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# $\alpha$-PRE-$E$-CONVEX AND $\alpha$-$E$-CONVEX FUNCTIONS WITH APPLICATIONS TO MATHEMATICAL PROGRAMMING

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Abstract. In this paper, we establish a new class of $E$-convexity, namely $\alpha$-pre-$E$-convexity. We define $\alpha$-pre-$E$-convex function (need not $E$-convex function), $\alpha$-prepseudo-$E$-convex function and discuss some properties of $\alpha$-pre-$E$-convex functions. We observe that the class of $\alpha$-prepseudo-$E$-convex functions is more general than the class of $\alpha$-pre-$E$-convex functions. We also introduce the class of $\alpha$-$E$-convex functions which we prove it is equivalent to the class of differentiable $\alpha$-pre-$E$-convex functions. Optimality conditions of these functions in mathematical programming problem are obtained.

Keywords: $E$-convex function, $\alpha$-pre-$E$-convex function, $\alpha$-prepseudo-$E$-convex function, $\alpha$-$E$-convex function, optimality conditions.

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## 1. Introduction

Convexity contains a broad range of significance in applied as well as pure mathematics. Today, the concept of convexity is not limited to the class of convex functions but it has extended to non-convex functions. A few years ago, the outcome of such work has proposed to several forms of non-convexity, like invexity, pre-invexity and $E$-convexity etc. For a family of real functions, Avriel established $r$-convexity [4] and obtained the solution of certain nonlinear programming problem with sum of r-convex functions [5]. After some years, Hanson proved the sufficiency of Kuhn-Tucker conditions for optimal solution of programming problem [15] and Craven obtained these conditions under the assumptions of K-invex functions [13]. Israel and Mond presented some new results for invexity and showed that invexity can be replaced for convexity in some certain problems [9]. Bector and Singh defined B-vex function and discussed some properties of B-vex functions [8]. By using the concept of invexity, Antczak introduced several (p, r)-invexity type nonlinear programming problems [2]. Antczak extended the concept of p-invex set and defined

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(p, r)-pre-invex function (non-differentiable) and (p, r)-invex function (differentiable) and obtained optimality conditions for nonlinear programming problem under the assumption of these functions [1]. Antczak defined r-preinvexity, r-invexity and obtained optimality criteria and duality relations for these functions in programming problem [3]. Antczak developed duality theorems for modified r-invex functions based on function $\eta$ [3]. Emam generalised B-invexity to roughly B-invexity and obtained some properties of roughly B-invex functions [12]. Ojha obtained optimality and duality results for programming problem under the assumption of generalised $(\psi, \rho)$-univexity [17].

Youness generalised the class of non-convexity and produced the concept of $E$-convexity. Firstly, $E$-convex functions and $E$-convex programming were defined by Youness [22]. Yang refined some results of $E$-convex programming [21], that were obtained by Youness [22]. Later, Youness obtained optimality criteria of $E$-convex programming and considered Fritz-John and Kuhn-Tucker problems for $E$-convexity [23]. Chen introduced the concept of semi-$E$-convex functions and obtained some basic properties of semi-$E$-convex functions [10]. Syau and Lee introduced $E$-quasiconvex function and established some characteristics of $E$-convex functions and $E$-quasiconvex functions [20]. By the concept of $E$-convexity, Fulga and Preda defined $E$-invex set, $E$-preinvex functions and discussed properties of E-preinvex functions [14]. For an optimal solution of nonlinear programming problem, the applications of $E$-preinvex and local $E$-preinvex functions are obtained in [14]. Jaiswal and Panda developed generalized $E$-convexity and $E$-invexity with the aid of differentiability and derived sufficient optimal condition for optimal solution of nonlinear programming problem [16]. Mishra et al. combined $E$-convexity with $b$-vexity and defined $E$-$b$-vex function [19].

A set $\zeta \subseteq \Re^n$ is said to be $E$-convex, if for all $p, q \in \zeta, \delta \in [0, 1]$, there is a map $E : \Re^n \to \Re^n$ such that $\delta E(p) + (1 - \delta)E(q) \in \zeta$ (see [22]).

A function $\psi : \Re^n \to \Re^n$ is called $E$-convex function on the set $\zeta \subseteq \Re^n$, if there is a map $E : \Re^n \to \Re^n$ such that $\zeta$ is an $E$-convex set and the following inequality is held

$$\psi(\delta E(p) + (1 - \delta)E(q)) \leq \delta\psi(E(p)) + (1 - \delta)\psi(E(q))$$

for all $p, q \in \zeta, \delta \in [0, 1]$. If this inequality has opposite sign for all $p, q \in \zeta, \delta \in [0, 1]$, then $\psi$ is called $E$-concave function. If the above inequality has strict sign, then $\psi$ is said to be strictly $E$-convex function (see [21, 22]).

In this paper, our main aim is to introduce $\alpha$-pre-$E$-convex function and $\alpha$-$E$-convex function and present applications of these functions in mathematical programming problem to obtain optimality related results.

In section 2, we define $\alpha$-pre-$E$- convex function, $\alpha$-pre-$E$- concave function and discuss some properties of these functions. We introduce $\alpha$-prepseudo-$E$-convex function and relate it with $\alpha$-pre-$E$-convex function. From the concept of these functions, optimality results for mathematical programming problem are obtained in section 3.
In section 4, we define $\alpha$-$E$-convex function and relate it with the differentiable $\alpha$-pre-$E$-convex function and present an application of $\alpha$-$E$-convex in problem (P) of section 3 to find optimality condition.

## 2. Definitions and Propositions

In this section, we introduce $\alpha$-pre-$E$-convex function and $\alpha$-pseudo-$E$-convex function, where $\alpha \in \Re$. we consider some examples and obtain some properties for $\alpha$-pre-$E$-convex functions.

**Definition 2.1** ($\alpha$-pre-$E$-convex function)**.** *Let $\zeta \subseteq \Re^n$ a nonempty $E$-convex set for the map $E : \Re^n \to \Re^n$, then the function $\psi : \zeta \to \Re$ is said to be $\alpha$-pre-$E$-convex function on $\zeta$, if the following inequality holds*

$$\psi(\delta E(p) + (1-\delta)E(q)) \leq \begin{cases} ln(\delta e^{\alpha\psi(E(p))} + (1-\delta)e^{\alpha\psi(E(q))})^{1/\alpha}, & \text{if } \alpha \neq 0 \\ \delta\psi(E(p)) + (1-\delta)\psi(E(q)), & \text{if } \alpha = 0 \end{cases} \tag{1}$$

*for all $p, q \in \zeta$ and $0 \leq \delta \leq 1$.*

**Definition 2.2** ($\alpha$-pre-$E$-concave function)**.** *Let $\zeta \subseteq \Re^n$ a nonempty $E$-convex set for the map $E : \Re^n \to \Re^n$, then the function $\psi : \zeta \to \Re$ is said to be $\alpha$-pre-$E$-concave function on $\zeta$, if the following inequality holds*

$$\psi(\delta E(p) + (1-\delta)E(q)) \geq \begin{cases} ln(\delta e^{\alpha\psi(E(p))} + (1-\delta)e^{\alpha\psi(E(q))})^{1/\alpha}, & \text{if } \alpha \neq 0 \\ \delta\psi(E(p)) + (1-\delta)\psi(E(q)), & \text{if } \alpha = 0 \end{cases} \tag{2}$$

*for all $p, q \in \zeta$ and $0 \leq \delta \leq 1$.*
*If the inequalities (1) and (2) have sharp sign, then function $\psi$ is said to be strict $\alpha$-pre-$E$-convex and strict $\alpha$-pre-$E$-concave function, respectively. It is well trued that every strict $\alpha$-pre-$E$-convex (strict $\alpha$-pre-$E$-concave) function is $\alpha$-pre-$E$-convex ($\alpha$-pre-$E$-concave) function.*

**Example 2.1.** *Let $\Re$ is an $E$-convex set for the map $E$ and $E$ is defined as*

$$E(p) = \begin{cases} (\frac{1+p}{2})^2, & p > 0 \\ (\frac{1-p}{2})^2, & p \leq 0 \end{cases}$$

*and function $\psi : \Re \to \Re$ is defined as*

$$\psi(p) = \begin{cases} 4p, & p > 0 \\ (p^2 - p + 1), & p \leq 0. \end{cases}$$

*Here, we see that $\psi$ is a convex function in its domain. For the above map $E$, function $\psi$ is $E$-convex as well as $\alpha$-pre-$E$-convex function on $\Re$ , for any $\alpha \in \Re$.*

**Example 2.2.** *Let $\Re$ is an $E$-convex set for the map $E$ and $E$ is defined as*

$$E(p) = \begin{cases} p^2 + 2p, & p > 0 \\ 0, & p \leq 0 \end{cases}$$

*and function $\psi : \Re \to \Re$ is defined as*

$$\psi(p) = \begin{cases} ln(p+1), & p > 0 \\ 0, & p \leq 0. \end{cases}$$

*We observe that function $\psi$ is concave in its domain and for the above map $E$, it is $E$-concave as well as $1/2$-pre-$E$-concave on $\Re$.*

**Example 2.3.** *Let $\Re^2$ is an $E$-convex set for the map $E$ and $E : \Re^2 \to \Re^2$ is defined as*

$$E(p, q) = (p^2, q^2), \forall p, q \in \Re$$

*and function $\psi : \Re \times \Re \to \Re$ is defined as*

$$\psi(p, q) = \begin{cases} \sqrt{pq} + 1, & pq > 0 \\ 1, & pq \leq 0. \end{cases}$$

*Then, we see that $\psi$ is not a convex function but when $\delta = 1/2$, it is $1$-pre-$E$-convex function and $E$-concave function at the points (1, 0) and (2, 3).*

From the definitions of $E$-convex and $\alpha$-pre-$E$-convex functions, it is well defined that every $E$-convex function is 0-pre-$E$-convex function but for all $\alpha \neq 0$, every $\alpha$-pre-$E$-convex function is need not $E$-convex function. For this, we see example 2.3. In which, function $\psi$ is 1-pre-$E$-convex but not $E$-convex for $\delta = 1/2$ at the points (1, 0) and (2,3).

## Properties of $\alpha$-pre-$E$-convex Functions

**Proposition 2.1.** *Let $\Re^n$ be an $E$-convex set for the map $E : \Re^n \to \Re^n$, then*

(1) *If $\psi : \Re^n \to \Re$ be a $\alpha$-pre-$E$-convex ($\alpha$-pre-$E$-concave) function on $\Re^n$, then for any $\beta \in \Re$, $\gamma \in \Re_+$, the functions $\psi + \beta$ and $\gamma\psi$ are $\alpha$-pre-$E$-convex ($\alpha$-pre-$E$-concave) and $\alpha/\gamma$-pre-$E$-convex ($\alpha/\gamma$-pre-$E$-concave) on $\Re^n$, respectively.*

(2) *If $\psi$, $\phi : \Re^n \to \Re$ be two $\alpha$-pre-$E$-convex ($\alpha$-pre-$E$-concave) functions on the set $\Re^n$ then for any $\nu_1, \nu_2 \in \Re_+$, the function $\omega$ defined as*

$$\omega(p) = \begin{cases} ln(\nu_1 e^{\alpha\psi(p)} + \nu_2 e^{\alpha\phi(p)})^{1/\alpha}, & if \alpha \neq 0 \\ \nu_1\psi(p) + \nu_2\phi(p), & if \alpha = 0 \end{cases}$$

*be a $\alpha$-pre-$E$-convex ($\alpha$-pre-$E$-concave) on the set $\Re^n$.*

(3) *If $\psi$ be a $\alpha$-pre-$E$-convex function on $\Re^n$ iff $-\psi$ is a $-\alpha$-pre-$E$-concave function on $\Re^n$.*

(4) *If $\psi$, $\phi : \Re^n \to \Re$ be two real valued functions defined on $\Re^n$ such that $\phi(p) = e^{\alpha\psi(p)}$, then function $\psi$ is $\alpha$-pre-$E$-convex ($\alpha$-pre-$E$-concave) on $\Re^n$, for all $\alpha \neq 0$ iff function $\phi$ is 0-pre-$E$-convex (0-pre-$E$-concave) on $\Re^n$.*

Now we prove part (2) and part (4) of proposition 2.1.

*Proof.* (2) For $\alpha \neq 0$, we assume that $\psi, \phi$ both are $\alpha$-pre-$E$-convex ($\alpha$-pre-$E$-concave) functions, then by definition 2.1 (2.2), for all $p, q, \in \zeta$ and $0 \leq \delta \leq 1$, we have

$$\psi(\delta E(p) + (1-\delta)E(q)) \leq (\geq) ln(\delta e^{\alpha\psi(E(p))} + (1-\delta)e^{\alpha\psi(E(q))})^{1/\alpha} \tag{3}$$

and

$$\phi(\delta E(p) + (1-\delta)E(q)) \leq (\geq) ln(\delta e^{\alpha\phi(E(p))} + (1-\delta)e^{\alpha\phi(E(q))})^{1/\alpha}. \tag{4}$$

Now we prove that $\omega$ be a $\alpha$-pre-$E$-convex ($\alpha$-pre-$E$-concave) function. For $\alpha \neq 0$, by the definition of $\omega$, we have for any $p, q \in \zeta$ and $0 \leq \delta \leq 1$,

$$\omega(\delta E(p) + (1-\delta)E(q)) = ln(v_1 e^{\alpha\psi(\delta E(p)+(1-\delta)E(q))} + v_2 e^{\alpha\phi(\delta E(p)+(1-\delta)E(q))})^{1/\alpha}. \tag{5}$$

Now by (3), (4) and (5), for all $p, q \in \zeta$ and $0 \leq \delta \leq 1$, we have

$$\begin{aligned} &ln(v_1 e^{\alpha\psi(\delta E(p)+(1-\delta)E(q))} + v_2 e^{\alpha\phi(\delta E(p)+(1-\delta)E(q))})^{1/\alpha} \\ &\leq (\geq) ln(v_1 e^{\alpha ln(\delta e^{\alpha\psi(E(p))}+(1-\delta)e^{\alpha\psi(E(q))})^{1/\alpha}} + v_2 e^{\alpha ln(\delta e^{\alpha\phi(E(p))}+(1-\delta)e^{\alpha\phi(E(q))})^{1/\alpha}})^{1/\alpha} \\ &= ln(v_1(\delta e^{\alpha\psi(E(p))} + (1-\delta)e^{\alpha\psi(E(q))}) + v_2(\delta e^{\alpha\phi(E(p))} + (1-\delta)e^{\alpha\phi(E(q))}))^{1/\alpha} \\ &= ln(\delta(v_1 e^{\alpha\psi(E(p))} + v_2 e^{\alpha\phi(E(p))}) + (1-\delta)(v_1 e^{\alpha\psi(E(q))} + v_2 e^{\alpha\phi(E(q))}))^{1/\alpha} \\ &= ln(\delta e^{\alpha\omega(E(p))} + (1-\delta)e^{\alpha\omega(E(q))})^{1/\alpha}. \end{aligned} \tag{6}$$

Then

$$\begin{aligned} &ln(v_1 e^{\alpha\psi(\delta E(p)+(1-\delta)E(q))} + v_2 e^{\alpha\phi(\delta E(p)+(1-\delta)E(q))})^{1/\alpha} \\ &\leq (\geq) ln(\delta e^{\alpha\omega(E(p))} + (1-\delta)e^{\alpha\omega(E(q))})^{1/\alpha}. \end{aligned} \tag{7}$$

By (5) and (7), for all $p, q \in \zeta$ and $0 \leq \delta \leq 1$, we get

$$\omega(\delta E(p) + (1-\delta)E(q)) \leq (\geq) ln(\delta e^{\alpha\omega(E(p))} + (1-\delta)e^{\alpha\omega(E(q))})^{1/\alpha}. \tag{8}$$

Now, for $\alpha = 0$, we consider the definition of $\alpha$-pre-$E$-convex ($\alpha$-pre-$E$-concave) function for $\psi$ and $\phi$, then for all $p, q \in \zeta$ and $0 \leq \delta \leq 1$,

$$\psi(\delta E(p) + (1-\delta)E(q)) \leq (\geq)\delta\psi(E(p)) + (1-\delta)\psi(E(q)) \tag{9}$$

and

$$\phi(\delta E(p) + (1-\delta)E(q)) \leq (\geq)\delta\phi(E(p)) + (1-\delta)\phi(E(q)). \tag{10}$$

We multiply (9) and (10) by $v_1$ and $v_2$, respectively, then adding them, we get

$$\begin{aligned} v_1\psi(\delta E(p) + (1-\delta)E(q)) + v_2\phi(\delta E(p) + (1-\delta)E(q)) \\ \leq (\geq)v_1(\delta\psi(E(p)) + (1-\delta)\psi(E(q))) + v_2(\delta\phi(E(p)) + (1-\delta)\phi(E(q))) \\ = \delta(v_1\psi(E(p)) + v_2\phi(E(p))) + (1-\delta)(v_1\psi(E(q)) + v_2\phi(E(q))) \\ = \delta\omega(E(p)) + (1-\delta)\omega(E(q)). \end{aligned} \tag{11}$$

Then, we have

$$\omega(\delta E(p) + (1-\delta)E(q)) \leq (\geq)\delta\omega(E(p)) + (1-\delta)\omega(E(q)) \tag{12}$$

Hence, for all $p, q \in \zeta$, $\delta \in [0,1]$, by (8) and (12), we show that $\omega$ is $\alpha$-pre-$E$-convex ($\alpha$-pre-$E$-concave) function on $\Re^n$. □

*Proof.* (4) For $\alpha \neq 0$, we assume that $\psi$ is a $\alpha$-pre-$E$-convex ($\alpha$-pre-$E$-concave) function on $R^n$. Then by the definition 2.1 (2.2), for all $p, q \in \zeta$ and $0 \leq \delta \leq 1$, we have

$$\psi(\delta E(p) + (1-\delta)E(q)) \leq (\geq) ln(\delta e^{\alpha\psi(E(p))} + (1-\delta)e^{\alpha\psi(E(q))})^{1/\alpha}.$$

Then for $\alpha \neq 0$, we have

$$e^{\alpha\psi(\delta E(p)+(1-\delta)E(q))} \leq (\geq)(\delta e^{\alpha\psi(E(p))} + (1-\delta)e^{\alpha\psi(E(q))}). \tag{13}$$

Now by the definition of function $\phi$, for all $p, q \in \zeta$ and $0 \leq \delta \leq 1$, (13) follows that

$$\phi(\delta E(p) + (1-\delta)E(q)) \leq (\geq)\delta\phi(E(p)) + (1-\delta)\phi(E(q)). \tag{14}$$

This show that $\phi$ is 0-pre-$E$-convex ( 0-pre-$E$-concave) function on $R^n$. Similarly its converse part can be easily proved. □

Now we define $\alpha$-prepseudo-$E$-convex function (may be non-differential function) and relate it with $\alpha$-pre-$E$-convex function.

**Definition 2.3** ($\alpha$-prepseudo-$E$-convex function)**.** *Let $\zeta \subseteq \Re^n$ be a nonempty $E$-convex set for the map $E : \Re^n \to \Re^n$. Then a function $\psi : \zeta \to \Re$ is said to be $\alpha$-prepseudo-$E$-convex function on $E$-convex set $\zeta$, if there is a function $\nu : \Re^n \times \Re^n \to \Re^n$ of strictly positive values such that the following inequality*

$$\begin{aligned} \psi(E(p)) < \psi(E(q)) \implies \psi(\delta E(p) + (1-\delta)E(q)) \\ \leq \begin{cases} ln(e^{\alpha\psi(E(q))} + \delta(\delta-1)\nu(E(p),E(q)))^{1/\alpha}, & \text{if } \alpha \neq 0 \\ \psi(E(q)) + \delta(\delta-1)\nu(E(p),E(q)), & \text{if } \alpha = 0 \end{cases} \end{aligned} \tag{15}$$

*holds, for all $p, q \in \zeta$ and $0 \leq \delta \leq 1$, where $e^{\alpha\psi(E(q))} + \delta(\delta-1)\nu(E(p),E(q)) > 0$.*

**Proposition 2.2.** *Let $\Re^n$ be an $E$-convex set for the map $E : \Re^n \to \Re^n$ and $\psi : \Re^n \to \Re$ be a $\alpha$-pre-$E$-convex function on $\Re^n$, then $\psi$ also be a $\alpha$-prepseudo-$E$-convex function on the same $E$-convex set $\Re^n$.*

*Proof.* We assume that $\psi : \Re^n \to \Re$ be a $\alpha$-pre-$E$-convex function on $E$-convex set $\Re^n$, such that $\psi(E(p)) < \psi(E(q))$, for all $p, q \in \Re^n$. Now we consider definition 2.1, for $\alpha > 0$, then, for all $p, q \in \Re^n$ and $0 \leq \delta \leq 1$

$$\psi(\delta E(p) + (1-\delta)E(q)) \leq ln(e^{\alpha\psi(E(q))} + \delta(e^{\alpha\psi(E(p))} - e^{\alpha\psi(E(q))}))^{1/\alpha}$$
$$\leq ln(e^{\alpha\psi(E(q))} + \delta(1-\delta)(e^{\alpha\psi(E(p))} - e^{\alpha\psi(E(q))}))^{1/\alpha}.$$

Now we put $\nu = e^{\alpha\psi(E(q))} - e^{\alpha\psi(E(p))}$, then

$$\psi(\delta E(p) + (1-\delta)E(q)) \leq ln(e^{\alpha\psi(E(q))} + \delta(\delta-1)\nu(E(p), E(q)))^{1/\alpha}.$$

Since, the condition $\psi(E(p)) < \psi(E(q))$ satisfies the definition of function $\nu(E(p), E(q))$ and $e^{\alpha\psi(E(q))} + \delta(\delta-1)\nu(E(p), E(q)) > 0$ is satisfied for all $p, q \in \Re^n$, $0 \leq \delta \leq 1$. Then by definition 2.3, we see that function $\psi$ be a $\alpha$-prepseudo-$E$-convex function on the same $E$-convex set $\Re^n$. For $\alpha = 0$, the proof has similarity so we omit this case. □

But converse of this proposition need not be true. i.e. If function $\psi$ is $\alpha$-prepseudo-$E$-convex function on $E$-convex set $\Re^n$, then $\psi$ is need not $\alpha$-pre-$E$-convex function on same set $\Re^n$ with respect to the same map $E$. For this we consider the following example,

**Example 2.4.** *Let $\Re_+$ is an $E$-convex set for the map $E$ and map $E : \Re_+ \to \Re_+$ is defined as $E(p) = p^2$ and function $\psi : \Re_+ \to \Re$ is defined as $\psi(p) = lnp$. We define the function $\nu : \Re_+ \times \Re_+ \to \Re_+$ as $\nu(p, q) = lnq - lnp$ and $q > p$, then we see that function $\psi$ is 0-prepseudo-$E$-convex but not 0-pre-$E$-convex for the map $E$ on $\Re_+$.*

Therefore we conclude that the class of $\alpha$-prepseudo-$E$-convex functions is wider than the class of $\alpha$-pre-$E$-convex functions.

## 3. Optimality Conditions in Mathematical Programming

Let $\zeta \subseteq \Re^n$ be an $E$-convex set for the map $E : \Re^n \to \Re^n$, then we consider the following mathematical programming problem (P)

$$\text{min. } \psi(E(\bar{p}))$$
$$\text{s.t. } \phi_i(E(\bar{p})) \leq 0,\ i = 1, 2, ...., n,$$

where $\bar{p} \in \zeta \subseteq \Re^n$ and $\psi, \phi_i : \zeta \to \Re, i = 1, 2, ..., n$ be not necessarily differentiable functions on $E$-convex set $\zeta$. Let

$$\Delta = \{\bar{p} \in \zeta : \phi_i(E(\bar{p})) \leq 0, i = 1, 2, ...., n\}$$

be a set that contains all feasible solutions of problem (P). Throughout this section, we consider $\alpha \neq 0$ and for $\alpha = 0$, all the proofs have similarity, so we omit this case.

**Theorem 3.1.** *Let $\phi_i : \zeta \to R$, $i = 1, 2, 3, ...n$ be $\alpha$-pre-$E$-convex functions with respect to the map $E$ on $\Delta$, then $\Delta \subseteq \zeta$ is an $E$-convex set.*

*Proof.* Let $\bar{p}, \bar{q}$ be any two feasible solutions of problem (P), then by the definition of $\Delta$, we have

$$\phi_i(E(\bar{p})), \phi_i(E(\bar{q})) \leq 0, i = 1, 2, 3, ...n. \tag{16}$$

Since $\phi_i$,$i = 1, 2, 3, ...n$ be $\alpha$-pre-$E$-convex functions on the set $\Delta$, then by definition 2.1, (1) is satisfied for all $\bar{p}, \bar{q} \in \Delta$ and $\delta \in [0, 1]$. Hence by (16), we get

$$\phi_i[\delta E(\bar{p}) + (1-\delta)E(\bar{q})] \leq ln[\delta e^{\alpha\phi_i(E(\bar{p}))} + (1-\delta)e^{\alpha\phi_i(E(\bar{q}))}]^{1/\alpha} \leq \frac{1}{\alpha} ln[\delta + (1-\delta)] = 0, \tag{17}$$

$i = 1, 2, 3, ...n$. Therefore for any $\bar{p}, \bar{q} \in \Delta, \delta \in [0, 1]$, we have

$$\delta E(\bar{p}) + (1-\delta)E(\bar{q}) \in \Delta. \tag{18}$$

This shows that $\Delta \subseteq \zeta$ is an $E$-convex set. □

**Theorem 3.2.** *Let $\Delta \subseteq \zeta$ be an $E$-convex set and $\psi : \zeta \to \Re$ be a $\alpha$-pre-$E$-convex function on $\Delta$. Then any local minimum $p*$ of problem (P) is a global minimum of problem (P).*

*Proof.* We assume that $p* \in \Delta$ is not a global minimum of (P), then there is a $\bar{p} \in \Delta$, with $p* \neq \bar{p}$, such that

$$\psi(E(\bar{p})) < \psi(E(p*)). \tag{19}$$

Since $\Delta$ be an $E$-convex set, then for any $\bar{p}, p* \in \Delta$ and $\delta \in [0,1]$, we have

$$\delta E(\bar{p}) + (1-\delta)E(p*) \in \Delta \tag{20}$$

and $\psi$ be a $\alpha$-pre-$E$-convex function on $\Delta$, then by definition 2.1, the following inequality

$$\psi[\delta E(\bar{p}) + (1-\delta)E(p*)] \leq ln[\delta e^{\alpha\psi(E(\bar{p}))} + (1-\delta)e^{\alpha\psi(E(p*))}]^{1/\alpha} \tag{21}$$

holds for all $\bar{p}, p* \in \Delta$ and $\delta \in [0,1]$. Now we use (19) and monotonocity for the logarithmic and exponential functions, then finally (21) gives that

$$\psi[\delta E(\bar{p}) + (1-\delta)E(p*)] < ln[\delta e^{\alpha\psi(E(p*))} + (1-\delta)e^{\alpha\psi(E(p*))}]^{1/\alpha} = \psi(E(p*)) \tag{22}$$

for all $\bar{p}, p* \in \Delta$ and $\delta \in [0,1]$. Thus we have

$$\psi[\delta E(\bar{p}) + (1-\delta)E(p*)] < \psi(E(p*))$$

for all $\bar{p}, p* \in \Delta$ and $\delta \in [0,1]$. This contradicts for $\delta \to 0$, hence our assumption is wrong. Therefore any local minimum $p* \in \Delta$ of problem (P) is also a global minimum of problem (P). □

**Theorem 3.3.** *Let $\Delta \subseteq \zeta$ be an $E$-convex set and $\psi$ be a $\alpha$-prepseudo-$E$-convex function on $\Delta$. Then any local minimum $p*$ of problem (P) is a global minimum of problem (P).*

*Proof.* We assume that $p* \in \Delta$ is not a global minimum of (P), then there is a $\bar{p} \in \Delta$, with $p* \neq \bar{p}$, such that

$$\psi(E(\bar{p})) < \psi(E(p*)). \tag{23}$$

Since $\Delta$ is an $E$-convex set, then for any $\bar{p}, p* \in \Delta$ and $\delta \in [0,1]$, we have

$$\delta E(\bar{p}) + (1-\delta)E(p*) \in \Delta \tag{24}$$

and $\psi$ is a $\alpha$-prepseudo-$E$-convex function on $\Delta$, then by definition 2.3, the following inequality

$$\psi[\delta E(\bar{p}) + (1-\delta)E(p*)] \leq ln[e^{\alpha\psi(E(p*))} + \delta(\delta-1)\nu(E(\bar{p}), E(p*))]^{1/\alpha} \tag{25}$$

holds, for all $\bar{p}, p* \in \Delta$ and $\delta \in [0,1]$. Now the function $\nu(E(\bar{p}), E(p*)) > 0$, by definition, then $\delta(\delta-1)\nu(E(\bar{p}), E(p*)) < 0$, for any $\delta \in (0,1)$ and we use monotonicity for logarithmic function, then

$$ln[e^{\alpha\psi(E(p*))} + \delta(\delta-1)\nu(E(\bar{p}), E(p*))]^{1/\alpha} < ln(e^{\alpha\psi(E(p*))})^{1/\alpha}. \tag{26}$$

Thus by (25) and (26), we get

$$\psi[\delta E(\bar{p}) + (1-\delta)E(p*)] < \psi(E(p*)),$$

for all $\bar{p}, p* \in \zeta$ and $\delta \in [0,1]$. This contradicts for $\delta \to 0$, hence our assumption is wrong. Therefore any local minimum $p* \in \Delta$ is also a global minimum of (P). □

**Theorem 3.4.** *Let $\Delta \subseteq \zeta$ be an $E$-convex set and $\psi : \zeta \to \Re$ be a strictly $\alpha$-pre-$E$-convex function on $\Delta$. If $p* \in \Delta$ be a local minimum of problem (P), then*

(1) *$p*$ is a global minimum of problem (P).*
(2) *$p*$ is a unique optimal solution of problem (P).*

*Proof.* First, we prove that $p*$ is a global minimum of problem (P). For this we consider the proof of theorem 3.2, with all the inequalities of strict sign, then we obtain required result (1).
Now we proof that $p* \in \Delta$ is a unique optimal solution of problem (P). We assume that $\bar{p} \in \Delta$ $(\bar{p} \neq p*)$ is a another optimal solution of (P), such that

$$\psi(E(\bar{p})) = \psi(E(p*)). \tag{27}$$

Since $\Delta \subseteq \zeta$ is an $E$-convex set, then (20) is satisfied, for all $\bar{p}, p* \in \Delta$ and $\delta \in [0,1]$ and $\psi$ be a strictly $\alpha$-pre-$E$-convex function on $\Delta$, then for all $\bar{p}, p* \in \Delta$ and $\delta \in [0,1]$, we have

$$\psi[\delta E(\bar{p}) + (1-\delta)E(p*)] < ln[\delta e^{\alpha\psi(E(\bar{p}))} + (1-\delta)e^{\alpha\psi(E(p*))}]^{1/\alpha}. \tag{28}$$

Thus by (27) and (28), we get

$$\psi[\delta E(\bar{p}) + (1-\delta)E(p*)] < ln[\delta e^{\alpha\psi(E(p*))} + (1-\delta)e^{\alpha\psi(E(p*))}]^{1/\alpha} = \psi(E(p*))$$

for all $\bar{p}, p* \in \Delta$ and $\delta \in [0,1]$. Then $\psi[\delta E(\bar{p}) + (1-\delta)E(p*)] < \psi(E(p*))$, for all $\bar{p}, p* \in \Delta$ and $\delta \in [0,1]$. This contradicts for $\delta \to 0$, hence our assumption is wrong in this case. This shows that $p*$ is a unique optimal solution of problem (P). □

## 4. $\alpha$-$E$-Convex Functions

In this section, we introduce the class of $\alpha$-$E$-convex functions. The functions in this class are differentiable and we show that this class is equivalent to the class of differentiable $\alpha$-pre-$E$-convex functions. Some optimality conditions for optimization problem determined by these functions are obtained.

**Definition 4.1** ($\alpha$-$E$-convex function)**.** *Let $\zeta \subseteq \Re^n$ be a nonempty $E$-convex set for the map $E : \Re^n \to \Re^n$. Then a differentiable function $\psi : \zeta \to \Re$ is said to be $\alpha$-$E$-convex at $q \in \zeta$ on $\zeta$ , if for all $p \in \zeta$, the following inequalities*

$$\frac{1}{\alpha}e^{\alpha\psi(E(p))} \geq \frac{1}{\alpha}e^{\alpha\psi(E(q))}(1 + \alpha(E(p) - E(q))^T \nabla\psi(E(q))), \qquad if \alpha \neq 0 \tag{29}$$

$$\psi(E(p)) - \psi(E(q)) \geq (E(p) - E(q))^T \nabla\psi(E(q)), \qquad if \alpha = 0 \tag{30}$$

*hold. If (29) and (30) are satisfied for all $p, q \in \zeta$, then $\psi$ be a $\alpha$-$E$-convex function on $\zeta$. If (29) and (30) have opposite for all $p, q \in \zeta$, then $\psi$ be a $\alpha$-$E$-concave function on $\zeta$. If (29) and (30) have strict sign for all $p, q \in \zeta$, with $E(p) \neq E(q)$, then $\psi$ be a strict $\alpha$-$E$-convex function on $\zeta$.*

**Theorem 4.1.** *Let $\zeta \subseteq \Re^n$ be a nonempty $E$-convex set for the map $E : \Re^n \to \Re^n$. Then a $\alpha$-$E$-convex function $\psi : \zeta \to \Re$ be also a $\alpha$-pre-$E$-convex function on $\zeta$.*

*Proof.* Let $\zeta$ be an $E$-convex set, then for any $p, q \in \zeta$ and $\delta \in [0,1]$, we have $\delta E(p) + (1-\delta)E(q) \in \zeta$. Now we assume that for any $\delta \in [0,1]$, $E(p^o) = \delta E(p) + (1-\delta)E(q) \in \zeta$, where $p^o \in \zeta$ and $\psi$ be a $\alpha$-$E$-convex function on $\zeta$. Then, for $\alpha \neq 0$, we consider the definition of $\alpha$-$E$-convex function $\psi$ on $\zeta$, we have

$$\frac{1}{\alpha}e^{\alpha\psi(E(p))} \geq \frac{1}{\alpha}e^{\alpha\psi(E(p^o))}(1 + \alpha(E(p) - E(p^o))\nabla\psi(E(p^o))) \tag{31}$$

and

$$\frac{1}{\alpha}e^{\alpha\psi(E(q))} \geq \frac{1}{\alpha}e^{\alpha\psi(E(p^o))}(1 + \alpha(E(q) - E(p^o))\nabla\psi(E(p^o))). \tag{32}$$

We multiply (31) and (32) by $\delta$ and $(1-\delta)$, respectively, then adding them , we get

$$\begin{aligned}\delta\frac{1}{\alpha}e^{\alpha\psi(E(p))}+(1-\delta)\frac{1}{\alpha}e^{\alpha\psi(E(q))}&\\ \geq \frac{1}{\alpha}e^{\alpha\psi(E(p^o))}\left[1+\alpha(\delta(E(p)-E(p^o))+(1-\delta)(E(q)-E(p^o)))\nabla\psi(E(p^o))\right]&\\ =\frac{1}{\alpha}e^{\alpha\psi(E(p^o))}\left[1+\alpha(\delta E(p)+(1-\delta)E(q)-E(p^o))\nabla\psi(E(p^o))\right].&\quad(33)\end{aligned}$$

Since $E(p^o)=\delta E(p)+(1-\delta)E(q)$ and a logarithmic function is an increasing function, then (33) follows that

$$\frac{1}{\alpha}ln(\delta e^{\alpha\psi(E(p))}+(1-\delta)e^{\alpha\psi(E(q))})\geq\psi(E(p^o)).\quad(34)$$

By (34) and $E(p^o)=\delta E(p)+(1-\delta)E(q)$, we have

$$\psi(\delta E(p)+(1-\delta)E(q))\leq ln[\delta e^{\alpha\psi(E(p))}+(1-\delta)e^{\alpha\psi(E(q))}]^{1/\alpha}.\quad(35)$$

Now for $\alpha=0$, we consider the definition of $\alpha$-$E$-convex function for $\psi$ on $\zeta$, then for all $p,q\in\zeta$, $\delta\in[0,1]$

$$\psi(E(p))-\psi(E(p^o))\geq(E(p)-E(p^o))\nabla\psi(E(p^o))\quad(36)$$

and

$$\psi(E(q))-\psi(E(p^o))\geq(E(q)-E(p^o))\nabla\psi(E(p^o))\quad(37)$$

We multiply (36) and (37) by $\delta$ and $(1-\delta)$, respectively, then adding them and by the definition of $E(p^o)$, finally we get

$$\delta\psi(E(p))+(1-\delta)\psi(E(q))-\psi(E(p^o))\geq 0.$$

Thus for all $p,q\in\zeta$ and $\delta\in[0,1]$, we have

$$\psi(\delta E(p)+(1-\delta)E(q))\leq\delta\psi(E(p))+(1-\delta)\psi(E(q)).\quad(38)$$

Hence, for all $p,q\in\zeta$, $\delta\in[0,1]$, by (35) and (38), we show that the function $\psi$ is $\alpha$-pre-$E$-convex on $\zeta$. □

**Theorem 4.2.** *Let $\zeta\subseteq\Re^n$ be a nonempty $E$-convex set for the map $E:\Re^n\to\Re^n$. Then a differentiable $\alpha$-pre-$E$-convex function $\psi:\zeta\to\Re$ is also a $\alpha$-$E$-convex function on $\zeta$.*

*Proof.* Let $\zeta$ be an $E$-convex set, then for any $p,q\in\zeta$ and $\delta\in[0,1]$, $\delta E(p)+(1-\delta)E(q)\in\zeta$. We assume that $\psi$ be a differentiable $\alpha$-pre-$E$-convex function on $\zeta$. Now for $\alpha\neq 0$, we consider definition 2.1, then for all $p,q\in\zeta$ and $\delta\in[0,1]$, we have

$$e^{\alpha[\psi(\delta E(p)+(1-\delta)E(q)]}\leq\delta e^{\alpha\psi(E(p))}+(1-\delta)e^{\alpha\psi(E(q))}=\delta(e^{\alpha\psi(E(p))}-e^{\alpha\psi(E(q))})+e^{\alpha\psi(E(q))},\quad(39)$$

then by (39)

$$\left(e^{\alpha[\psi(\delta E(p)+(1-\delta)E(q))]}-e^{\alpha\psi(E(q))}\right)\delta^{-1}\leq e^{\alpha\psi(E(p))}-e^{\alpha\psi(E(q))}.\quad(40)$$

Now we arrange (40) in the following form

$$\left(e^{\alpha[\psi(\delta E(p)+(1-\delta)E(q))-\psi(E(q))]}-1\right)e^{\alpha\psi(E(q))}\delta^{-1}\leq e^{\alpha\psi(E(p))}-e^{\alpha\psi(E(q))}.$$

Taking lim $\delta\to 0$, then we have the following inequality

$$\frac{1}{\alpha}e^{\alpha\psi(E(p))}\geq\frac{1}{\alpha}e^{\alpha\psi(E(q))}(1+\alpha(E(p)-E(q))^T\nabla\psi(E(q))).\quad(41)$$

Now, we consider definition 2.1, for $\alpha = 0$. Then for all $p, q \in \zeta$, $\delta \in [0,1]$, we have

$$\psi(\delta E(p) + (1-\delta)E(q)) \leq \delta\psi(E(p)) + (1-\delta)\psi(E(q)). \tag{42}$$

Then (42) can be written as

$$[\psi(\delta E(p) + (1-\delta)E(q)) - \psi(E(q))]\,\delta^{-1} \leq \psi(E(p)) - \psi(E(q)).$$

Taking lim $\delta \to 0$, then we have the following inequality

$$\psi(E(p)) - \psi(E(q)) \geq (E(p) - E(q))^T \nabla\psi(E(q)). \tag{43}$$

Hence, for all $p, q \in \zeta$, $\delta \in [0,1]$, by (41) and (43), we show that the function $\psi$ is $\alpha$-$E$-convex on $\zeta$. □

By theorem 4.2, we conclude that differentiability is a sufficient condition for $\alpha$-pre-$E$-convex function to be a $\alpha$-$E$-convex function.

**Theorem 4.3.** *Let $\zeta \subseteq \Re^n$ be a nonempty $E$-convex set for the map $E : \Re^n \to \Re^n$ and $\psi : \zeta \to \Re$ be a $\alpha$-$E$-convex ($\alpha$-$E$-concave) function on $\zeta$. Then every stationary point of the function $\psi$ is its global minimum (maximum).*

*Proof.* Assume that $\psi$ be a $\alpha$-$E$-convex ($\alpha$-$E$-concave) function on the set $\zeta$ and any $q \in \zeta$ is its stationary point. Then, we have

$$\nabla\psi(E(q)) = 0 \implies \psi(E(p)) \geq \psi(E(q))(\psi(E(p)) \leq \psi(E(q))).$$

Therefore $q \in \zeta$ is a global minimum (maximum) of the function $\psi$. □

Now we consider the previous programming problem (P) (section 3), in which we assume that $\psi$ and $\phi_i, i = 1, 2, 3, ...n$ are differentiable functions. Then, we obtain the following optimality condition under the assumption of $\alpha$-$E$-convex functions.

**Theorem 4.4.** *Let $\Delta \subseteq \zeta$ be a nonempty $E$-convex set for the map $E : \Re^n \to \Re^n$ and $\psi, \phi : \zeta \to \Re$ be $\alpha$-$E$-convex functions on $\Delta$, such that*

$$\nabla[\psi(E(\bar{p})) + \gamma^t\phi(E(\bar{p}))] = 0, \tag{44}$$

$$\gamma^t\phi(E(\bar{p})) = 0, \tag{45}$$

$$\gamma \geq 0, \gamma \in \Re^n, \tag{46}$$

*where $\bar{p} \in \Delta$ is a feasible solution of (P). Then $\bar{p}$ is also optimal solution (global minimum) of problem (P).*

*Proof.* We assume that $\bar{p} \in \Delta$ be a feasible solution of problem (P) and $\psi$, $\phi$ be two $\alpha$-$E$-convex functions. Let for any $\gamma \geq 0, p \in \Delta$, $\psi(p) + \gamma^t\phi(p)$ be a $\alpha$-$E$-convex function. Therefore by the definition 4.1 and (44), we have the following inequality

$$\frac{1}{\alpha}e^{\alpha[\psi(E(p))+\gamma^t\phi(E(p))]} \geq \frac{1}{\alpha}e^{\alpha[\psi(E(\bar{p}))+\gamma^t\phi(E(\bar{p}))]} \tag{47}$$

for all $p, \bar{p} \in \Delta$. This follows that

$$\psi(E(p)) + \gamma^t\phi(E(p)) \geq \psi(E(\bar{p})) + \gamma^t\phi(E(\bar{p})). \tag{48}$$

By (45) and (48), we have

$$\psi(E(p)) - \psi(E(\bar{p})) \geq -\gamma^t\phi(E(p)). \tag{49}$$

Since $\phi(E(p)) \leq 0$ and $\gamma \geq 0$, then $\gamma^t\phi(E(p)) \leq 0$, therefore $-\gamma^t\phi(E(p)) \geq 0$. Then by (49), we have

$$\psi(E(p)) \geq \psi(E(\bar{p})).$$

Hence $\bar{p}$ is optimal solution (global minimum) of problem (P). □

**Example 4.1.** *Let $\Re$ be an $E$-convex set for the map $E : \Re \to \Re$, such that $E(p) = p + 1$ and $\psi, \phi : \Re \to \Re$ be two differentiable $\alpha$-pre-$E$-convex functions on $\Re$ defined as $\psi(p) = p - 1$ and $\phi(p) = -p$, respectively. Then by theorem 4.2, $\psi$ and $\phi$ be $\alpha$-$E$-convex functions on $\Re$ and we consider the following programming problem:*

$$\begin{aligned} &min.\ p \\ &s.t. -(p+1) \leq 0. \end{aligned}$$

*Here $\Delta = \{p \in \Re : p \geq -1\}$ and $p = -1$ be a feasible set and a feasible solution for this problem, respectively. For $\gamma = 1$, $p = -1$ satisfies the optimality conditions of theorem 4.4. Hence $p = -1$ be also an optimal solution.*

## 5. Conclusions

- In this paper, we established the existence of relation between $\alpha$-pre-$E$-convex function and $\alpha$-$E$-convex function and also proved that every $\alpha$-pre-$E$-convex function is $\alpha$-prepseudo-$E$-convex function but its converse is not true.
- If we take $E(p) = p$, then putting $\eta(p, q) = p - q$ in [3], then results of [3] can be obtained as special case.

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