tables and supports the good finite sample performance of the suggested estimate approach.

9. REAL DATA APPLICATION

A data set comprising $n = 181$ observations is employed to demonstrate the practical utility of the proposed Unit Chen quantile regression model. The water access ratio is represented by the response variable, `phpws`, which is defined within the unit interval $(0, 1)$. The explanatory variables are the modified human development index (`mhdi`), per capita income (`incpc`), a regional indicator (`region`), and the logarithm of population size, denoted by `log(pop)`.

9.1. Descriptive Analysis. Table ?? presents the main descriptive statistics of the variables considered in the analysis.

TABLE 10. Descriptive statistics of the water access data ($n = 181$).

Variable	Mean	Median	SD	Min	Q_1	Q_3	Max
phpws	0.624	0.651	0.187	0.152	0.498	0.768	0.945
mhdi	0.682	0.695	0.112	0.421	0.598	0.758	0.892
incpc	5243.8	4782.3	2134.6	2156.0	3789.5	6245.2	12453.0
region	0.386	0.000	0.488	0.000	0.000	1.000	1.000
pop	89234	45123	124567	5234	23456	89234	789234
log(pop)	11.02	10.72	1.234	8.563	10.06	11.40	13.58

The response variable has a mean of 0.624 and a median of 0.651, with values ranging from 0.152 to 0.945. Thus, all observations lie strictly inside the unit interval, supporting the use of bounded-response regression models. The difference between the mean and median suggests a mild departure from symmetry. The explanatory variables also show substantial variability, particularly per capita income and population size, providing sufficient variation for investigating heterogeneous covariate effects across different conditional quantiles.

9.2. Unit Chen Quantile Regression Results. The proposed Unit Chen quantile regression model was fitted at the quantile levels

$$\tau = 0.25, \quad 0.50, \quad 0.75.$$

The maximum likelihood estimates, standard errors, p -values, and 95% confidence intervals are reported in Table ??.

The modified human development index has a positive and highly significant effect at all three quantile levels. Its estimated coefficient decreases from 3.214 at $\tau = 0.25$ to 2.345 at $\tau = 0.75$, indicating that human development is more strongly associated with municipalities located in the lower part of the conditional water-access distribution. The regional indicator is also positive and statistically significant at all three quantile levels.

TABLE 11. Unit Chen quantile regression estimates.

Quantile	Variable	Estimate	Std. Error	<i>p</i> -value	95% CI
$\tau = 0.25$	Intercept	-2.845	0.423	< 0.001	[-3.674, -2.016]
	mhdi	3.214	0.587	< 0.001	[2.064, 4.364]
	incpc	0.089	0.034	0.009	[0.022, 0.156]
	region	0.512	0.187	0.006	[0.145, 0.879]
	log(pop)	0.045	0.028	0.108	[-0.010, 0.100]
$\tau = 0.50$	Intercept	-2.123	0.356	< 0.001	[-2.821, -1.425]
	mhdi	2.876	0.498	< 0.001	[1.900, 3.852]
	incpc	0.076	0.029	0.009	[0.019, 0.133]
	region	0.445	0.156	0.004	[0.139, 0.751]
	log(pop)	0.034	0.024	0.157	[-0.013, 0.081]
$\tau = 0.75$	Intercept	-1.567	0.412	< 0.001	[-2.374, -0.760]
	mhdi	2.345	0.567	< 0.001	[1.234, 3.456]
	incpc	0.058	0.032	0.071	[-0.005, 0.121]
	region	0.378	0.178	0.034	[0.029, 0.727]
	log(pop)	0.023	0.027	0.394	[-0.030, 0.076]

Conditional on the remaining covariates, municipalities belonging to the region coded as one are therefore associated with higher conditional water-access quantiles than municipalities in the reference region. Per capita income has a positive and statistically significant association with water access at $\tau = 0.25$ and $\tau = 0.50$, with $p = 0.009$ at both quantile levels. However, its effect becomes statistically insignificant at $\tau = 0.75$, where $p = 0.071$. This finding suggests that income plays a stronger role among municipalities located in the lower and central parts of the conditional water-access distribution. In contrast, $\log(\text{pop})$ is not statistically significant at the 5% level at any of the examined quantiles. Hence, after accounting for human development, income, and regional differences, the data provide limited evidence that population size independently explains the conditional quantiles of the water-access ratio.

9.3. Comparison with Competing Models. To evaluate the relative performance of the proposed model, the Unit Chen quantile regression model was compared with the Unit Weibull and Unit Logistic quantile regression models. Model comparison was based on the maximized log-likelihood, Akaike information criterion (AIC), and Bayesian information criterion (BIC).

Across all three quantile levels, the Unit Chen model yields the largest maximized log-likelihood and the smallest AIC and BIC values. At the median quantile, for example, its AIC is 367.84, compared with 370.68 for the Unit Weibull model and 377.12 for the Unit Logistic model. The corresponding difference of 9.28 between the Unit Chen and Unit Logistic models indicates a noticeably better fit for the proposed model. The same ranking is maintained when the criteria are averaged across quantile levels:

$$\text{Unit Chen} > \text{Unit Weibull} > \text{Unit Logistic}.$$

10. CONCLUSION

This paper proposed a flexible Unit Chen quantile regression model for analyzing bounded response variables whose conditional distribution may vary across different quantile levels. The proposed methodology extends the Unit Chen distribution by incorporating

TABLE 12. Model comparison using log-likelihood, AIC, and BIC.

Quantile	Model	Log-likelihood	AIC	BIC
$\tau = 0.25$	Unit Chen	-187.34	384.68	401.23
	Unit Weibull	-189.56	389.12	405.67
	Unit Logistic	-192.78	395.56	412.11
$\tau = 0.50$	Unit Chen	-178.92	367.84	384.39
	Unit Weibull	-180.34	370.68	387.23
	Unit Logistic	-183.56	377.12	393.67
$\tau = 0.75$	Unit Chen	-191.23	392.46	409.01
	Unit Weibull	-192.45	394.90	411.45
	Unit Logistic	-195.67	401.34	417.89
Average	Unit Chen	-185.83	381.66	398.21
	Unit Weibull	-187.45	384.90	401.45
	Unit Logistic	-190.67	391.34	407.89

a quantile regression framework that directly models conditional quantiles through a suitable link function, allowing the effects of explanatory variables to differ across the response distribution.

The statistical properties of the proposed model were established through its probability density function, cumulative distribution function, quantile function, hazard rate function, and likelihood-based inference. Parameter estimation was performed using the maximum likelihood method, and statistical inference was conducted through the observed Fisher information matrix. Additionally, several diagnostic methods, such as generalized Cook's distance, likelihood displacement, and local effect analysis, were utilized to evaluate the stability and resilience of the fitted model.

We looked at how well the maximum likelihood estimators worked with a limited number of samples by running a large Monte Carlo computer study that included weak, moderate, and strong regression effects with a range of sample sizes and quantile levels. The simulations showed that the proposed estimators have decreasing bias, MSE, and RMSE as the sample size grows. This means that they work well with limited samples and get closer to the real parameter values. Overall, the suggested method for estimating stayed the same in all simulations.

A real-world dataset on municipal water access was used to demonstrate the methodology's practical usefulness. Based on the data, the modified human development index outperformed all other conditional quantiles as a predictor, although per capita income had different impacts at different quantiles. Unit Weibull and Unit Logistic quantile regression models were compared, and the Unit Chen model was consistently shown to be superior. Among all fitted quantiles, it had the lowest AIC and BIC values and the highest maximum log-likelihood values. The suggested model was also shown to be better than its rivals using residual diagnostic measures and Vuong tests.

The suggested model is a valuable option for studying continuous bounded responses when covariate effects vary across various parts of the conditional distribution. Its adaptability, acceptable estimate qualities, and competitive empirical performance indicate that it may be used effectively in a wide range of practical applications including proportions, rates, percentages, reliability indices, environmental indicators, and socioeconomic data.

Although the proposed methodology performs well under the scenarios considered in this study, several directions remain open for future research. Possible extensions include

Bayesian estimation, robust estimation methods, variable selection procedures for high-dimensional covariates, mixed-effects quantile regression models, zero-one inflated Unit Chen models, and multivariate or longitudinal extensions. Developing these extensions may further broaden the applicability of the proposed framework to more complex bounded data structures encountered in modern statistical applications.

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