TWMS J. App. and Eng. Math. V.16, N.11, 2026, pp. 1297-1306

A CHARACTERIZATION OF TRIANGLE-FREE CYCLIC GRAPHS WITH SELF-LOOPS OF RANK 3

JOHNNY LIM[1,*] §

Abstract. Let G_S be a self-loop graph as the graph obtained by attaching a self-loop at every vertex in $S \subseteq V(G)$ of a simple graph G. If $G = C_n$ is the cycle graphs of order n and $S \neq \emptyset$, we show that there are no rank 3 self-loop graphs $(C_n)_S$ for $n \geq 5$. As a consequence, we determine and construct all possible rank 3 triangle-free self-loop cyclic graph of order at least 4 from $(C_4)_S$ via graph join operations. This provides a partial solution to the characterization problem of rank 3 self-loop graphs.

Keywords: Graphs with self-loops, triangle-free, characterization, rank.

AMS Subject Classification: 05C50, 05C90, 05C92.

1. Introduction

Let G be a simple (i.e., without self-loops and multiple edges) undirected graph of order $n = |V(G)|$, where $V(G) = \{v_1, v_2, ..., v_n\}$ is the set of vertices in G. In this paper, we consider only *connected* graphs, for which there exists a path between any two vertices. Let $A(G) = (a_{ij})_{n\times n}$ be the adjacency matrix of G, whose the (i, j)-entry is defined by $a_{ij} = 1$ if v_i is adjacent to v_j where $i \neq j$ and $a_{ij} = 0$ otherwise. For a matrix $A \in \mathrm{Mat}_n(\mathbb{R})$, the *rank* of A, $\mathrm{rank}(A)$, is the dimension of the column space of A. Thus, the rank of a graph G, $\mathrm{rank}(G)$, is defined to be $\mathrm{rank}(A(G))$.

Let $S \subseteq V(G)$ be a non-empty subset and $|S| = \sigma$. Denote by G_S the *self-loop graph* of G at S, as a graph obtained from G by attaching a self-loop at each vertex in S. If $\sigma = 0$, then $G_S = G$. If $\sigma = n$, we denote the graph with full-loops by $\widehat{G}$. The adjacency matrix $A(G_S) = (a_{ij})_{n\times n}$ of G_S is then defined by $a_{ij} = 1$ if v_i is adjacent to v_j with possibly $i = j$, and $a_{ij} = 0$ otherwise. Thus, $A(G_S) = A(G) + I_\sigma$, where I_σ is the diagonal matrix with σ many 1's on the diagonal and zero otherwise. The rank of G_S is defined similarly by $\mathrm{rank}(G_S) := \mathrm{rank}(A(G_S))$. Throughout this paper, we work primarily with the rank of graphs. In particular, it is classically known that:

Proposition 1.1. [8] *If A is a submatrix of B, then* $\mathrm{rank}(A) \leq \mathrm{rank}(B)$.

An immediate consequence is the following that will be used frequently henceforth:

[1] School of Mathematical Sciences, Universiti Sains Malaysia, Penang, Malaysia.
e-mail: johnny.lim@usm.my; ORCID: 0000-0002-4562-1869.

* Corresponding author.

§ Manuscript received: June 07, 2025; accepted: September 22, 2025.
TWMS Journal of Applied and Engineering Mathematics, Vol.16, No.11; © Işık University, Department of Mathematics, 2026; all rights reserved.

Corollary 1.1. *If H is a subgraph of G_S, then* $\operatorname{rank}(H) \leq \operatorname{rank}(G_S)$.

Throughout this paper, we adopt the standard convention by denoting K_n, P_n, and C_n to be the *complete graph*, *path graph*, and *cycle graph* of order n, respectively. For our purpose, $(P_n)_S$ will be the self-loop path graph of order n, and thus denote

$$(P_n)_{\alpha_1\alpha_2\cdots\alpha_i}, \quad 1 \leq \alpha_1 \leq \cdots \leq \alpha_i \leq \cdots \leq n,$$

to represent each loop located at vertices $\alpha_1, \ldots, \alpha_i$, respectively.[1] For example,



Figure 1. $(P_4)_1$

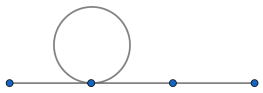


Figure 2. $(P_4)_2$

In some proofs, we utilize the following two notions (cf. [5]): the first is the *distance* between vertices. For vertices v_i and v_j, denote by $\operatorname{dist}(v_i, v_j)$ to be the length (i.e., number of edges) of a shortest path between v_i and v_j. If U and U' are non-empty subsets $V(G)$, then the distance $\operatorname{dist}(U, U')$ is defined to be the minimum distance between all vertices U and U', i.e.,

$$\operatorname{dist}(U, U') := \min\{\operatorname{dist}(v, v') \mid v \in U, v' \in U'\}.$$

The second is the graph joins. The *join* of (disjoint) graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph obtained by from the union of disjoint copies of G_1 and G_2 by joining each vertex of G_1 to each vertex of G_2. For our purpose, if $A \subseteq V(G_1)$, we write $G_1 \underset{A}{\vee} G_2$ to mean the graph join of G_1 and G_2 *over* the set A, that is for which each vertex of A is joined with each vertex of G_2 and none of $V(G_1)\backslash A$ is joined with vertices of G_2.

The study of spectral properties and energy of self-loop graphs is very recent. It was initiated by Gutman et. al. [7] in 2021. From the perspective of Molecular Chemistry, self-loop graphs represent heteroatoms and its graph energy is closely related to the total π-electron energy of conjugated molecules, cf. [6] and references therein. In 2023, the author and team [2] have explored some spectral results on self-loop graphs, as well as a characterization of self-loop graphs of order n whose eigenvalues are all positive or non-negative, and for any graphs with a few distinct eigenvalues. On the other hand, characterizations of simple graphs in terms of low rank are well-studied in the literature. In particular, Sciriha [9, Sect.6] proved the characterization of simple graphs of rank 0, 1, 2, and 3:

(i) A simple graph is of rank 0 if and only if it is an empty graph.
(ii) There are no simple graphs of rank 1.
(iii) The only simple graphs of order n and rank 2 are the complete bipartite graphs $K_{r,n-r}$, for $1 \leq r \leq \lfloor \frac{n}{2} \rfloor$.
(iv) The only simple graphs of rank 3 are the complete tripartite graphs $K_{a,b,c}$.

Characterization results for rank 4 and 5 were then obtained by [3] and [4], respectively. It is then natural to ask the question: *what about the characterization of self-loop graphs with low rank*? This motivated the study on characterization of self-loop graphs in [1], for which those with rank 1 and 2 are completely characterized.

[1]The notational consistency up to isomorphism is implicitly assumed. For example in Fig. 1 and 2 one may interpret the right-most vertex as the starting point, thus $(P_4)_1 \cong (P_4)_4$ and $(P_4)_3 \cong (P_4)_2$.

Theorem 1.1. [1] *Let G_S be a connected self-loop graph of order n. Then,*

(i) G_S is of rank 1 if and only if $G_S \cong \widehat{K_n}$.
(ii) G_S is of rank 2 if and only if $G_S \cong \widehat{K_\sigma} \vee (n-\sigma)K_1$ for $1 \leq \sigma \leq n-1$.

Some results on rank 3 characterizations of connected self-loop graphs have been obtained and the research is currently in progress. It is the purpose of this paper to establish a *partial* result for rank 3 characterizations. The organization of this paper is as follows. In Sect. 2, we prove that it is not possible to obtain rank 3 graphs that contain the self-loop cycle graphs of order $n \geq 5$. As a consequence, we provide a characterization of rank 3 triangle-free self-loop cyclic graphs of order *at least* 4 in Sect. 3, more precisely they must be isomorphic to one of the graphs in Fig. 16, Fig. 17, Fig. 22, Fig. 29, Fig. 30, and Fig. 31. Lastly, we propose some interesting open problems that are not pursued in this article in Sect. 5.

2. Initial results on self-loop cycle graphs of order at least five

2.1. **Consecutive loops.** We begin to investigate the possibility of obtaining rank 3 self-loop graphs based on the position of loops in $(C_n)_S$ for $n \geq 5$. The discussion will be separated into two scenarios, one for which all σ loops are consecutive, and another that are not. More specifically, if $\ell_1, \ell_2, \ldots, \ell_\sigma$ are the σ loops, then consecutive loops means $\operatorname{dist}(v_{\ell_i}, v_{\ell_{i+1}}) = 1$ for all $1 \leq i \leq \sigma - 1$.

Lemma 2.1. *For $n \geq 5$, there are no rank 3 cycle graphs $(C_n)_S$ with consecutive loops.*

Proof. For $n \geq 5$, if $1 \leq \sigma \leq n-3$, then $(C_n)_S$ contains the subgraph $(P_4)_1$ (cf. Fig. 1) of rank 4. Thus, it suffices to consider only consecutive loops with $\sigma \geq n-2$. It is clear that $(C_5)_S$ with $\sigma = 3, 4, 5$ will be of rank $5, 5, 4$, respectively. For $n \geq 6$, for $\sigma \geq 4$, $(C_n)_S$ contains the subgraph $\widehat{P_4}$ (cf. Fig. 3) of rank 4. By Corollary 1.1, there are no rank 3 cycle graphs of order $n \geq 5$ with consecutive loops. □

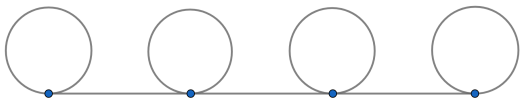


Figure 3. $\widehat{P_4}$

2.2. **Non-consecutive loops.** It follows from the proof of Lemma 2.1 that $(C_n)_S$ with consecutive loops of $\sigma \geq 4$ will not be of rank 3. Let's consider a *cluster* of loops: $\mathscr{C}_i, i = 1, 2, 3$, to be the clusters of one, two, and three *consecutive* loops, respectively. For "non-consecutive" to be meaningful, we require $\operatorname{dist}(\mathscr{C}_i, \mathscr{C}_j) \geq 2$ for $1 \leq i, j \leq 3$. Thus, it suffices to consider $S = \bigcup_{i=1}^{3} a_i \mathscr{C}_i$ where $a_i \in \mathbb{N} \cup \{0\}$ is the number of such clusters on $(C_n)_S$, so that $\sigma = |S| = a_1 + 2a_2 + 3a_3$. Some examples of different clusters of loops are given below for better visualization.

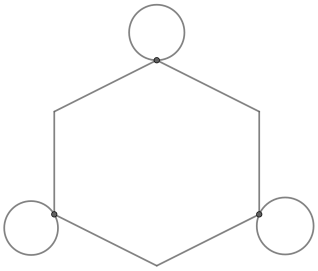


Figure 4. $3\mathscr{C}_1$

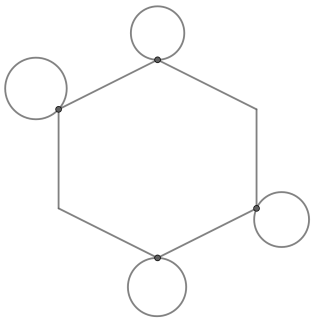


Figure 5. $2\mathscr{C}_2$

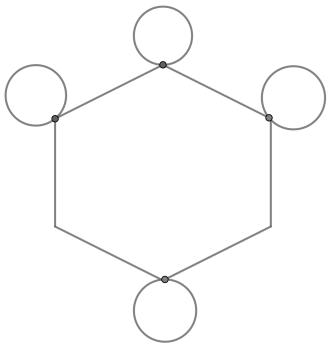


Figure 6. $\mathscr{C}_1 \cup \mathscr{C}_3$

Lemma 2.2. *For $n \geq 5$, there are no rank 3 cycle graphs $(C_n)_S$ with non-consecutive loops.*

Proof. We split the proof into discussing several cases of 'gaps' between $\mathscr{C}_i$.

(i) Let $\text{dist}(\mathscr{C}_i, \mathscr{C}_j) = 2$. Observe that any pair $(\mathscr{C}_i, \mathscr{C}_j)$ for $1 \leq i, j \leq 3$, except for $(\mathscr{C}_1, \mathscr{C}_1)$, contains the subgraph $(P_4)_{124}$ of rank 4.



Figure 7. $(P_4)_{124}$

For $(\mathscr{C}_1, \mathscr{C}_1)$, since $n \geq 5$, there exists a vertex v_0 such that $(C_n)_S$ contains the subgraph $(P_4)_{13}$ of rank 4.

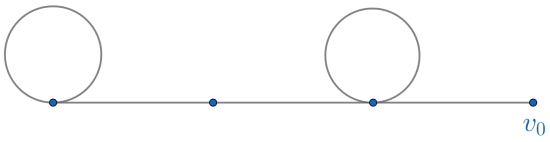


Figure 8. $(P_4)_{13}$

(ii) Let $\text{dist}(\mathscr{C}_i, \mathscr{C}_j) = 3$. Then any pair $(\mathscr{C}_i, \mathscr{C}_j)$ for $1 \leq i, j \leq 3$, except for $(\mathscr{C}_1, \mathscr{C}_1)$, contains the subgraph $(P_5)_{145}$ of rank 5.

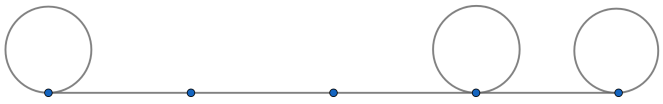


Figure 9. $(P_5)_{145}$

For $(\mathscr{C}_1, \mathscr{C}_1)$ and $n \geq 6$, $(C_n)_S$ contains the subgraph $(P_5)_{14}$ of rank 5;



Figure 10. $(P_5)_{14}$

if $n = 5$, the graph contains the subgraph $(P_4)_{13}$ of rank 4, similar to Fig. 8.

(iii) Let $\text{dist}(\mathscr{C}_i, \mathscr{C}_j) \geq 4$. Then, $(C_n)_S$ contains the subgraph $(P_4)_1$ (cf. Fig. 1) of rank 4.

By applying Corollary 1.1, we reach at the desired claim. □

These two lemmas can be concluded in one theorem.

Theorem 2.1. *Let $(C_n)_S$ be the cycle graph of order $n \geq 5$. For any nonempty $S \subseteq V(C_n)$, then* $\text{rank}((C_n)_S) \geq 4$.

3. Main results

From the results established above, one ought to consider cycles graphs of low order to possibly obtain rank 3 self-loop graphs. In fact, considerable effort has been put forth in [1] where rank 1 and rank 2 connected self-loop graphs are completely characterized. Here, we focus on getting partial results for rank 3 connected self-loop graphs by working with 4-cycle graphs.

Theorem 3.1. *Up to isomorphisms, all rank 3 triangle-free connected cyclic graphs with one self-loop constructed from $G_S = (C_4)_S$ with $|S| = 1$ are*

$$H_1 = G_S \underset{\mathcal{V}_I}{\vee} W \quad \text{and} \quad H_2 = G_S \underset{V(G_S)\setminus\mathcal{V}_I}{\vee} W,$$

where $\mathcal{V}_I$ is the independent set of loopless vertices in $V(G_S)$, and $W \nsubseteq V(G_S)$ is an independent set of K_1's.

Proof. Let $\mathcal{V}_I = \{v_a, v_b\}$. If $v \in W$, then $v \neq v_a, v_b$. If v is adjacent to v_a (or v_b) and at least one vertex in $N(a)$ (or $N(b)$), then at least one triangle is formed. Thus, v is adjacent to either

(i) only one vertex of G_S, or
(ii) only v_a and v_b, or
(iii) only $N(v_a)$ and $N(v_b)$.

For (i), the resulting graph is either of the following three graphs:

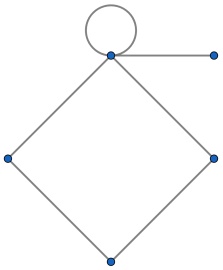


Figure 11

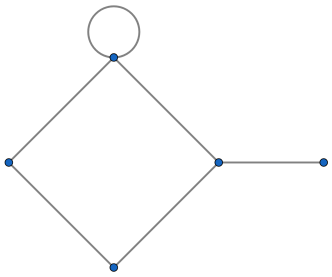


Figure 12

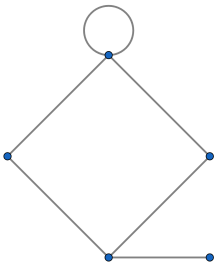


Figure 13

all of which contain either of the two subgraphs $(P_4)_1$ and $(P_4)_2$ of rank 4, so case (i) can be ignored. The remaining cases (ii) and (iii) give two non-isomorphic graphs of rank 3:

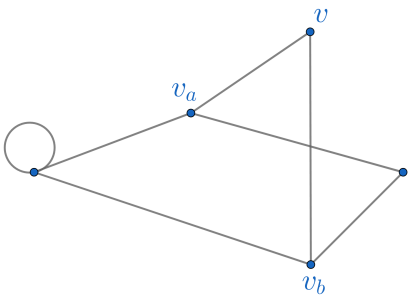


Figure 14

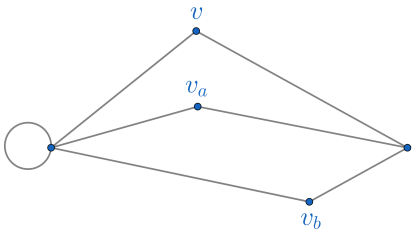


Figure 15

By induction on the graph joins, we obtain the following graphs of rank 3, which we refer as H_1 and H_2, respectively

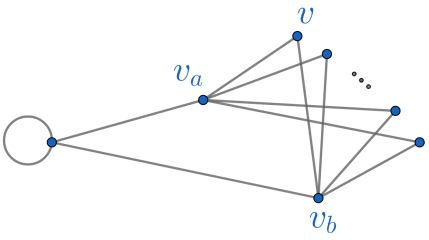


Figure 16. H_1

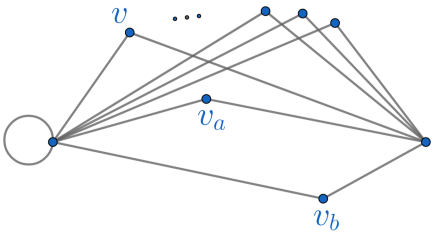


Figure 17. H_2

Note that the edge sets $E(H_1)$ and $E(H_2)$ are maximal with respect to rank 3. For if there is an edge $e' \notin E(H_i), i = 1, 2$, such that e' is incident with any two vertices in $V(H_i)$, then either at least a triangle is formed, or the new graph contains a subgraph of rank 4, a contradiction. Thus, H_1 and H_2 are the only triangle-free connected self-loop graphs of rank 3 when adding independent loopless vertices. □

Remark 3.1. *It can be verified that the only self-loop 4-cycle $(C_4)_S$ with $2 \leq |S| \leq 4$ of rank 3 is $(C_4)_S$ with $|S| = 2$. Moreover, such S must be an independent set in C_4. For any other cases, $(C_4)_S$ has rank 4.*

Theorem 3.2. *Up to isomorphisms, all rank 3 triangle-free connected cyclic graphs with two self-loops constructed from $G_S = (C_4)_S$ with $|S| = 2$ are*

$$H_2' = G_S \underset{V(G_S)\setminus \mathcal{V}_I}{\vee} W,$$

where $\mathcal{V}_I$ is the independent set of loopless vertices in $V(G_S)$, and $W \nsubseteq V(G_S)$ is an independent set of K_1's.

Proof. By Remark 3.1, it suffices to consider only the graph $(C_4)_S$ with $\sigma = 2$ and S is an independent set in C_4. The approach is similar to that of Theorem 3.1, there are three adjacency cases (i), (ii), and (iii). Suppose W is an independent set of isolated vertices possibly with loops. Then, up to isomorphisms,

(i) There are two sub-cases, each corresponds to the graphs below, respectively.

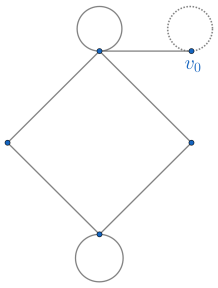


Figure 18

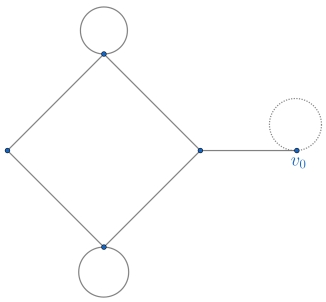


Figure 19

However, regardless of whether v_0 has a loop or not, the left is of rank 4 and the right is of rank 5.

(ii) In this case, the corresponding graph is Fig. 20, which has rank 4, regardless of whether v_0 has a loop or not.

(iii) The last case corresponds to the graph in Fig. 21, which has rank 3 if v_0 has no loop and rank 4 otherwise.

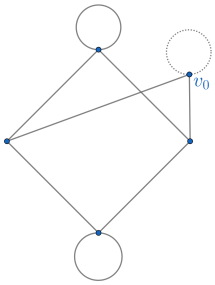


Figure 20

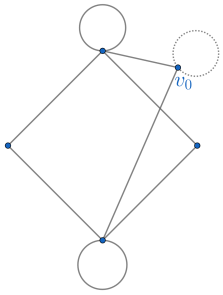


Figure 21

Thus, only Case (iii) is possible to obtain rank 3 graphs and W must contain only loopless vertices. By induction on graph joins with W, the graph H_2' (Fig. 22) is of rank 3 and two loops. The maximality of $E(H_2')$ is similar to the proof of Theorem 3.1. □

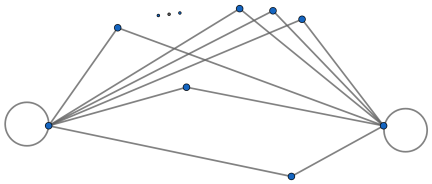


Figure 22. H_2'

Remark 3.2. *Observe that the graph in Fig. 23 (resp. 24) obtained by adding a loop at v in Fig 14 (resp. 15) are both of rank 4. If W is an independent set of isolated vertices possibly with loops, then one verifies that the n-graph joins of Fig. 23 and Fig. 24 with W cannot be of rank 3 if W contains at least one $\widehat{K_1}$. However, removing such looped vertex from W in the graph joins may create new classes of triangle-free connected self-loop cyclic graphs of rank 3, as shown in the next theorem.*

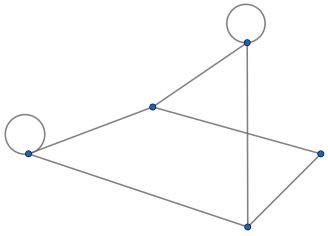


Figure 23

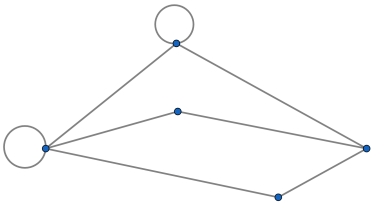


Figure 24

Theorem 3.3. *Let F_S and F_S' be the two graphs obtained from $G_S = (C_4)_S$, $|S| = 1$, by adding an one-sided looped pendent edge as in Fig. 25 and Fig. 26, respectively. Then, up to isomorphisms, all rank 3 triangle-free connected cyclic graphs with two self-loops constructed from F_S and F_S' are*

$$H_3 = F_S \underset{\mathcal{V}_I}{\vee} W, \qquad H_4 = F_S \underset{V(G_S)\setminus\mathcal{V}_I}{\vee} W, \qquad H_5 = F_S' \underset{\mathcal{V}_I}{\vee} W,$$

where $\mathcal{V}_I$ is the independent set of $V(G_S) \setminus S$ and $W \nsubseteq V(G_S) \cup \{v_0\}$ is an independent set of $K_1's$.

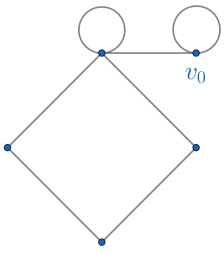


Figure 25. F_S

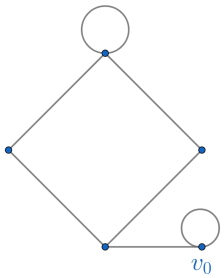


Figure 26. F_S'

Proof. Let $\mathcal{V}_I = \{v_a, v_b\}$ be the independent set in $V(G_S) \setminus S$. If $v_0 = \widehat{K_1}$ is adjacent to all vertices in $V(G_S) \setminus \mathcal{V}_I$, then the resulting graph is of rank 4. Thus, v_0 is only adjacent

to either vertex in $V(G_S) \setminus \mathcal{V}_I$. This produces F_S and F'_S of rank 3. Observe that adding a distinct vertex (with or without a loop) to F_S with one edge will produce a graph that contains a subgraph of rank at least 4. As such, the only possible rank 3 graphs of desired are as follows in Fig. 27 and Fig. 28

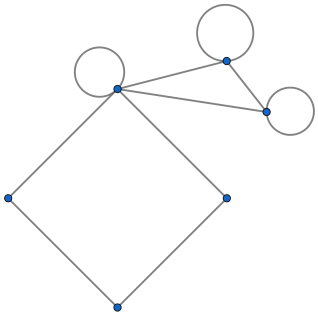


Figure 27

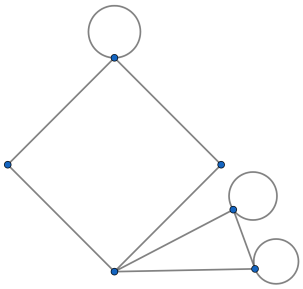


Figure 28

which, however, clearly contain a triangle. Thus, the added vertices must be loopless. By Corollary 1.1 and applying graph joins, the only possible desired graphs are H_3, H_4, and H_5, respectively:

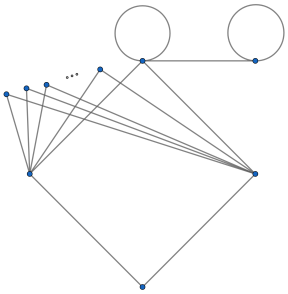


Figure 29. H_3

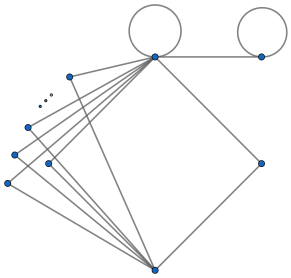


Figure 30. H_4

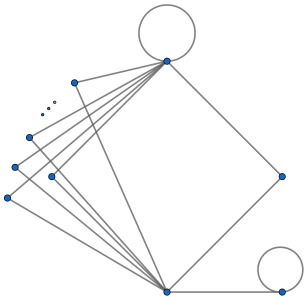


Figure 31. H_5 □

Remark 3.3. *Let H'_3, H'_4, and H'_5 be the corresponding graphs with $V(G_S)\setminus\mathcal{V}_I \subseteq S$. Then, all these graphs have rank 4:*

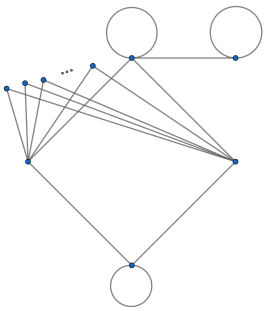


Figure 32. H'_3

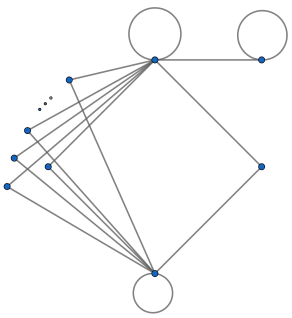


Figure 33. H'_4

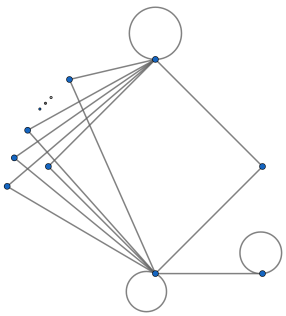


Figure 34. H'_5

where, of course, H'_4 is isomorphic to H'_5.

4. Conclusion

We have shown that if a triangle-free connected self-loop cyclic graph is not isomorphic to either of the six graphs in Fig. 16, Fig. 17, Fig. 22, Fig. 29, Fig. 30, and Fig. 31, then such graph cannot be of rank 3. The necessary condition of the characterization is then obtained by the contrapositive argument. A sufficient condition is straightforward by checking the rank of their adjacency matrix: if a triangle-free connected self-loop cyclic graph is isomorphic to either of the six graphs aforementioned, then it is of rank 3. Thus, we have provided the desired characterization and a partial solution to the complete characterizations of rank 3 connected self-loop graphs.

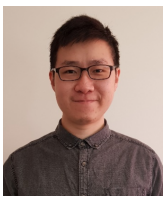
5. Open problem

(i) It is evident from Fig. 27 and Fig. 28 there are more rank 3 connected self-loop graphs of order at least 4 that *contain a triangle*. Thus, it would be interesting to characterize all such graphs.
(ii) One can also investigate the relationship between the rank and energy of self-loop graphs.

Acknowledgement. The author would like to thank the anonymous referees, Miin Huey Ang, and Kevin Fung for their helpful comments and suggestions in improving the manuscript.

References

[1] Abd Aziz, N. A., Akbari, S., Ang, M. H., Li, X. Y., and Lim, J., (2026), The Characterizations of Rank One and Rank Two Self-Loop Graphs, AIP Conference Proceedings 3444 (1), 050014.
[2] Akbari, S., Al Menderj, H., Ang, M. H., Lim, J., and Ng, Z. C., (2023), Some results on spectrum and energy of graphs with loops, Bull. Malays. Math. Sci. Soc. 46, no. 3, Paper No. 94, 18.
[3] Chang, G. J., Huang, L-H., and Yeh, H-G., (2011), A characterization of graphs with rank 4, Linear Algebra Appl. 434, no. 8, 1793-1798.
[4] Chang, G. J., Huang, L-H., and Yeh, H-G., (2012), A characterization of graphs with rank 5, Linear Algebra Appl. 436, no. 11, 4241-4250.
[5] Cvetković, D. M., Rowlinson, P., and Simić, S., (2010), An introduction to the theory of graph spectra, London Mathematical Society Student Texts, vol. 75, Cambridge University Press, Cambridge.
[6] Gutman, I., (1990), Topological studies on heteroconjugated molecules. VI. Alternant systems with two heteroatoms, Zeitschrift für Naturforschung A 45, no. 9-10, 1085-1089.
[7] Gutman, I., Redžepović, I., Furtula, B., and Sahal, A., (2021), Energy of Graphs with Self-Loops, MATCH Commun. Math. Comput. Chem. 87, 645-652.
[8] Horn, R. A., and Johnson, C. R., (2013), Matrix analysis, second ed., Cambridge University Press, Cambridge.
[9] Sciriha, I., (1999), On the rank of graphs, Combinatorics, graph theory, and algorithms, Vol. I, II (Kalamazoo, MI, 1996), New Issues Press, Kalamazoo, MI, pp. 769-778.



Johnny Lim received his BSc degree in pure math from Universiti Sains Malaysia (Malaysia) in 2014, his MSc degree in pure math from Australian National University (Australia) in 2017 and his PhD in pure math from University of Adelaide (Australia) in 2019. He is currently working as a senior lecturer at the School of Mathematical Sciences, Universiti Sains Malaysia (Malaysia). His research area includes Index Theory, K-theory, and more recently, Algebraic/Spectral Graph Theory.