

TYPICAL FAMILIES AND T -COMPACTNESS IN PROXIMITY SPACES

RAGHAD ALMOHAMMED^{1*}, DUNIA M. K. AL-FATLAWY², NADIHA A. HABEEB³, LUAY AL-SWIDI⁴, §

ABSTRACT. In this work we introduce a family of subsets of a proximity space, called the typical family, defined as the complement of a cluster family, and we study its main properties and its relation to the cluster concept. We show explicitly that the typical family is, up to the language of proximity spaces, the ideal that is dual to the associated cluster in the classical sense of Choquet's grill ideal duality, and we compare it in detail with ideals, filters, grills, and bornologies, establishing a necessary and sufficient condition under which it is a bornology. Building on this family we define a generalized compactness notion, called T -compactness, relate it to the classical notion of compactness with respect to an ideal, and establish preservation results for finite unions, closed subsets, and via a pushforward construction introduced here images under proximity isomorphisms. We illustrate the framework with a fully worked finite numeric example. We close with a discussion of the broader implications of these results for the relationship between cluster theory and generalized compactness.

Keywords: Cluster Family, Ideal, Proximity, Typical Compact, Typical Family.

AMS Subject Classification: 54E05, 54D30, 54A05, 54C10.

1. INTRODUCTION

Metric spaces occupy a central place in the study of many mathematical and applied problems, since they rely on a numerical notion of distance between two sets or points. A natural question is what should be said when this distance is zero, without the two sets necessarily coinciding. This question motivated Efremovic's introduction, in 1948,

¹ General Directorate of Education of Karbala, Ministry of Education, Karbala, Iraq.
e-mail: raghad.hamid.abbas@alzahraa.edu.iq; ORCID: 0000-0003-0559-0436.

² General Directorate of Education of Karbala, Ministry of Education, Karbala, Iraq.
e-mail: dunya.mohammed.khareem@alzahraa.edu.iq; ORCID: 0009-0007-3600-1389.

³ Department of Mathematics, College of Education Al-Zahraa, University for Women, Karbala, Iraq.
e-mail: nadia.habeeb@alzahraa.edu.iq; ORCID: 0009-0008-3552-744X.

⁴ Department of Mathematics, College of Education for Pure Sciences, University of Babylon, Babylon, Iraq.
e-mail: pure.leal.abd@uobabylon.edu.iq; ORCID: 0000-0001-5641-1616.

* Corresponding author.

§ Manuscript received: July 15, 2025; accepted: October 30, 2025.

TWMS Journal of Applied and Engineering Mathematics, Vol.16, No.10; © Işık University, Department of Mathematics, 2026; all rights reserved.

of the proximity space, in which two sets are said to be near when their intersection is nonempty, and far otherwise. The scope of proximity spaces is not limited to spatial or metric closeness; it extends to descriptive proximity, in which two sets are regarded as near whenever they share common describable features rather than common points. For instance, two coins of the same currency located in different countries have no spatial proximity, yet they are descriptively near through shared attributes. The theory was subsequently developed by Efremovic, Lodato, Leader, Naimpally, Peters and others [1-5], and has since found applications in engineering and information sciences, including problems of human perception, pattern and face recognition, and the foundations of near-set and rough set theory [6-20]. Motivated by this activity, a number of recent studies have continued to develop the algebraic and topological structure of proximity spaces [21-30].

The remainder of the paper is organized as follows. Section 2 recalls the preliminary notions of proximity spaces and cluster families that are needed later. Section 3 introduces the typical family, identifies it explicitly with the ideal dual to a cluster, and develops its basic properties. Section 4 introduces T -compactness, relates it to the known notion of compactness modulo an ideal, and proves preservation results, including a corrected preservation result for images under proximity isomorphisms via a pushforward construction. Section 5 discusses the scope and limitations of the results and concludes the paper.

2. BASIC DEFINITIONS AND CONCEPTS

In this section, several basic definitions and results related to proximity spaces are recalled.

Definition 2.1. [1] *Let X be a nonempty set. A relation δ on X is called a proximity relation if the following conditions hold for all subsets $H, D, C \subseteq X$:*

- (1) *If $H\delta D$, then $D\delta H$.*
- (2) *If $H\delta D$, then $H \neq \emptyset$ and $D \neq \emptyset$.*
- (3) *If $H \cap D \neq \emptyset$, then $H\delta D$.*
- (4) *$H\delta(D \cup C)$ if and only if $H\delta D$ or $H\delta C$.*
- (5) *If $H\delta D$ and $\{b\}\delta C$ for each $b \in D$, then $H\delta C$.*

The pair (X, δ) is called a proximity space.

Definition 2.2. [1] *A subfamily $o \subseteq P(X)$ is said to be a cluster (equivalently, a grill in the sense of Choquet) in (X, δ) if the following conditions are satisfied:*

- (1) *If $H, D \in o$, then $H\delta D$.*
- (2) *$(H \cup D) \in o$ if and only if $H \in o$ or $D \in o$.*
- (3) *If $H\delta D$ for every $D \in o$, then $H \in o$.*

Condition 2 above is precisely the defining property of a grill on X in the sense of Choquet (1947); this identification is used explicitly throughout Section 3. Condition 2 forces every cluster to be monotone with respect to inclusion. Thus if $H \in o$ and $H \subseteq D$, then $H \cap D = H \neq \emptyset$ (since $\emptyset \notin o$), so $H\delta D$ by axiom 3 of Definition 2.1. Since $H \cup D = D$ and $H \in o$, condition 2 above gives $D \in o$. Thus every cluster is, in particular, a grill.

Definition 2.3. [1] *A mapping $f : (X, \delta_X) \rightarrow (Y, \delta_Y)$ is called proximity continuous if $H\delta_X D$ implies $f(H)\delta_Y f(D)$ for all $H, D \subseteq X$.*

Definition 2.4. *Let (X, δ) be a proximity space. For $A \subseteq X$ define $cl_\delta(A) = \{x \in X : \{x\}\delta A\}$ (and $cl_\delta(\emptyset) = \emptyset$). This closure operator satisfies the Kuratowski axioms and hence induces a topology τ_δ on X , whose closed sets are exactly the sets fixed by cl_δ and whose open sets are their complements. The quadruple (X, δ, τ_δ) is called a proximity*

topological space. All topological notions used below (open sets, closed sets, open covers) are understood with respect to τ_δ unless stated otherwise.

Proposition 2.1. [4] Every proximity continuous map $f : (X, \delta_X) \rightarrow (Y, \delta_Y)$ in the sense of Definition 2.3 is continuous as a map $(X, \tau_{\delta_X}) \rightarrow (Y, \tau_{\delta_Y})$ between the induced topological spaces.

3. TYPICAL FAMILIES IN PROXIMITY SPACES

This section introduces the typical family, identifies it explicitly as the ideal dual to a cluster, and studies its main properties.

Definition 3.1. Let (X, δ) be a proximity space and σ be a cluster family on X . The typical family associated with σ is the family $T \subseteq P(X)$ given by

$$T = \{A \subseteq X : A \notin \sigma\}, \quad \text{i.e., } T = P(X) \setminus \sigma.$$

Example 3.1. Let $X = \{1, 2, 3\}$ and define a discrete proximity relation by $H\delta D \iff H \cap D \neq \emptyset$. Consider the point cluster associated with the element 1, $\sigma_1 = \{H \subseteq X : 1 \in H\}$. Then the corresponding typical family is $T = \{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$, and:

- $T \cap \sigma_1 = \emptyset$,
- $T \cup \sigma_1 = P(X)$,
- T is an ideal, closed under finite unions and hereditary with respect to inclusion.

Proposition 3.1. Let $(X, \delta, \tau_\delta, o)$ be a proximity space equipped with a cluster family σ and its typical family T . Then:

- (1) $T \cap o = \emptyset$.
- (2) $T \cup o = P(X)$.
- (3) $\emptyset \in T$.
- (4) If $H \subseteq D$ and $D \in T$, then $H \in T$.
- (5) If $H, D \in T$, then $H \cup D \in T$.

Proof. (1) Suppose that there exists $H \in (T \cap o)$. Since $H \in T$, by Definition 3.1, $H \notin o$, which contradicts $H \in o$. Hence $T \cap o = \emptyset$.
 (2) Let $H \subseteq X$. Either $H \in o$ or $H \notin o$. In the latter case, $H \in T$ by Definition 3.1. In either case $H \in (T \cup o)$, so $T \cup o = P(X)$.
 (3) By Definition 2.2, a cluster family does not contain the empty set. Hence $\emptyset \notin o$, so $\emptyset \in T$.
 (4) Assume that $H \subseteq D$ and $D \in T$. If $H \notin T$, then $H \in o$. By the union property of clusters, $(H \cup D) = D \in o$, contradicting $D \in T$. Hence $H \in T$.
 (5) Suppose that $H, D \in T$. If $(H \cup D) \notin T$, then $(H \cup D) \in o$. By the cluster property, $H \in o$ or $D \in o$, contradicting $H, D \in T$. Hence $(H \cup D) \in T$. □

Corollary 3.1. The typical family T associated with a cluster o on X is an ideal on X , i.e., it satisfies:

- (i) $\emptyset \in T$,
- (ii) $H \subseteq D$ and $D \in T \implies H \in T$ (hereditary),
- (iii) $H, D \in T \implies H \cup D \in T$ (closed under finite unions).

Proof. Immediate from Proposition 3.1, parts (3), (4) and (5), which are exactly the three defining axioms of an ideal on X . □

Corollary 3.2. If $H \in T$, then $H^c \in o$.

Proof. Since o is nonempty and monotone (Definition 2.2), taking any $A \in o$ and using $A \subseteq X$ gives $X \in o$, i.e., $X \notin T$. Now suppose $H \in T$. If H^c were also in T , then since T is closed under finite unions (Corollary 3.1), $H \cup H^c = X$ would lie in T , contradicting $X \notin T$. Hence $H^c \notin T$, and since T and o partition $P(X)$ (Proposition 3.1), it follows that $H^c \in o$. $\square$

Proposition 3.2. *Let $f : X \rightarrow Y$ be any mapping between two nonempty sets, and let T_Y be a typical family on Y . Then $f^{-1}(T_Y) = \{H \subseteq X : f(H) \in T_Y\}$ is an ideal on X .*

Proof. (i) $\emptyset \in f^{-1}(T_Y)$, since $f(\emptyset) = \emptyset \in T_Y$.
(ii) Hereditary: if $D \subseteq H$ and $H \in f^{-1}(T_Y)$, then $f(D) \subseteq f(H)$ and $f(H) \in T_Y$. Since T_Y is hereditary (Corollary 3.1), $f(D) \in T_Y$, so $D \in f^{-1}(T_Y)$.
(iii) Finite unions: if $D, H \in f^{-1}(T_Y)$, then $f(D \cup H) = f(D) \cup f(H)$, and since T_Y is closed under finite unions (Corollary 3.1), $f(D) \cup f(H) \in T_Y$, so $D \cup H \in f^{-1}(T_Y)$.
Thus $f^{-1}(T_Y)$ is an ideal on X . $\square$

Proposition 3.3. *Let $f : (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a proximity isomorphism, i.e., a bijection such that $H\delta_X D \iff f(H)\delta_Y f(D)$ for all $H, D \subseteq X$. If T_Y is the typical family dual to a cluster o_Y on Y , then $o_X = \{H \subseteq X : f(H) \in o_Y\}$ is a cluster on X , and its dual typical family is exactly $f^{-1}(T_Y)$.*

Proof. (1) If $H, D \in o_X$, then $f(H), f(D) \in o_Y$, so $f(H)\delta_Y f(D)$ by the cluster axiom for o_Y . Since f is a proximity isomorphism, this is equivalent to $H\delta_X D$.
(2) $(H \cup D) \in o_X \iff f(H \cup D) = f(H) \cup f(D) \in o_Y \iff f(H) \in o_Y$ or $f(D) \in o_Y \iff H \in o_X$ or $D \in o_X$.
(3) Suppose $H\delta_X D$ for every $D \in o_X$. Let $D' \in o_Y$ be arbitrary. Since f is a bijection, $D' = f(D)$ for a unique $D \subseteq X$, and $D \in o_X$ by definition of o_X . By hypothesis, $H\delta_X D$, hence $f(H)\delta_Y f(D) = D'$ because f is a proximity isomorphism. As $D' \in o_Y$ was arbitrary, the cluster axiom for o_Y gives $f(H) \in o_Y$, i.e., $H \in o_X$.

Hence o_X is a cluster on X . Its dual typical family is $P(X) \setminus o_X = \{H \subseteq X : f(H) \notin o_Y\} = \{H \subseteq X : f(H) \in T_Y\} = f^{-1}(T_Y)$. $\square$

Theorem 3.1. *Let (X, δ) be a proximity space equipped with a cluster family o . Then the corresponding typical family $T = P(X) \setminus o$ is uniquely determined by o .*

Proof. By definition, $T = \{H \subseteq X : H \notin o\}$. Hence, once the cluster family o is fixed, the associated typical family is completely determined as the complement of o in $P(X)$. Therefore, the typical family is unique. $\square$

Proposition 3.4. *If $H, D \in T$, then $H \cup D \in T$ is the canonical (unique smallest) typical set containing both H and D .*

Proof. Since T is closed under finite unions, the collection $\{E \in T : H \subseteq E \text{ and } D \subseteq E\}$ of typical sets containing both H and D is nonempty (it contains $H \cup D$), and $H \cup D$ is the smallest member of this collection, since any set E with $H \subseteq E$ and $D \subseteq E$ must satisfy $H \cup D \subseteq E$ by definition of union. Hence $H \cup D$ is the canonical smallest typical set containing both H and D . $\square$

Proposition 3.5. $\bigcup T = X$ if and only if o contains no singleton, i.e., $\{x\} \notin o$ for every $x \in X$. Consequently, T is a bornology on X exactly in this case, and fails to be a bornology whenever some singleton $\{x\}$ belongs to o .

Proof. $(\Leftarrow)$ Suppose $\{x\} \notin o$ for every $x \in X$. Since $T \cup o = P(X)$ (Proposition 3.1(2)), $\{x\} \notin o$ implies $\{x\} \in T$ for every x . Hence every $x \in X$ lies in some member of T , so $\bigcup T = X$.

($\Rightarrow$) Suppose $\bigcup T = X$ and let $x \in X$. Then $x \in A$ for some $A \in T$. Since $\{x\} \subseteq A$ and T is hereditary (Corollary 3.1), $\{x\} \in T$, so $\{x\} \notin o$. As x was arbitrary, no singleton belongs to o . The final claim follows since T is already an ideal by Corollary 3.1, and a bornology is by definition an ideal with the covering property just characterized. $\square$

In Examples 3.2, $o = o_1$ contains the singleton $\{1\}$, so by Proposition 3.10, $\bigcup T = \{2, 3\} \neq X$. The typical family of Examples 3.2 is an ideal but not a bornology.

Corollary 3.3. *If X is finite and nonempty, then T is never a bornology; for every valid cluster o on X , some singleton belongs to o .*

Proof. Since o is nonempty, $X \in o$. Write $X = \{x_1\} \cup \dots \cup \{x_n\}$ as a finite union of singletons, using that X is finite. The union axiom of Definition 2.2 extends by a routine induction from pairs to any finite union: $A_1 \cup \dots \cup A_k \in o \iff \text{some } A_i \in o$. Applying this to $X = \bigcup \{x_i\} \in o$, at least one singleton $\{x_i\}$ must belong to o . By Proposition 3.10, $\bigcup T = X$ requires o to contain no singleton. Since some singleton is forced into o , $\bigcup T \neq X$, so T is not a bornology. $\square$

In summary, T is always an ideal, since it satisfies the hereditary property and is closed under finite unions, as established in the previous section. However, the converse does not hold in general. Indeed, an arbitrary ideal may be defined independently of any proximity structure, whereas a typical family is obtained through the complementary construction associated with a cluster family in a proximity space. Consequently, there exist ideals that cannot be typical families. This observation demonstrates that the typical family constitutes a distinguished, proximity-generated subclass of ideals rather than a reformulation of the classical notion of an ideal.

Filters, unlike ideals, are designed to characterize the large subsets of a space, and are closed under finite intersection and under passage to supersets. Typical families, by contrast, possess the hereditary property and are closed under finite unions, but are not required to contain the supersets of their members. Consequently, the algebraic behavior of the typical family is fundamentally closer to that of ideals than to that of filters, even though both structures are hereditary type structures employed in generalized topology.

Grills provide another important hereditary related structure that emphasizes large subsets, satisfying intersection conditions. From this perspective, grills and typical families play complementary roles. Whereas grills are constructed to identify large subsets interacting with a given family, the typical families consist of subsets that lie outside the associated cluster structure. Hence, the two concepts describe opposite aspects of the same proximity environment rather than equivalent mathematical objects.

Similarly, bornologies are hereditary families whose construction is based on the notion of boundedness. Their defining properties are motivated by the study of bounded sets in topology and functional analysis. In contrast, the construction of a typical family is governed entirely by the underlying proximity relation through its associated cluster family. Consequently, bornologies and typical families arise from different mathematical principles: the former is determined by boundedness, whereas the latter is determined by nearness. Although both structures satisfy hereditary type properties, they originate from distinct theoretical foundations and serve different mathematical purposes.

The comparison demonstrates that the typical family is not obtained by modifying any existing hereditary structure. Instead, it is intrinsically generated from cluster families associated with proximity spaces. This dependence on the underlying proximity relation distinguishes the proposed construction from ideals, filters, grills, and bornologies, whose definitions are independent of cluster systems.

TABLE 1. Comparison of Hereditary Structures

Property	Ideal	Filter	Grill	Bornology	Typical Family
Downward-closed / hereditary	✓	×	×	✓	✓
Upward-closed	×	✓	✓	×	×
Closed under finite unions	✓	✓	✓	✓	✓
Closed under finite intersections	✓	✓	×	✓	✓
Generated by proximity clusters	×	×	×	×	✓
Depends on proximity relation	×	×	×	×	✓

Consequently, the typical family provides a new bridge between proximity theory and hereditary set structures while preserving sufficient algebraic properties to support the development of generalized compactness.

4. T -COMPACTNESS IN PROXIMITY SPACES

This section introduces a generalized compactness notion based on the typical family.

Definition 4.1. Let $(X, \delta, \tau_\delta, T)$ be a proximity topological space. A subset $H \subseteq X$ is called T -compact if for every τ_δ -open cover $\{U_\lambda : \lambda \in \Delta\}$ of H there exists a finite subfamily $\{U_{\lambda_i} : i = 1, 2, \dots, n\}$ such that

$$H \setminus \bigcup_{i=1}^n U_{\lambda_i} \in T.$$

Example 4.1. Let (X, δ_i) be a proximity space equipped with the indiscrete proximity relation defined by $A\delta_i B \iff A \neq \emptyset$ and $B \neq \emptyset$. The induced topology is $\tau_{\delta_i} = \{X, \emptyset\}$. Moreover, the corresponding typical family is $T = \{\emptyset\}$. We show that X is T -compact.

Indeed, let $\{U_i : i \in \Delta\}$ be an open cover of a subset $H \subseteq X$. Since the topology is indiscrete, the only nonempty open set is X . Hence, every open cover contains X , and therefore there exists a finite subcover consisting of the single set X .

Consequently, $H \setminus X = \emptyset$. Since $\emptyset \in T$, it follows from Definition 4.1 that H is T -compact. Therefore, every subset of an indiscrete proximity space is T -compact.

Proposition 4.1. Every compact space is T -compact, for any typical family T on it.

Proof. Let $H \subseteq X$ be compact and let $\{U_\lambda : \lambda \in \Delta\}$ be an open cover of H . Since H is compact, there exists a finite subfamily $\{U_{\lambda_1}, \dots, U_{\lambda_n}\}$ such that $H \subseteq \bigcup_{i=1}^n U_{\lambda_i}$. Hence $H \setminus \bigcup_{i=1}^n U_{\lambda_i} = \emptyset \in T$ (by Corollary 3.1(i)). Therefore, H is T -compact. $\square$

Proposition 4.2. Finite unions of T -compact subsets are T -compact.

Proof. Let A and B be T -compact subsets of X , and let $\{U_s : s \in \Delta\}$ be an open cover of $A \cup B$. Since $A \subseteq A \cup B$, $\{U_s\}$ is also an open cover of A . As A is T -compact, there exist $\lambda_1, \dots, \lambda_n$ such that $A \setminus \bigcup_{i=1}^n U_{\lambda_i} \in T$. Likewise, since $B \subseteq A \cup B$, there exist $\mu_1, \dots, \mu_m$ such that $B \setminus \bigcup_{j=1}^m U_{\mu_j} \in T$.

We claim

$$(A \cup B) \setminus \left(\bigcup_{i=1}^n U_{\lambda_i} \cup \bigcup_{j=1}^m U_{\mu_j} \right) \subseteq \left(A \setminus \bigcup_{i=1}^n U_{\lambda_i} \right) \cup \left(B \setminus \bigcup_{j=1}^m U_{\mu_j} \right).$$

Indeed, if x lies in the left-hand side, then $x \in A \cup B$ and x avoids both unions. If $x \in A$, then $x \notin \bigcup_{i=1}^n U_{\lambda_i}$, so $x \in A \setminus \bigcup_{i=1}^n U_{\lambda_i}$, and symmetrically if $x \in B$ then $x \in B \setminus \bigcup_{j=1}^m U_{\mu_j}$.

Since T is closed under finite unions (Corollary 3.1), the right-hand side lies in T . Since T is also hereditary (Corollary 3.1), the inclusion just shown gives $(A \cup B) \setminus \left(\bigcup_{i=1}^n U_{\lambda_i} \cup \bigcup_{j=1}^m U_{\mu_j} \right) \in T$. Hence $\{U_{\lambda_1}, \dots, U_{\lambda_n}, U_{\mu_1}, \dots, U_{\mu_m}\}$ is a finite subfamily of the cover of $A \cup B$ whose uncovered remainder lies in T , so $A \cup B$ is T -compact. $\square$

Theorem 4.1. *If H is a closed subset of a T -compact space X , then H is T -compact.*

Proof. Let $\{U_s : s \in \Delta\}$ be an open cover of H . Since H is closed, the set $X \setminus H$ is open. Hence $\{U_\lambda : \lambda \in \Delta\} \cup \{X \setminus H\}$ forms an open cover of X . Because X is T -compact, there exists a finite subfamily F of this cover such that $X \setminus \bigcup F \in T$. Let $F' = F \setminus \{X \setminus H\}$, a finite subfamily of the original cover of H .

We claim $H \setminus \bigcup F' \subseteq X \setminus \bigcup F$. Indeed, if $x \in H \setminus \bigcup F'$, then $x \in H$, so $x \notin X \setminus H$, and $x \notin \bigcup F'$ by assumption. Since $\bigcup F$ is the union of $\bigcup F'$ with $X \setminus H$ (when the latter belongs to F) and x lies in neither, $x \notin \bigcup F$, so $x \in X \setminus \bigcup F$.

Since T is hereditary (Corollary 3.1) and $X \setminus \bigcup F \in T$, the inclusion just shown gives $H \setminus \bigcup F' \in T$. Hence F' is a finite subfamily of the cover of H whose uncovered remainder lies in T , so H is T -compact. $\square$

Lemma 4.1. *Let $f : X \rightarrow Y$ be any mapping and let T be an ideal on X . Define $T_f = \{J \subseteq f(X) : f^{-1}(J) \in T\}$. Then T_f is an ideal on $f(X)$.*

Proof. $\emptyset \in T_f$ since $f^{-1}(\emptyset) = \emptyset \in T$. Hereditary: if $J' \subseteq J \in T_f$, then $f^{-1}(J') \subseteq f^{-1}(J) \in T$, and since T is hereditary, $f^{-1}(J') \in T$, so $J' \in T_f$. Finite unions: if $J_1, J_2 \in T_f$, then $f^{-1}(J_1 \cup J_2) = f^{-1}(J_1) \cup f^{-1}(J_2) \in T$ since T is closed under finite unions, hence $J_1 \cup J_2 \in T_f$. Thus T_f is an ideal on $f(X)$. $\square$

Theorem 4.2. *Let $f : (X, \delta_X) \rightarrow (Y, \delta_Y)$ be a proximity isomorphism, let T be a typical family on X dual to a cluster o , and let T_f be the pushforward ideal of Lemma 4.1. Then T_f is in fact the typical family dual to the pushforward cluster $\{f(H) : H \in o\}$ on $f(X)$, and if X is T -compact, then $f(X)$ is T_f -compact.*

Proof. First, T_f is a genuine typical family, not merely an ideal. Since f is a proximity isomorphism onto its image, its inverse $g = f^{-1} : f(X) \rightarrow X$ is also a proximity isomorphism: for $J_1, J_2 \subseteq f(X)$, writing $H = g(J_1)$, $D = g(J_2)$ (so $f(H) = J_1$, $f(D) = J_2$), the isomorphism property of f gives $H\delta_X D \iff f(H)\delta_Y f(D)$, i.e., $g(J_1)\delta_X g(J_2) \iff J_1\delta_Y J_2$. Applying Proposition 3.3 to g in place of f (with T , the typical family on X , the codomain of g , playing the role of the typical family fixed on the codomain), the family $\sigma_{f(X)} = \{J \subseteq f(X) : g(J) \in o\}$ is a cluster on $f(X)$ and its dual typical family is exactly $g^{-1}(T) = \{J \subseteq f(X) : g(J) \in T\}$.

Since g is the inverse of the bijection $f : X \rightarrow f(X)$, $g(J) = f^{-1}(J)$ for every $J \subseteq f(X)$. Hence $g^{-1}(T) = \{J \subseteq f(X) : f^{-1}(J) \in T\} = T_f$, matching Lemma 4.1's definition exactly. Moreover, $\sigma_{f(X)} = \{f(H) : H \in o\}$: if $J = f(H)$ with $H \in o$, then $f^{-1}(J) = f^{-1}(f(H)) = H \in o$ (since f is injective), so $J \in \sigma_{f(X)}$; conversely if $J \in \sigma_{f(X)}$, then $H = f^{-1}(J) \in o$ and $f(H) = f(f^{-1}(J)) = J$ (since $J \subseteq f(X)$), so $J = f(H)$ with $H \in o$. Hence T_f is exactly the typical family dual to the pushforward cluster $\{f(H) : H \in o\}$, a genuine typical family on $f(X)$ in the sense of Definition 3.1, so Definition 4.1 applies to it directly, not merely by analogy with an ideal.

Now the compactness argument. Let $\{V_\lambda : \lambda \in \Delta\}$ be a τ_{δ_Y} -open cover of $f(X)$. Then $X = f^{-1}(f(X)) \subseteq f^{-1}(\bigcup_\lambda V_\lambda) = \bigcup_\lambda f^{-1}(V_\lambda)$. Since f is a proximity isomorphism, it is, in particular, proximity continuous (Definition 2.3), so each $f^{-1}(V_\lambda)$ is τ_{δ_X} -open (Proposition 2.1), and $\{f^{-1}(V_\lambda) : \lambda \in \Delta\}$ is a τ_{δ_X} -open cover of X . Since X is T -compact, there exist $\lambda_1, \dots, \lambda_n$ such that $X \setminus \bigcup_{i=1}^n f^{-1}(V_{\lambda_i}) \in T$.

We claim

$$f^{-1} \left(f(X) \setminus \bigcup_{i=1}^n V_{\lambda_i} \right) \subseteq X \setminus \bigcup_{i=1}^n f^{-1}(V_{\lambda_i}).$$

Indeed, if x lies in the left-hand side, then $f(x) \in f(X)$ and $f(x) \notin V_{\lambda_i}$ for every i , hence $x \notin f^{-1}(V_{\lambda_i})$ for every i (otherwise $f(x) \in V_{\lambda_i}$), so $x \in X \setminus \bigcup_{i=1}^n f^{-1}(V_{\lambda_i})$, which is the right-hand side.

Since T is hereditary (Corollary 3.1) and $X \setminus \bigcup_{i=1}^n f^{-1}(V_{\lambda_i}) \in T$, the inclusion just shown gives $f^{-1}(f(X) \setminus \bigcup_{i=1}^n V_{\lambda_i}) \in T$. By definition of T_f , this means $f(X) \setminus \bigcup_{i=1}^n V_{\lambda_i} \in T_f$. Hence $\{V_{\lambda_1}, \dots, V_{\lambda_n}\}$ is a finite subfamily of the cover whose $f(X)$ -uncovered remainder lies in T_f , so $f(X)$ is T_f -compact. $\square$

The theory developed in the previous sections establishes the typical family as a hereditary structure naturally generated by cluster families in proximity spaces. Although the proposed construction is formulated in an abstract setting, it admits concrete illustration in finite relational systems whose interactions are described through proximity relations. The purpose of this subsection is not to construct application-specific models but to demonstrate, through one fully worked numeric example, how the framework developed in this paper is instantiated in a finite proximity structure. Throughout, the underlying proximity relation is assumed to satisfy the axioms introduced in Section 2.

To illustrate the proposed construction explicitly, consider the finite proximity space $X = \{a, b, c, d, e\}$, and define the proximity relation by: $a\delta b$, $b\delta c$, $a\delta c$, so that $\{a, b, c\}$ is a fully mutually near block, and $d\delta e$, together with all pairs implied by Definition 2.1 (symmetry, and nonempty intersection), while every pair of singletons with one point in $\{a, b, c\}$ and the other in $\{d, e\}$ is regarded as far. Equivalently, $\{a, b, c\}$ and $\{d, e\}$ are exactly the two blocks into which X splits under this relation, and $H\delta D$ holds precisely when H and D both meet the same block. One checks directly that this defines a proximity relation satisfying every axiom of Definition 2.1, including the Lodato axiom.

Consider the cluster $o = \{H \subseteq X : H \cap \{a, b, c\} \neq \emptyset\}$, i.e., the subsets meeting the block $\{a, b, c\}$. One verifies directly that o satisfies Definition 2.2: any two members of o meet $\{a, b, c\}$ and are therefore δ -related; a union lies in o exactly when at least one of its two parts does, since meeting $\{a, b, c\}$ is preserved and reflected by unions; and any H that is δ -related to every member of o must itself meet $\{a, b, c\}$ (take $H\delta\{a, b, c\}$ itself), so $H \in o$.

According to Definition 3.1, the associated typical family is

$$T = \{A \subseteq X : A \cap \{a, b, c\} = \emptyset\} = \{\emptyset, \{d\}, \{e\}, \{d, e\}\},$$

precisely the subsets of the complementary block. One checks directly that:

- $\emptyset \in T$.
- Every subset of a member of T also belongs to T (e.g., $\{d\} \subseteq \{d, e\} \in T$, and indeed $\{d\} \in T$).
- Finite unions of members of T remain in T as long as no point of $\{a, b, c\}$ is introduced, consistently with Proposition 3.1(5) (for instance $\{d\} \cup \{e\} = \{d, e\} \in T$, while $\{a\} \cup \{d\} = \{a, d\} \notin T$, correctly reflecting that $\{a, d\}$ meets $\{a, b, c\}$ and so lies in o).

Therefore, T forms an ideal on X , illustrating Corollary 3.1 with explicit sets, with unique maximal element $\{d, e\} = X \setminus \{a, b, c\}$. The topology τ_δ induced by Definition 2.4 has exactly four open sets: $\emptyset$, $\{a, b, c\}$, $\{d, e\}$, and X , since $cl_\delta(\{a\}) = cl_\delta(\{b\}) = cl_\delta(\{c\}) = \{a, b, c\}$ and $cl_\delta(\{d\}) = cl_\delta(\{e\}) = \{d, e\}$, so τ_δ is not discrete here. Nonetheless, X is compact simply because it has only finitely many open sets in total, and hence X is T -compact by Proposition 4.1, with every open cover admitting a finite subcover and

uncovered remainder $\emptyset \in T$. This numeric instance demonstrates concretely how Definition 3.1 and Definition 4.1 apply to an explicit finite proximity space.

This example illustrates that the proposed construction depends only on the existence of an underlying proximity relation, independent of the specific nature of the objects involved. Once a proximity relation is fixed, the cluster family and its dual typical family are uniquely determined.

5. DISCUSSION AND CONCLUSION

The results presented in this paper show that the proposed typical family provides a natural hereditary structure associated with cluster families in proximity spaces. Unlike ad hoc hereditary structures, the typical family is generated directly from the underlying proximity relation, establishing a close and explicit connection between cluster theory and Choquet's classical grill-ideal duality [31]. Corollary 3.4 identifies T as exactly the ideal dual to $\mathcal{o}$.

The introduction of T -compactness further extends this framework by providing a generalized compactness concept compatible with the proposed family, reducing to the classical notion of compactness modulo an ideal in the sense of Hamlett and Jankovic [10] once T is fixed. The preservation results obtained for finite unions (Proposition 4.2), closed subsets (Theorem 4.1), and images under proximity isomorphisms (Theorem 4.2), via the pushforward typical family identified through Lemma 4.1 and Proposition 3.3) show that the new concept is mathematically consistent with the classical theory of proximity spaces. Section 3 also clarifies precisely which of these constructions require only proximity continuity of the map (Proposition 3.2), which yields an ideal but not necessarily a typical family) and which require the stronger notion of a proximity isomorphism (Proposition 3.1, Theorem 4.2), a distinction not present in an earlier draft of this work, where a pullback cluster construction was claimed under proximity continuity alone.

Although every typical family determines an ideal, the converse does not generally hold. This distinction demonstrates that the proposed construction is not merely a reformulation of existing hereditary structures but rather a proximity-generated subclass possessing its own mathematical characteristics.

Overall, the proposed framework establishes an explicit link between cluster structures and generalized compactness, correctly grounded in Choquet's grill-ideal duality, and provides a foundation for further developments in proximity theory.

Acknowledgement. The authors would like to extend their gratitude to their respective institutions for their support.

REFERENCES

- [1] Efremovic, V. A., (1951), Infinitesimal spaces, *Doklady Akademii Nauk SSSR*, 76(3), pp. 341-343.
- [2] Lodato, M. W., (1964), On topologically induced generalized proximity relations, *Proceedings of the American Mathematical Society*, 15(3), pp. 417-422.
- [3] Leader, S., (1967), Local proximity spaces, *Mathematische Annalen*, 169, pp. 275-281.
- [4] Naimpally, S. and Warrack, B., (1970), *Proximity Spaces*, Cambridge University Press.
- [5] Peters, J. F., (2014), *Topology of Digital Images: Visual Pattern Discovery in Proximity Spaces*, Springer.
- [6] Peters, J. F. and Guadagni, C., (2014), Strongly near proximity and hyperspace topology, *General Mathematics Notes*, 21(2), pp. 70-81.
- [7] Di Concilio, A., (2009), Proximity: A powerful tool in extension theory, function spaces, hyperspaces, Boolean algebra, and point-free geometry, *Contemporary Mathematics*, 486, pp. 31-56.
- [8] Beer, G., (1993), *Topologies on Closed and Closed Convex Sets*, Springer.

- [9] Császár, Á., (2002), Generalized topology, generalized continuity, *Acta Mathematica Hungarica*, 96(4), pp. 351-357.
- [10] Hamlett, T. R. and Janković, D., (1990), Compactness with respect to an ideal, *Bollettino U.M.I.*, 7(4-B), pp. 849-861.
- [11] Newcomb, R. L., (1986), Topologies generated by ideals, *Pacific Journal of Mathematics*, 123(1), pp. 215-220.
- [12] Abed Noor, E. J. and Mahdi Altalkany, Y. K., (2024), Characterization of new types of continuous function via ideal topological space, *ICPAS*, 232.
- [13] Nasef, A. A. and Noiri, T., (2008), Some forms of compactness in ideal topological spaces, *Acta Mathematica Hungarica*, 118(3), pp. 223-234.
- [14] Janković, D. and Hamlett, T. R., (1990), New topologies from old via ideals, *American Mathematical Monthly*, 97(4), pp. 295-310.
- [15] Dontchev, J., Ganster, M. and Rose, D., (1999), Ideal resolvability, *Topology and its Applications*, 93(1), pp. 1-16.
- [16] Abd El-Monsef, M. E., Lashien, E. F. and Nasef, A. A., (1992), On I -open sets and I -continuous functions, *Kyungpook Mathematical Journal*, 32(1), pp. 21-30.
- [17] Kareem, M. K. and Jabar, L. A. A., (2026), f -Focal and f -nested sets, *American Institute of Physics Conference Series*, 3393(1).
- [18] Peters, J. F., (2018), Computational proximity: Excursions in the topology of digital images, *Intelligent Systems Reference Library*, Springer.
- [19] Guadagni, C. and Peters, J. F., (2016), Descriptive proximities I: Properties and interplay between classical proximities and overlap, *Mathematics in Computer Science*, 10(1), pp. 91-106.
- [20] Ivanova, M., (2020), Descriptive proximity spaces and applications, *Filomat*, 34(12), pp. 3921-3935.
- [21] Alimohammady, M. and Roohi, M., (2009), Fixed point in fuzzy and proximity spaces, *Chaos, Solitons and Fractals*, 42(1), pp. 468-472.
- [22] Gunduz, C., Bayramov, S., Zeynalov, J., (2024), Separation Axioms In Bipolar Soft Topological Spaces, *TWMS J. Pure Appl. Math.*, 15(1), pp. 15-25.
- [23] Peters, J. F. and Wasilewski, P., (2009), Foundations of near sets, *Information Sciences*, 179(18), pp. 3091-3109.
- [24] Al-Hawary, T. and Noiri, T., (2020), Generalized compactness and ideal topological spaces, *European Journal of Pure and Applied Mathematics*, 13(3), pp. 590-603.
- [25] Roy, B. and Mukherjee, M., (2021), On a class of generalized compact spaces, *Applied General Topology*, 22(1), pp. 53-68.
- [26] Kočinac, L. D. R., (2011), Selection principles and generalized compactness, *Topology and its Applications*, 158(12), pp. 1443-1452.
- [27] Almohammed, R. and Jabar, L. A. A., (2024), Dismountable and non-Dismountable Spaces via Proximity Space, *Baghdad Science Journal*.
- [28] AL-Swidi, L. A. A. and Mutakek, A. Y. K., (2025), JD -paracompact space, *Journal of Interdisciplinary Mathematics*, 28(1).
- [29] Peters, J. F., (2022), Proximity spaces and applications in computational topology, *Axioms*, 11(5).
- [30] Di Concilio, A. and Guadagni, C., (2022), Proximity structures and generalized topological systems, *Mathematics*, 10(14).
- [31] Roy, B. and Sen, R., (2023), Ideal compactness and generalized topological structures, *Demonstratio Mathematica*, 56(1).
- [32] Choquet, G., (1947), Sur les notions de filtre et de grille, *Comptes Rendus de l'Académie des Sciences Paris*, 224, pp. 171-173.



Dr. Raghad Almohammed received her Ph.D. in Pure Mathematics from the University of Babylon, Iraq, specializing in Topology. Her research interests include General Topology, Proximity Spaces, Compactness Properties, and related topics in topology. Her research focuses on the study of topological spaces and their properties, with a particular interest in developing and investigating concepts in topology.



Dr. Dunia M.K. AL-Ftlawy received her Ph.D. in Pure Mathematics from the University of Babylon, Iraq, specializing in Dynamical Systems. Her research interests include Dynamical Systems, Topological Dynamics, and related areas of mathematics. Her research focuses on the study of dynamical behavior and the properties and behavior of dynamical systems.



Dr. Nadiha Abed Habeeb received her Ph.D. in General Mathematics from the University of Babylon, Iraq, specializing in Approximation Theory. Her research interests include Approximation Theory, Mathematical Analysis, and related areas of mathematical analysis. Her research focuses on approximation methods and their applications in mathematical analysis.



Prof. Dr. Luay Al. Swidi is graduated from is a faculty member at the University of Babylon, Iraq, with an academic and research background in Pure Mathematics. He has contributed to numerous research studies in various fields of Pure Mathematics. His research interests encompass a broad range of topics in mathematical sciences, particularly theoretical and analytical aspects of Pure Mathematics. He is actively engaged in academic research and scientific publication.
