TWMS J. App. and Eng. Math. V.16, N.11, 2026, pp. 1346-1362

# BEAL'S FUZZY NORMAL SUBGROUP STRUCTURES WITH RESPECT TO (MIN, MAX) – NORMS

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Abstract. In this paper, we study a new kind of Beal's fuzzy subgroup with respect to suitable norms. we strongest relation, cosets and middle cosets of beal's fuzzy subgroup of a group with respect to norms. Also, we investigate equivalent characterizations of them and we construct a new group called Beal's fuzzy normal subgroup induced by them. Finally, we present the homomorphism theorem and discuss the preimage between them.

Keywords: Fuzzy set, Beal's fuzzy set, Beal's fuzzy subgroup, Normal subgroup, Relations, coset, middle coset, homomorphism, pre-image, norms.

AMS Subject Classification: 20N25, 03E72, 03F55.

## 1. Introduction

"Groups are vital in modern algebra, their basic structure can be found in many mathematical aspects". Groups can be found in geometry, representing phenomena such as symmetry and certain types of transformations. Introduction of group, in modern algebra the study of groups, which are systems consisting of a set of elements and a binary operation that can be applied to two elements of the set, which together satisfy certain axioms. Group theory has applications in physics, chemistry, computer science and even puzzles like Rubik's cube. In 1965, Zadeh [19] introduced the notion of fuzzy sets. In 1971, Rosenfeld introduced fuzzy group and formulated the concepts of subgroups of a group. "An increasing number of properties from classical group theory have been generalized". Many authors have worked on fuzzy group theory [5, 6, 12]. Especially, some authors consider the fuzzy subgroups with respect to t-norm and gave some amazing results. [1, 9, 12]. The concept of intuitionistic fuzzy set was introduced by Atanassov [1] as a generalization of the notion of fuzzy set. The theory of intuitionistic fuzzy set is expected to play an important role in modern mathematics in general as it represents a generalization of fuzzy set.

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§ Manuscript received: July 01, 2025; accepted: May 22, 2026.
TWMS Journal of Applied and Engineering Mathematics, Vol.16, No.11; © Işık University, Department of Mathematics, 2026; all rights reserved.

Yager [16] presented the concept of Pythagorean fuzzy set which is the generalization of intuitionistic fuzzy sets and it is a more effective tool to solve uncertain problems. Ibrahim et.al [3] defined another type of generalized Pythagorean fuzzy set is called (3,2) fuzzy sets. Fermatean fuzzy sets studied by some authors and they also introduced fundamental operations on the fermatean fuzzy sets. Murad et.al [7] presented new kind of fuzzy sets is called (3, 4)-fuzzy sets which is deal with more uncertain situations than (3, 2) – fuzzy. A generalization of the Beal's conjecture and prize problem defined by [8] in 1997. In this paper, we construct a new kind of Beal's fuzzy normal subgroup with respect to suitable norms. we define strongest relation, cosets and middle cosets of beal's fuzzy subgroup of a group with respect to norms. Also, we investigate equivalent characterizations of them and we construct a new group induced by them. Finally, we analyse the homomorphism theorem, preimage between them.

## 2. Basic and Preliminaries Concepts

**Definition 2.1.** *A group is a non-empty set $G$ on which there is a binary operation $(x, y) \to xy$ such that*

*(i) If $x$ and $y$ belong to $G$, then $xy$ is also in $G$ (Closure Law)*
*(ii) $x(yz) = (xy)z$ for all $x, y, z \in G$. (Associative Law)*
*(iii) There is an element $e \in G$ such that $x * e = e * x = x$ for all $x \in G$. (Identity Law)*
*(iv) If $x \in G$, then there is an element $x^{-1} \in G$ such that $x * x^{-1} = x^{-1} * x = e$. (Inverse Law)*

*One can easily check that, this implies the unicity of the identity and of the inverse. A group $G$ is called an abelian if the binary operation is commutative (ie) $x * y = y * x$ for all $x, y \in G$.*

**Remark 2.1.** *There are two standard notations for the binary group operation either the additive notation, that is $(x, y) \to x + y$ in which case the identity is represented by 0, or the multiplicative notation, that is $(x, y) \to xy$ for which the identity is denoted by $e$.*

**Proposition 2.1.** *Let $G$ be a group. Let $H$ be a non-empty subset of $G$. The following are equivalent:*

*(i) $H$ is a subgroup of $G$.*
*(ii) $x, y \in H$ implies $xy^{-1} \in H$ for all $x$ and $y$.*

**Definition 2.2.** *Let $G$ be an arbitrary group with a multiplicative binary operation and identity $e$.*

*A fuzzy subset of $G$, we mean a function from $G$ into [0,1]. The set of all fuzzy subsets of $G$ is called the [0,1] – power set of $G$ and is denoted by $[0, 1]^G$.*

**Definition 2.3.** *For set $X$, $Y$ and $Z$, $f = (f_1, f_2) : X \to Y \times Z$ is called a complex mapping if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.*

**Definition 2.4** (Beal's Fuzzy Set)**.** *A Beal's fuzzy set (BFS) on a universal set $X$ is a set of 3-tuples of the form*

$$A = \{(x, J_A(x), K_A(x))/ x \in X\} \tag{1}$$

*where $J_A(x)$ and $K_A(x)$ represent the membership and non-membership degrees of $x \in X$ and satisfy the condition $0 \leq J_A(x)^m + K_A(x)^n \leq 1$ for all $x \in X, m, n \in N$ if $m, n \geq 4$ and the degree of indeterminancy of $X$ to $A$ is*

$$\prod_A (x) = \sqrt[m+n]{1 - [(J_A(x)^m) + (K_A(x)^n)]}.$$

**Definition 2.5** ( Beal's Fuzzy Subgroup)**.** *Let $A = (X, J_A,\ K_A)$ be a Beal's fuzzy set of a group $G$. Then $A$ is known as Beal's fuzzy subgroup (BFSG) of $G$ if the following conditions hold:*

$$(BFG_1): J_A^m(xy) = \min\{J_A^m(x),\ \ J_A^m(y)\} \text{ and}$$
$$K_A^n(xy) = \max\{K_A^n(x),\ K_A^n(y)\}, \forall x, y \in G.$$
$$(BFG_2): J_A^m\left(x^{-1}\right) = J_A^m(x) \text{ and } K_A^n\left(x^{-1}\right) = K_A^n(x),\ \forall\ x \in G.$$

In this paper, we often use Beal's fuzzy subgroup as BFSG.

**Definition 2.6.** *Let $X$ be a non-empty set and let $D_1 = (J_{D_1}, K_{D_1})$ and $D_2 = (J_{D_2}, K_{D_2})$ be BFSs in $X$. Then*

*(i) Inclusion: $D_1 \subseteq D_2$ if and only if $J_{D_1} \leq J_{D_2}$ and $K_{D_1} \geq K_{D_2}$.*
*(ii) Equality: $D_1 = D_2$ if and only if $D_1 \subseteq D_2$ and $D_2 \subseteq D_1$.*

**Definition 2.7.** *A t-norm $T$ is a function $T : [0,1] \times [0,1] \to [0,1]$ having the following four properties*

$(T_1)$ $T(x,1) = x$ *(Neutral element)*
$(T_2)$ $T(x,y) \leq T(x,z)$, *if* $y \leq z$ *(Monotonicity)*
$(T_3)$ $T(x,y) = T(y,z)$ *(Commutative)*
$(T_4)$ $T(x,\ T(y,z)) = T(T(x,y),z)$ *(Associativity) for all* $x, y, z \in [0,1]$.

**Corollary 2.1.** *Let $T$ be a $t-$ norms. Then for all $x \in [0,1]$,*

*(i)* $T(x,0) = 0$
*(ii)* $T(0,0) = 0$.

**Example 2.1.** *(i) Standard intersection $t-$ norm* $T_m(x,y) = \min\{x,y\}$.
*(ii) Bounded sum $t-$ norms* $T_b(x,y) = \max\{0,\ x+y-1\}$.
*(iii) Algebraic product $t-$ norms* $T_p(x,y) = xy$
*(iv) Drastic $t-$ norm*
$$T_D(x,y) = \begin{cases} y, & \text{if } x = 1 \\ x, & \text{if } y = 1 \\ 0 & , \text{ otherwise} \end{cases}$$
*(v) Nilpotent minimum $t-$ norm*
$$T_{nM}\ (x,y) = \begin{cases} \min\{x,y\}, & \text{if } x+y > 1 \\ 0, & \text{elsewhere} \end{cases}$$
*(vi) Hamacher product $t-$ norm*
$$T_{H0}(x,y) = \begin{cases} 0, & \text{if } x = y = 0 \\ \dfrac{xy}{x+y-xy}, & \text{elsewhere} \end{cases}.$$

*The drastic $t-$ norm is the pointwise smallest $t-$ norm and the minimum is the pointwise largest $t-$ norms: $T_D(x,y) \leq T(x,y) \leq T_{min}(x,y)$ for all $x, y \in [0,1]$.*

**Lemma 2.1.** *Let $T$ be a $t-$ norm. Then $T(T(x,y),\ T(w,z)) = T(T(x,w),\ T(y,z))$ for all $x, y, w, z \in [0,1]$.*

**Definition 2.8.** *An $S-$norm is a function $S : [0,1] \times [0,1]$ having the following four properties*

$(S_1)$ $S(x,0) = x$
$(S_2)$ $S(x,y) \leq S(x,z)$ *if* $y \leq z$
$(S_3)$ $S(x,y) = S(y,x)$
$(S_4)$ $S(x, S(y,z)) = S(S(x,y),z)$ *for all* $x, y, z \in [0,1]$.

**Corollary 2.2.** *Let $S$ be a $S-$ norm. Then for all $x \in [0,1]$*

*(i)* $S(x,1)=1$
*(ii)* $S(0,0)=0$

**Example 2.2.** *(i) Standard union s− norm* $S_m(x,y)=\max\{x,y\}$.
*(ii) Bounded sum s− norm* $S_b(x,y)=\min\{1,\ x+y\}$
*(iii) Algebraic sum s−norm* $S_p(x,y)=x+y-xy$
*(iv) Drastic s−norm* $s_D(x,y)=\begin{cases}y, & \text{if } x=1\\ x, & \text{if } y=1\\ 0, & \text{otherwise}\end{cases}$ *dual to the drastic s− norm.*
*(v) Nilpotent maximum s− norm, dual to the nilpotent minimum T− norm*

$$S_{n,M}(x,y)=\begin{cases}\max\{x,y\}, & \text{if } x+y<1\\ 1, & \text{otherwise}.\end{cases}$$

*Note that all S− norms are bounded by the maximum and the drastic s−norm* $S_{\max}(x,y)\le S(x,y)\le S_D(x,y)$ *for any s− norm* $'S'$ *and all* $x,y\in[0,1]$.

**Lemma 2.2.** *Let S be a s− norm. Then* $S(S(x,y)),\ S(w,z))=S(S(x,w),\ S(y,z))$ *for all* $x,y,w,z\in[0,1]$.

## 3. Standard Results Based on Beal's Fuzzy subgroup

**Proposition 3.1.** *Let* $D=(J_D\ ,\ K_D)\in BFSG$ *w.r.t norms and* $\gamma,\delta\in[0,1]$. *If T, S are idempotent. Then* $D_{\gamma,\delta}=\{x\in G/D(x)\supseteq(\gamma,\delta)\}$ *is a subgroup of G.*

*Proof.* Let $x,y\in D_{\gamma,\delta}$. Then $D(x)\supseteq(\gamma,\delta)$ and $(y)\supseteq(\gamma,\delta)$. Thus $J_D^m(x)\ ,\ J_D^m(y)\ge\gamma$ and $K_D^n(x)\ ,\ K_D^n(y)\le\delta$. As

$$\begin{aligned}J_D^m\left(xy^{-1}\right)&\ge T\left\{\ J_D^m(x),\ J_D^m\left(y^{-1}\right)\right\}\\ &\ge T\ \{\ J_D^m(x),\ J_D^m(y)\}\\ &\ge T\ \{\ \gamma,\ \gamma\}\\ &\ge\gamma \text{ and}\\ K_D^n\left(xy^{-1}\right)&\le S\left\{\ K_D^n(x),\ K_D^n\left(y^{-1}\right)\right\}\\ &\le S\ \{\ K_D^n(x),\ K_D^n(y)\}\\ &\le S\ \{\delta,\ \delta\}\\ &\le\delta\end{aligned}$$

So $J_D^m\left(xy^{-1}\right)\ge\gamma$ and $K_D^n\left(xy^{-1}\right)\le\delta$, which implies that $x*y^{-1}\in D_{\gamma,\delta}$. Hence $D_{\gamma,\delta}$ is a subgroup of $G$. □

**Proposition 3.2.** *Let* $D=(J_D,\ K_D)\in BFSG$ *w.r.t. norms and* $\gamma,\delta\in[0,1]$, *for* $i=1,2$ *such that* $D(e_G)\supseteq(\gamma_i,\ \delta_i)$ *for* $i=1,2$ *with* $\gamma_1>\gamma_2$ *and* $\delta_1<\delta_2$. *Then* $D_{\gamma_1,\delta_1}=D_{\gamma_2,\delta_2}$ *if and only if there is no* $x\in G$ *such that* $(\gamma_2,\delta_2)\subset D(x)\subset\ (\gamma_1\ ,\ \delta_1)$.

*Proof.* Let $D_{\gamma_1,\delta_1}=D_{\gamma_2\ ,\ \delta_2}$ and their exists an $'x'$ such that $(\gamma_2,\delta_2)\subset D(x)\subset\ (\gamma_1\ ,\ \delta_1)$. Then $D_{\gamma_1,\ \delta_1}\subseteq D_{\gamma_2\ ,\ \delta_2}$ and so $x\in D_{\gamma_2,\delta_2}$ but $x\notin\ D_{\gamma_1,\ \delta_1}$. Thus, there is no $x\in G$ such that $(\gamma_2,\delta_2)\subset D(x)\subset\ (\gamma_1\ ,\ \delta_1)$.

Conversely, if there is no $x\in G$ such that $(\gamma_2,\delta_2)\subset D(x)\subset\ (\gamma_1,\delta_1)$ , then $D_{\gamma_1,\delta_1}=D_{\gamma_2,\delta_2}$. □

**Proposition 3.3.** *Let* $D=(J_D,\ K_D)\in BFSG$ *w.r.t. norm* $D_{\gamma,\delta}$ *be a subgroup of G for all* $\gamma,\delta\in[0,1]$ *and* $D(e_G)\ \supseteq(\gamma,\delta)$. *Then* $D=(J_D,\ K_D)\in BFSG$ *w.r.t norm.*

*Proof.* Let $x, y \in G$ with $D(x) = (\gamma_1, \delta_1)$ and $D(y) = (\gamma_2,\ \delta_2)$ and then $x \in D_{\gamma_1,\delta_1}$ and $y \in D_{\gamma_2\ ,\delta_2}$. Now we analyse the following axioms:

(i) If $\gamma_1 < \gamma_2$ and $\delta_1 > \delta_2$, then $y \in D_{\gamma_1,\delta_1}$ and as $D_{\gamma_1,\delta_1}$ is a subgroup of $G$. So $x * y^{-1} \in D_{\gamma_1,\delta_1}$. Now,

$$\begin{aligned} J_D^m(x*y) &\geq \ \gamma_1 = T(\gamma_1,\ \gamma_2) = \ T\{\ J_D^m(x),\ J_D^n(y)\} \\ K_D^n(x*y) &\leq S(\delta_1,\ \delta_2) = \ S\{\ K_D^n(x),\ K_D^n(y)\} \\ J_D^m\left(x^{-1}\right) &\ \geq\ \gamma_1 = J_D^m(x) \text{ and } K_D^n\left(x^{-1}\right) \leq \delta_1 = \ K_D^n(x) \end{aligned}$$

Thus $D = (J_D,\ K_D) \in BFSG$ w.r.t norm

(ii) If $\gamma_1 = \gamma_2$ *and* $\delta_1 > \delta_2$, or $\delta_1 = \delta_2$ *and* $\gamma_1 < \gamma_2$, then the proof is similar to $(i)$.

(iii) If $\gamma_1 < \gamma_2$ *and* $\delta_2 > \delta_1$, then $x \in D_{\gamma_2,\delta_2}$ and as $D_{\gamma_2,\delta_2}$ is a subgroup of $G$. So $x * y\ ,\ x^{-1} \in D_{\gamma_2,\delta_2}$. Now,

$$\begin{aligned} J_D^m(x*y) &\geq \gamma_2 = T(\gamma_2,\gamma_1) = \ T(\gamma_1,\ \gamma_2) = T\{J_D^m(x),\ \ J_D^m(y)\} \\ K_D^n(x*y) &\leq \ \delta_2 = S(\delta_2,\ \delta_1) = \ S(\delta_1\ ,\ \delta_2) = S\{K_D^n(x),\ K_D^n(y)\} \\ J_D^m\left(x^{-1}\right) &\geq \ \gamma_2 = J_D^m(x) \\ K_D^n\left(x^{-1}\right) &\leq \delta_2 = K_D^n(x) \end{aligned}$$

Then $A = (J_D,\ K_D) \in BFSG$ w.r.t norms.

(iv) If $\gamma_1 = \ \gamma_2$ and $\delta_2 > \delta_1$, or if $\delta_1 = \ \delta_2$ and $\gamma_1 > \gamma_2$, the proof is similar to $(iii)$.

(v) If $\gamma_2 = \ \gamma_1,$ *and* $\delta_2 = \ \delta_1$, then it is trivial.

□

**Proposition 3.4.** *Let $D = (J_D,\ K_D) \in BFSG$ w.r.t norms and $\gamma_i,\ \delta_i \in [0,1]$ for $i = 1, 2$. If $D_{\gamma_1,\delta_1}$ and $D_{\gamma_2\ ,\ \delta_2}$ are two subgroups in G, then $D_{\gamma_1,\delta_1} \bigcap D_{\gamma_2\ ,\ \delta_2}$ also a subgroup in $G$.*

*Proof.* Let $D(e_G) \supseteq (\gamma_1,\ \delta_1)$ and $D(e_G) \supseteq (\gamma_1,\ \delta_1)$ *and* $D(e_G) \supseteq (\gamma_2,\ \delta_2)$ and $x \in G$. Then

(i) If $(\gamma_1, \delta_1) \subseteq D(x) \subset\ (\gamma_2, \delta_2)$, then $D_{\gamma_2,\delta_2} \subseteq D_{\gamma_1,\delta_1}$ and so $D_{\gamma_1,\delta_1} \bigcap D_{\gamma_2\ ,\ \delta_2}$ and as $D_{\gamma_2\ ,\ \delta_2}$ is a subgroup in G, so $D_{\gamma_1,\delta_1} \bigcap D_{\gamma_2\ ,\ \delta_2}$ also a subgroup in $G$.

(ii) If $(\gamma_2,\ \delta_2) \subset D(x) \subset (\gamma_1,\ \delta_1)$, then $D_{\gamma_1,\delta_1} \subseteq D_{\gamma_2\ ,\ \delta_2}$ and so $D_{\gamma_2\ ,\ \delta_2} \bigcup D_{\gamma_1,\delta_1} = D_{\gamma_2,\delta_2}$ is a subgroup in $G$, so $D_{\gamma_2,\delta_2} \bigcup D_{\gamma_1,\delta_1}$, also a subgroup in $G$.

(iii) If $(\gamma_1, \delta_1) = (\gamma_2, \delta_2)$, then $D_{\gamma_1,\delta_1} = D_{\gamma_2,\delta_2}$ and then $D_{\gamma_1,\delta_1} \bigcup\ D_{\gamma_2,\delta_2}$ also a subgroup in $G$.

□

**Corollary 3.1.** *Let $D = (J_D,\ K_D) \in BFSG$ w.r.t norms and $\{\gamma_i, \delta_i\}_{i\in I} \in [0,1]$. If $D_{\gamma_i,\delta_i}$ are subgroup in $G$. Then $\bigcup D_{\gamma_i,\delta_i}$ also a subgroup in $G$.*

**Proposition 3.5.** *Let $T, S$ be idempotent norms. Then any subgroup $H$ of a group $G$ can be realized as a level subgroup of $D = (J_D,\ K_D) \in BFSG$ w.r.t norms.*

*Proof.* Let $D = (J_D,\ K_D) \in BFSG$ w.r.t norms defined by

$$D(x) = (J_D(x),\ K_D(x)) = \begin{cases} (\gamma, 0), & \text{if } x \in H \text{ and } \gamma \in [0,1] \\ (0, \gamma), & \text{if } x \notin H \text{ and } \beta \in [0,1] \end{cases}.$$

we show that $D \in BFSG$ w.r.t norms. Let $x, y \in G$ and now we consider that following conditions.

(i) If $x, y \in H$, then as $H$ is a subgroup of $G$. So $x * y^{-1} \in H$. Thus

$$\begin{aligned} J_D^m(x) &= J_D^m(y) \\ &= J_D^m\left(xy^{-1}\right) \\ &= \gamma \text{ and} \\ K_D^m(x) &= K_D^n(y) = K_D^n\left(x * y^{-1}\right) = 0. \end{aligned}$$

Then, $J_D^m\left(x * y^{-1}\right) = \gamma \geq \gamma = T(\gamma, \gamma) = T\{(J_D^m(x), J_D^m(y)\}$ and

$$\begin{aligned} K_D^n\left(x * y^{-1}\right) &= 0 \leq 0 = S(0,0) \\ &= S(K_D^n(x), K_D^n(y)) \end{aligned}$$

and from Definition 2.5, we get that $D = (J_D, K_D) \in BFSG$ w.r.t norm.

(ii) If $x \in H$ and $y \notin H$, then $x * y^{-1} \notin H$ and then
$J_D^m(x) = \gamma$ $J_D^m(y) = J_D^m\left(x * y^{-1}\right) = 0$ ; $K_D^n(x) = 0$ and
$K_D^n(y) = K_D^n\left(x * y^{-1}\right) = \delta$.
Thus, $J_D^m\left(x * y^{-1}\right) = 0 \geq 0 = T(\gamma, 0) = T(J_D^m(x), J_D^m(y))$ and
$K_D^n\left(x * y^{-1}\right) = \delta \leq \delta = S(0, \delta) = S(K_D^n(x), K_D^n(y)$ ,from Definition 2.5, we obtain that $D \in BFSG$ w.r.t norms.

(iii) If $x, y \notin H$, then $J_D^m(x) = J_D^m(y) = 0$ and $K_D^n(x) = K_D^n(y) = \delta$, then $x * y^{-1}$ may or may not belong to $H$. If $x * y^{-1} \in H$, then

$$\begin{aligned} J_D^m\left(x * y^{-1}\right) &= \gamma \geq 0 = T(0,0) = T(J_D^m(x), J_D^m(y)) \text{ and} \\ K_D^n\left(x * y^{-1}\right) &= \delta \leq \delta = S(\delta, \delta) = S(K_D^n(x), K_D^n(y)) \end{aligned}$$

as Definition 2.5, we will have $D \in BFSG$ w.r.t norms. If $x * y^{-1} \in H$, then

$$\begin{aligned} J_D^m\left(x * y^{-1}\right) &= 0 \geq 0 = T(0,0) = T(J_D^m(x), J_D^m(y)) \text{ and} \\ K_D^n\left(x * y^{-1}\right) &= \delta \leq \delta = S(\delta, \delta) = S(K_D^n(x), K_D^n(y)) \end{aligned}$$

as Definition 2.5 we get $D \in BFSG$ w.r.t norms. Thus in all cases $D \in BFSG$ w.r.t norms.

□

**Proposition 3.6.** *Let $D = (J_D, K_D) \in BFSG$ w.r.t norms such that $T$ and $S$ be idempotent. Then $D(e_G) \supseteq D(x)$ for all $x \in G$.*

*Proof.* Let $D = (J_D, K_D) \in BFSG$ w.r.t norms and $x \in G$. Then

$$\begin{aligned} J_D^m(G) &= J_D^m\left(x^{-1}x\right) \\ &\geq T\left\{J_D^m(x), J_D^m\left(x^{-1}\right)\right\} \\ &\geq T(J_D^m(x), J_D^m(x)) \\ &= J_D^m(x) \end{aligned} \tag{2}$$

and so $J_D^m(e_G) \geq J_D^m(x)$.
Also,

$$\begin{aligned} K_D^n(e_G) &= K_D^n\left(x^{-1}x\right) \\ &\leq S\left(K_D^n\left(x^{-1}\right), K_D^n(x)\right) \\ &\leq S(K_D^n(x), K_D^n(x)) \\ &= K_D^n(x) \end{aligned} \tag{3}$$

and then $K_D^n(e_G) \leq K_D^n(x)$.

Thus from (2) and (3) we have $D(e_G) = (J_D(e_G),\ K_D(e_G)) \supseteq (J_D(x),\ K_D(x)) = D(x)$. □

**Proposition 3.7.** *Let $D = (J_D,\ K_D) \in BFSG$ w.r.t norms such that $T$ and $S$ be idempotent and $D\left(xy^{-1}\right) = D(e_G)$. Then $D(x) = D(y)$, for all $x, y \in G$.*

*Proof.* Let $x, y \in G$. Then

$$\begin{aligned}
J_D^m(x) &= J_D^m\left(x * y^{-1} * y\right) \\
&\geq T\left(J_D^m\left(x * y^{-1}\right),\ J_D^m(y)\right) \\
&= T(J_D^m(e_G),\ J_D^m(y)) \\
&\geq T(J_D^m(y),\ J_D^m(y)) \\
&= J_D^m(x) \\
&= J_D^m\left(y * x^{-1} * x\right) \\
&\geq T\left(J_D^m\left(y * x^{-1}\right),\ J_D^m(x)\right) \\
&= T\left(J_D^m\ \left(x * y^{-1}\right),\ J_D^m(x)\right) \\
&= T(J_D^m(e_G),\ J_D^m(x)) \\
&\geq T(J_D^m(y),\ J_D^m(y)) \\
&= J_D^m(x).
\end{aligned}$$

Thus $J_D^m(x) = J_D^m(y)$. Also,

$$\begin{aligned}
K_D^n(x) &= K_D^n\left(x * y^{-1} * y\right) \\
&\leq S(K_D^n\left(x * y^{-1}\right),\ K_D^n(y)) \\
&= S(\ K_D^n(e_G),\ K_D^n(y)) \\
&\leq S(K_D^n(y),\ K_D^n(y)) \\
&= K_D^n(y) \\
&= K_D^n(y * x^{-1} * x) \\
&= S\left(K_D^n\left(\left(x * y^{-1}\right)^{-1}\right),\ K_D^n(x)\right) \\
&= S(K_D^n\left(x * y^{-1}\right),\ K_D^n(x)) \\
&\leq S(K_D^n(e_G),\ K_D^n(x)) \\
&= S(K_D^n(y),\ K_D^n(y)) \\
&= K_D^n(y).
\end{aligned}$$

Therefore $D(x) = D(y)$.
Then $K_D^n(x) = K_D^n(y)$. □

**Proposition 3.8.** *Let $T$ and $S$ be idempotent. Then $D = (J_D, K_D) \in BFSG$ w.r.t norms if and only if $D\left(xy^{-1}\right) \supseteq D(T(J_D^m,\ K_D^m(y)),\ S(K_D^n(x),\ K_D^n(y))$ for all $x, y \in G$.*

*Proof.* Let $D = (J_D, K_D) \in BFSG$ w.r.t norms and $x, y \in G$. Then

$$\begin{aligned}
J_D^m\left(x * y^{-1}\right) &\geq T(J_D^m(x),\ J_D^m\left(y^{-1}\right)) \\
&\geq T(\ J_D^m(x),\ J_D^m(y))
\end{aligned}$$

and

$$\begin{aligned} K_D^n(\left(x * y^{-1}\right) &\leq S(K_D^n(x),\ K_D^n(y)) \\ &\leq S\left(K_D^n(x),\ K_D^n(y)\right) \end{aligned}$$

and then

$$\begin{aligned} D\left(x * y^{-1}\right) &= \left(J_D^m\left(x * y^{-1}\right),\ K_D^n\left(x * y^{-1}\right)\right) \\ &\supseteq D(T(J_D^m(x), J_D^m(y), S(K_D^n(x), K_D^n(y)) \end{aligned}$$

Conversely, Let $D\left(x * y^{-1}\right) \supseteq D(T(J_D^m(x), J_D^m(y)), S(K_D^n(x),\ K_D^n(y))$. Then

$$\begin{aligned} J_D^m\left(x^{-1}\right) &= J_D^m\left(e_G * x^{-1}\right) \\ &\geq T(J_D^m(e_G),\ J_D^m(x)) \\ &\geq T\left(J_D^m(x),\ J_D^m(x)\right) \\ &= J_D^m(x) \end{aligned}$$

and

$$\begin{aligned} K_D^n\left(x^{-1}\right) &= K_D^n(e_G * x^{-1}) \\ &\leq S(K_D^n(e_G),\ K_D^n(x)) \\ &\leq S(K_D^n(x),\ K_D^n(x)) \\ &= K_D^n(x) \end{aligned}$$

and then

$$D\left(x^{-1}\right) \supseteq D(x) \tag{4}$$

Also,

$$\begin{aligned} J_D^m(x * y) = J_D^m\left(x * \left(y^{-1}\right)^{-1}\right) &\geq T(J_D^m(x),\ J_D^m(y^{-1}) \\ &\geq T(J_D^m(x),\ J_D^m(y)) \end{aligned}$$

and

$$\begin{aligned} K_D^n(x * y) = K_D^n\left(x * \left(y^{-1}\right)^{-1}\right) &\leq S\left(K_D^n(x),\ K_D^n\left(y^{-1}\right)\right) \\ &\leq S(K_D^n(x),\ K_D^n(y)) \end{aligned}$$

Hence

$$D(x * y) \supseteq D(T\left(J_D^m(x),\ J_D^m(y)\right),\ S(K_D^n(x),\ K_D^n(y)) \tag{5}$$

Therefore from (4) and (5), we have $D = (J_D,\ K_D) \in BFSG$ w.r.t norms. □

**Proposition 3.9.** *Let $J_D, K_D \in [0,1]^G$ such that $D(e_G) = (1,0)$ and $D(x * y^{-1}) \supseteq (T\left(J_D^n(x),\ J_D^n(y)\right),\ S\left(K_D^n(x),\ K_D^n(y)\right)$ for all $x, y \in G$. Then $D = (J_D,\ K_D) \in BFSG$ w.r.t norm.*

*Proof.* Let $x, y \in G$, then

$$\begin{aligned} J_D^m\left(x^{-1}\right) = J_D^m\left(e_G * x^{-1}\right) &\geq T(J_D^m(e_G),\ J_D^m(x)) \\ &= T\left(1,\ J_D^m(x)\right) = J_D^m(x) \end{aligned}$$

and

$$\begin{aligned} K_D^n\left(x^{-1}\right) = K_D^n\left(e_G * x^{-1}\right) &\leq S\left(K_D^n(e_G),\ K_D^n(x)\right) \\ &= S\left(D,\ K_D^n(x)\right) = K_D^n(x) \end{aligned}$$

and so,

$$D\left(x^{-1}\right) = \left(J_D^m\left(x^{-1}\right),\ K_D^n\left(x^{-1}\right)\right) \supseteq \left(J_D^m(x),\ \ K_D^n(x)\right) = D(x) \tag{6}$$

Also,

$$\begin{aligned} J_D^m(x*y) = J_D^m\left(x*\left(y^{-1}\right)^{-1}\right) &\geq T\left(J_D^m(x),\ J_D^m\left(y^{-1}\right)\right) \\ &\geq T\left(J_D^m(x),\ J_D^m(y)\right) \end{aligned}$$

and

$$\begin{aligned} K_D^n(x*y) = K_D^n(x*\left(y^{-1}\right)^{-1}) &\leq S(K_D^n(x),\ K_D^n(y^{-1})) \\ &\leq S(K_D^n(x),\ K_D^n(y)) \end{aligned}$$

and so

$$D(x*y) = (J_D^m(x*y),\ K_D^n(x*y) \supseteq D\left(T\left(J_D^m(x),\ J_D^m(y)\right),\ S\left(K_D^n(x),\ K_D^n(y)\right)\right) \quad (7)$$

Therefore from (6) and (7) we have $D = (J_D,\ K_D) \in BFSG$ w.r.t norm. □

**Proposition 3.10.** *If $A = (J_D,\ K_D) \in BFSG$ w.r.t norms. Then $H = \{x \in G/D(x) = (1,0)\}$ is a subgroup of $G$.*

*Proof.* Let $x, y \in H$. Then $J_D^m(x) = J_D^m(y) = 1$ and $K_D^n(x) = K_D^n(y) = 0$.

Since, $D = (J_D,\ K_D) \in BFSG$ w.r.t norms. So $J_D^m\left(x*y^{-1}\right) \geq T\{J_D^m(x),\ J_D^m(y)\} = T(1,1) = 1$ and this implies that $J_D^m\left(x*y^{-1}\right) = 1$. Also,

$$K_D^n\left(x*y^{-1}\right) \leq S\left(K_D^n(x),\ K_D^n(y)\right) = S(0,0) = 0$$

and so $K_D^n\left(x*y^{-1}\right) = 0$. Then

$$D\left(x*y^{-1}\right) = \left(J_D^m\left(x*y^{-1}\right), K_D^n\left(x*y^{-1}\right)\right) = (1,0)$$

and then $x*y^{-1} \in H$. So from Definition we see that $H$ is a subgroup of $G$. □

**Proposition 3.11.** *Let $D = (J_D, K_D) \in BFSG$ w.r.t norms if $D\left(x*y^{-1}\right) = (0,1)$. Then either $D(x) = (0,1)$ or $D(y) = (0,1)$ for all $x, y \in G$.*

*Proof.* Let $D = (J_D, K_D) \in BFSG$ w.r.t norms then for all $x, y \in G$. we see that

$$D = J_D^m\left(x*y^{-1}\right) \geq T(J_D^m(x),\ J_D^m(y))$$

and then either $J_D^m(x) = 0$ *or* $J_D^m(y) = 0$.

Also, $1 = K_D^n\left(x*y^{-1}\right) \leq S\left(K_D^n(x),\ K_D^n(y)\right)$ and either $K_D^n(x) = 1$ or $K_D^n(y) = 1$. Therefore either $D(x) = (J_D^m(x),\ K_D^n(y)) = (0,1)$ and

$$D(y) = (J_D^m(y),\ K_D^n(x)) = (0,1).$$

□

**Proposition 3.12.** *Let $D = (J_D, K_D) \in BFSG$ w.r.t norms. Then $D(x*y) = D(y*x)$ if and only if $D(x) = D(y^{-1}*x*y)$ for all $x, y \in G$.*

*Proof.* Let $x, y \in G$ and $D(x*y) = D(y*x)$. Then

$$\begin{aligned} J_D^m\left(y^{-1}*x*y\right) &= J_D^m\left(y^{-1}*(x*y)\right) \\ &= J_D^m\left(x*y*y^{-1}\right) = J_D^m(x*e_G) = J_D^m(x) \end{aligned}$$

and

$$\begin{aligned} K_D^n\left(y^{-1}*x*y\right) &= K_D^n\left(y^{-1}*(x*y)\right) \\ &= K_D^n\left(x*y*y^{-1}\right) = K_D^n(x*e_G) = K_D^n(x) \end{aligned}$$

and so

$$\begin{aligned} D\left(y^{-1}*x*y\right) &= \left(J_D^m\left(y^{-1}*x*y\right),\ K_D^n(\left(y^{-1}*x*y\right)\right. \\ &= \left(J_D^m\left(x\right),\ K_D^n\left(x\right)\right) = D\left(x\right). \end{aligned}$$

Conversely, let $D\left(x\right) = D\left(y^{-1}*x*y\right)$. Then

$$\begin{aligned} J_D^m\left(x*y\right) &= J_D^m\left(x\left(y*x\right)x^{-1}\right) = J_D^m\left(y*x\right) \\ K_D^n\left(x*y\right) &= K_D^n\left(x\left(y*x\right)x^{-1}\right) = K_D^n\left(y*x\right) \end{aligned}$$

and then

$$D\left(x*y\right) = \left(J_D^m\left(xy\right),\ K_D^n\left(xy\right)\right) = \left(J_D^m\left(yx\right),\ K_D^n\left(yx\right)\right) = D\left(y*x\right).$$

□

## 4. Relations among the Beal's Fuzzy Set with Cosets

In this section, we study the concept of strongest relations cosets and middle cosets of Beal's fuzzy set.

**Definition 4.1.** *Let $D = (J_D, K_D) \in BFSG$ w.r.t norms and $A = (J_A,\ K_A) \in BFS(G\times G)$ w.r.t norms. we say that $A$ is the strongest relation of $G$ w.r.t $D$ if $A\left(x,y\right) = \left(J_A^m\left(x,y\right),\ J_A^m\left(x,y\right)\right)$ such that $J_A^m\left(x,y\right) = T(J_A^m\left(x\right),\ J_A^m\left(y\right))$ and*

$$K_A^n\left(x,y\right) = S\left(K_A^n\left(x\right),\ K_A^n\left(y\right)\right) \text{ for all } (x,y) \in G\times G.$$

**Proposition 4.1.** *Let $D = (J_D, K_D) \in BFSG$ and $A = (J_A,\ K_A) \in BFS(G\times G)$ such that $A$ be the strongest relation of $G$ w.r.t $D$. Then $D \in BFSG$ w.r.t norms if and only if $A \in BFS(G\times G)$ w.r.t norms.*

*Proof.* Let $D = (J_D, K_D) \in BFSG$ w.r.t norms. First we show that $J_A^m((x_1,\ x_2)\left(y_1,y_2\right)) \geq T\{J_A^m\left(x_1,y_1\right),\ J_A^m\left(x_2,y_2\right)\}$ for all $\left(x_1,x_2\right),\ \left(y_1,y_2\right) \in G\times G$.

$$\begin{aligned} J_A^m\left(\left(x_1,x_2\right)\left(y_1,y_2\right)\right) &= J_A^m\left(\left(x_1y_1,\ x_2y_2\right)\right) \\ &= T(J_D^m\left(x_1,y_1\right),\ J_D^m(x_2,y_2)) \\ &\geq T(T\left(J_D^m\left(x_1\right),\ J_D^m\left(x_2\right)\right),\ T\left(J_D^m\left(y_1\right)\ ,\ J_D^m\left(y_2\right)\right)) \\ &= T(T\left(J_D^m\left(x_1\right),\ J_D^m\left(y_1\right)\right),\ T(J_D^m\left(x_2\right),\ J_D^m\left(y_2\right)) \\ &= T(J_A^m((x_1,y_1),\ J_A^m\left(x_2,y_2\right)) \end{aligned}$$

Similarly, we will have,

$$K_A^n((x_1,x_2)\left(y_1,y_2\right)) \leq S(K_A^n\left(x_1,y_1\right),\ K_A^n\left(x_2,y_2\right))$$

for all $\left(x_1,x_2\right),\ \left(y_1,y_2\right) \in G\times G$.

We now show that $J_A^m\left(\left(x,y\right)^{-1}\right) \geq J_A^m\left(x_1,y_1\right),$ for all $(x,y) \in G\times G$

$$\begin{aligned} \left(x,y\right)^{-1} &= J_A^m\left(x^{-1},\ y^{-1}\right) \\ &= T\left(J_D^m\left(x^{-1}\right),\ J_D^m\left(y^{-1}\right)\right) \\ &\geq T(J_D^m\left(x\right),\ \ J_D^m\left(y\right)) \\ &= J_A^m(x,y) \end{aligned}$$

Similarly, $K_A^n(x,y)^{-1} \leq K_A^n\ \ (x,y)$, for all $(x,y) \in G\times G$. Hence by Definition $A = (J_A,\ K_A) \in BFS(G\times G)$ w.r.t norms.

Conversely, suppose that $A = (J_A,\ K_A) \in BFS\left(G\times G\right)$ w.r.t norms.

(i) Let $x_1, x_2, y_1, y_2 \in G$ with $x_2 = y_2 = e_G$. Then W.K.T $D(e) \supseteq D(x_1, y_1)$. Then

$$\begin{aligned}
J_D^m(x_1, y_1) &= T\left(J_D^m(x_1, y_1),\ J_D^m(e_G)\right) \\
&= T(J_D^m(x_1, y_1),\ J_D^m(x_2, y_2), \\
&= J_A^m((x_1, y_1,\ x_2, y_2) \\
&\geq T(J_A^m(x_1, x_2),\ J_A^m(y_1, y_2)) \\
&= T(T\left(J_D^m(x_1),\ J_D^m(x_2)\right),\ T\left(J_D^m(y_1),\ J_D^m(y_2)\right)) \\
&= T(T\left(J_D^m(x_1),\ J_D^m(e)\right),\ T\left(J_D^m(y_1),\ J_D^m(e)\right)) \\
&\geq T(T\left(J_D^m(x_1),\ J_D^m(x_1)\right),\ T\left(J_D^m(y_1),\ J_D^m(y_1)\right)) \\
&= T(J_D^m(x_1),\ J_D^m(y_1)) \\
J_D^m(x_1, y_1) &\geq T(J_D^m(x_1),\ J_D^m(y_1))
\end{aligned}$$

(ii) Let $x, y \in G$ with $y = e_G$. we obtain

$$\begin{aligned}
J_D\left(x^{-1}\right) &= T\left(J_D^m\left(x^{-1}\right)\ ,\ \ J_D^m(e_G)\right) \\
&\geq T(J_D^m\left(x^{-1}\right),\ J_D^m\left(y^{-1}\right)) \\
&= J_D^m(x^{-1},\ y^{-1}) \\
&= J_D^m\left((x, y)^{-1}\right) \\
&\geq J_D^m((x, y)) \\
&= T\left(J_D^m(x),\ J_D^m(y)\right) \\
&= T\left(J_D^m(x),\ J_D^m(e)\right) \\
&\geq T(J_D^m(x),\ J_D^m(x) \\
&= J_D^m(x)
\end{aligned}$$

and thus

$$J_D\left(x^{-1}\right) \geq J_D(x).$$

Also

$$\begin{aligned}
K_D^n\left(x^{-1}\right) &= S(K_D^n\left(x^{-1}\right),\ K_D^n(e_G)) \\
&\leq S(K_D^n\left(x^{-1}\right),\ K_D^n(e_G)) \\
&= K_D^n(x^{-1}, y^{-1}) \\
&= K_D^n((x, y))^{-1} \\
&\leq K_D^n((x, y)) \\
&= S(K_D^n(x),\ K_D^n(e)) \\
&\leq S(K_D^n(x),\ K_D^n(x)) \\
&= K_D^n(x)
\end{aligned}$$

Thus $K_D^n\left(x^{-1}\right) \leq K_D^n(x)$.

Therefore from (i) and (ii) we have that $D = (J_D,\ K_D) \in BFSG$ w.r.t norms. □

**Definition 4.2.** *Let $D = (J_D,\ K_D) \in BFSG$ w.r.t norms. Then the middle coset $aDb : G \longrightarrow [0, 1]$ is defined by $(aDb)(x) = (aJ_Db\ ,\ aK_Db)(x) = \left(J_D\left(a^{-1}xb^{-1}\right),\ K_D\left(a^{-1}xb^{-1}\right)\right)$ for all $x \in G$ and $a, b \in G$.*

**Proposition 4.2.** *Let $D = (J_D,\ K_D) \in BFSG$ w.r.t norms. Then $aDa^{-1} \in BFSG$ w.r.t norms for any $a \in G$.*

*Proof.* Let $a, x, y \in G$. Then

$$\begin{aligned}\left(aJ_D^m a^{-1}\right)(x*y) &= J_D^m\left(a^{-1}xya\right)\\ &= J_D^m\left(a^{-1}xaa^{-1}ya\right)\\ &\geq T\left(J_D^m a^{-1}xa\right),\ J_D^m(a^{-1}ya))\\ &= T\left(\left(aJ_D^m a^{-1}\right)(x),\ \left(aJ_D^m a^{-1}\right)(y)\right)\end{aligned}$$

Then,

$$\left(aJ_D^m a^{-1}\right)(xy) \geq T\left(aJ_D^m a^{-1}\right)(x),\ \left(aJ_D^m a^{-1}\right)(y)).$$

Also,

$$\begin{aligned}\left(aK_D^n a^{-1}\right)(xy) &= K_D^n\left(a^{-1}xya\right)\\ &= K_D^n\left(a^{-1}xaa^{-1}ya\right)\\ &\leq S(K_D^n\left(a^{-1}xa\right),\ K_D^n(a^{-1}ya))\\ &= S(\left(aK_D^n a^{-1}\right)(x),\ (a\ K_D^n a^{-1})(y))\end{aligned}$$

Then

$$\left(aK_D^n a^{-1}\right)(xy) \leq\ S(\left(aK_D^n a^{-1}\right)(x),\ (a\ K_D^n a^{-1})(y))$$

and

$$\begin{aligned}\left(aJ_D^m a^{-1}\right)\left(x^{-1}\right) &= J_D^m\left(a^{-1}x^{-1}a\right)\\ &= J_D^m\left(\left(a^{-1}xa\right)^{-1}\right)\\ &\geq J_D^m\ (a^{-1}xa)\\ &= \left(aJ_D^m a^{-1}\right)(x)\end{aligned}$$

Thus

$$\left(aJ_D^m a^{-1}\right)\left(x^{-1}\right) \geq\ \left(aJ_D^m a^{-1}\right)(x).$$

Moreover,

$$\begin{aligned}\left(aK_D^n a^{-1}\right)\left(x^{-1}\right) &= K_D^n\left(a^{-1}x^{-1}a\right)\\ &= K_D^n\left(\left(a^{-1}xa\right)^{-1}\right)\\ &\leq K_D^n\ (a^{-1}xa)\\ &= \left(aK_D^n a^{-1}\right)(x)\end{aligned}$$

Thus

$$\left(aK_D^n a^{-1}\right)\left(x^{-1}\right) \leq\ \left(aK_D^n a^{-1}\right)(x).$$

Thus $aDa^{-1} \in BFSG$ w.r.t norms for any $a \in G$. □

**Definition 4.3.** *Let $D = (J_D,\ K_D) \in BFSG$. Then coset $aD = (aJ_D\ ,\ aK_D) : G \longrightarrow [0,1]$ is defined by $(aD)(x) = (aJ_D,\ aK_D)(x) = J_D^m\left(a^{-1}x\right),\ K_D^n\left(a^{-1}x\right) =\ D\left(a^{-1}x\right)$ for all $x \in G$ and $a \in G$.*

**Proposition 4.3.** *Let $D = (J_D,\ K_D) \in BFSG$ and $T, S$ be idempotent norms. Then $xD = yD$ if and only if $D\left(x^{-1}y\right) =\ D\left(y^{-1}x\right) = D\left(e_G\right)$, for all $x, y \in G$.*

*Proof.* Let $x, y, g \in G$.
If $xD = yD$, then $xD(x) = yD(x)$, $D\left(x^{-1}x\right) = D\left(y^{-1}x\right)$ and so $D\left(e_G\right) = D\left(y^{-1}x\right)$.
Also as $xD = yD$ so $xD(y) = yD(y)$ and then $\left(x^{-1}y\right) = D(y^{-1}x)$.

Then $D\left(x^{-1}y\right) = D\left(e_G\right)$. Therefore $D\left(x^{-1}y\right) = D\left(y^{-1}x\right) = D\left(e_G\right)$. Conversely, let $D\left(x^{-1}y\right) = D\left(y^{-1}x\right) = D\left(e_G\right)$. Then

$$\begin{aligned}
xJ_D^m\left(g\right) &= J_D^m(x^{-1}g) \\
&= J_D^m\left(x^{-1}yy^{-1}g\right) \\
&\geq T(J_D^m\left(x^{-1}y\right),\ J_D^m\left(y^{-1}g\right)) \\
&= T\left(J_D^m\left(e_G\right),\ J_D^m\left(y^{-1}g\right)\right) \\
&\geq T(J_D^m\left(y^{-1}g\right)\ ,\ J_D^m\left(y^{-1}g\right)) \\
&= J_D^m\left(y^{-1}g\right) \\
&= yJ_D^m(y) \\
&= J_D^m\left(y^{-1}g\right) \\
&= J_D^m\left(y^{-1}xx^{-1}g\right) \\
&\geq T(J_D^m\left(y^{-1}x\right),\ J_D^m\left(x^{-1}g\right)) \\
&= T(J_D^m\left(e_G\right),\ J_D^m(x^{-1}g) \\
&\geq T(J_D^m\left(x^{-1}g\right),\ J_D^m\left(x^{-1}g\right)) \\
&= J_D^m(x^{-1}g) \\
&= xJ_D^m(g)
\end{aligned}$$

Also,

$$\begin{aligned}
xK_D^n\left(g\right) &= K_D^n(x^{-1} \\
&= K_D^n\left(x^{-1}yy^{-1}g\right) \\
&\geq S(J_D^m\left(x^{-1}y\right),\ J_D^m\left(y^{-1}g\right)) \\
&= S\left(J_D^m\left(e_G\right),\ J_D^m\left(y^{-1}g\right)\right) \\
&\leq S(J_D^m\left(y^{-1}g\right)\ ,\ J_D^m\left(y^{-1}g\right)) \\
&= K_D^n\left(y^{-1}g\right) \\
&= yK_D^n(y) \\
&= K_D^n\left(y^{-1}g\right) \\
&= K_D^n\left(y^{-1}xx^{-1}g\right) \\
&\leq S(K_D^n\left(y^{-1}x\right),\ K_D^n\left(x^{-1}g\right)) \\
&= S(K_D^n\left(e_G\right),\ K_D^n(x^{-1}g) \\
&\leq S(K_D^n\left(x^{-1}g\right),\ K_D^n\left(x^{-1}g\right)) \\
&= K_D^n(x^{-1}g) \\
&= xK_D^n(g)
\end{aligned}$$

Therefore

$$\begin{aligned}
xD\left(g\right) &= \left(xJ_D\left(g\right),\ xJ_D\left(g\right)\right) = \left(yJ_D\left(g\right),\ yK_D\left(g\right)\right) \\
&= yD\left(g\right),
\end{aligned}$$

Then

$$xD = yD.$$

Also let $x, y, g \in G$.
If $xD = yD$, then $xD(x) = yD(x)$,.
Then $D\left(x^{-1}x\right) = D\left(y^{-1}x\right)$ and so $D(e_G) = D\left(y^{-1}x\right)$.
Also as $xD = yD$ so $xD(y) = yD(y)$
and then $D\left(x^{-1}y\right) = D(y^{-1}x)$.
Then $D\left(x^{-1}y\right) = D(e_G)$.
Therefore $D\left(x^{-1}y\right) = D\left(y^{-1}x\right) = D(e_G)$. □

**Proposition 4.4.** *Let $D = (J_D, K_D) \in BFSG$ w.r.t norms. If $xD = yD$, then $D(x) = D(y)$ for all $x, y \in G$.*

*Proof.* As $xD = yD$ by Proposition (4.3) gives us that $D\left(x^{-1}y\right) = D\left(y^{-1}x\right) = D(e_G)$ for all $x, y \in G$. Now,

$$\begin{aligned}
J_D^m(x) =& J_D^m\left(yy^{-1}x\right)\\
&\geq T\left(J_D^m(y),\ J_D^m\left(y^{-1}\right)\right)\\
&= T(J_D^m(y),\ J_D^m(e_G))\\
&\geq T(J_D^m(y),\ J_D^m(y))\\
&= J_D^m(y)\\
&= J_D^m\left(xx^{-1}y\right)\\
&\geq T(J_D^m(x),\ J_D^m\left(x^{-1}\right))\\
&= T(J_D^m(x),\ J_D^m(e_G))\\
&\geq T(J_D^m(x),\ J_D^m(x))\\
&= J_D^m(e)
\end{aligned}$$

we get that

$$J_D^m(x) = J_D^m(y).$$

Also,

$$\begin{aligned}
K_D^n(x) &= K_D^n\left(yy^{-1}x\right)\\
&\leq S\left(K_D^n(y),\ K^n\left(y^{-1}\right)\right)\\
&= S(K_D^n(y),\ K_D^n(e_G))\\
&\leq T(K_D^n(y),\ K_D^n(y))\\
&= K_D^n(y)\\
&= K_D^n\left(xx^{-1}y\right)\\
&\leq S(K_D^n(x),\ K_D^n\left(x^{-1}\right))\\
&= S(K_D^n(x),\ K_D^n(e_G))\\
&\leq T(K_D^n(x),\ K_D^n(x))\\
&= K_D^n(e)
\end{aligned}$$

we get that

$$D(x) = (J_D^m(x),\ K_D^n\ (x)) = (J_D^m(y),\ K_D^n(y)) = D(y).$$

□

**Definition 4.4.** *We say that* $D = (J_D\ , K_D) \in BFSG$ *w.r.t norm is a normal if* $J_D^m\left(xyx^{-1}\right) = J_D^m(y)$ *and* $K_D^n\left(xyx^{-1}\right) = K_D^n(y)$ *for all* $x, y \in G$. *we denote by BFNSG the set of all normal Beal's fuzzy subgroup of* $G$ *w.r.t norms.*

**Proposition 4.5.** *If* $D = (J_D\ , K_D) \in BFNSG$, *then the set* $\frac{G}{D} = \{xD/x \in G\}$ *is a group with the operation* $(xD)(yD) = (xy)D$..

*Proof.* The result is straight forward. □

**Remark 4.1.** *(i) If* $x, y \in G$, *then* $xy \in G$.
*If* $xD, yD \in \frac{G}{A}$, *then* $(xD)(yD) = (xy)D \in \frac{G}{D}$ *(Closure Law)*
*(ii) Let* $x, y, z \in G$. *Then* $x(yz) = (xy)z$..
*Now let* $, yD, zD \in \frac{G}{D}$. *So*

$$\begin{aligned}(xD)[(yD)(zD)] &= (xD)(yzD)\\ &= (xyz)D\\ &= [(xyD)](zD)\\ &= [(xD)(yD)](zD) \qquad \text{(Associative Law)}\end{aligned}$$

*(iii) Let* $x \in G$. *Then* $xe_G = e_Gx = x$.
*Thus* $(xD)(e_G) = (xe_{_G}D) = (e_{_G}xD) = xD$ *(Identity Law)*
*(iv) If* $x \in G$, *then there is an element* $x^{-1} \in G$ *such that* $xx^{-1} = x^{-1}x = e_G$. *If* $xD \in \frac{G}{D}$, *then there is an element* $(xD)^{-1} = x^{-1}D \in \frac{G}{D}$ *such that*

$$(xD)(xD)^{-1} = (xD)\left(x^{-1}D\right) = \left(xx^{-1}D\right) = \left(x^{-1}xD\right) = e_GD = D. \qquad \text{(Inverse Law)}$$

*Hence* $G/D$ *is a group.*

**Proposition 4.6.** *Let* $\phi : G \longrightarrow H$ *be a homomorphism of group and let* $A = (J_A, K_A\ ) \in BFNSG(H)$ *and* $D = (J_D,\ K_D) \in BFNSG(G)$ *be a homomorphic pre-image of A. Then* $\chi : \frac{G}{D} \longrightarrow \frac{H}{A}$ *such that* $\chi(xD) = \phi(x)A$, *for every* $x \in G$, *is an isomorphism of groups.*

*Proof.* Firstly, we prove that $\chi$ is a group homomorphism. Let $x, y \in G$. Then

$$\begin{aligned}\chi((xD)(yD)) = \chi((xy)D) &= \phi(xy)A \qquad \text{(Since } \phi \text{ is a homomorphism)}\\ &= \phi(x)\phi(y)A\\ &= \phi(x)A\ \phi(y)A\\ &= \chi(xD)\chi(yD)\end{aligned}$$

and so $\chi$ is a group homomorphism.
Clearly, $\chi$ is onto and we prove that $\chi$ is 1-1. If $\chi(\ xD) = \chi(yD)$, then $\phi(x)A = \phi(y)A$
from this, we clearly know that $A(\phi(x)^{-1}\phi(y) = A(\phi\left(y^{-1}\phi(x)\right) = A(\phi(e_G))$
and so $A(\phi(x)^{-1}\phi(y) = A(\phi\left(y^{-1}\phi(y)\right) = A(\phi(e_G))$
and then $A(\phi(x^{-1}y)) = A(\phi\left(y^{-1}x\right) = A(\phi(e_G))$
which implies that $D\left(x^{-1}y\right) = D\left(y^{-1}x\right) = D(e_G)$ and thus by Proposition 4.1 gives that $xD = yD$ which implies that $\chi$ is an isomomorphism of group. □

## Conclusion

Here, we studied the new kind of Beal's Fuzzy Subgroups with respect to norms.Beals fuzzy subgroup structure with min-max norms have important application in artifical integellience, decision making system, pattern recoginition and uncertainity modelling, where vague and imprecise information must be handled efficency. Beals fuzzy subgroup

also applicable in various mathematical structures especially in quantum structures with combination of hilbert algebras and hope algebra.

**Funding Declaration.** This manuscript not supported by any Funding agency.

**Future work.** One can obtain the various structures like fuzzy set such as soft set, picture fuzzy set and Neutrosophic set of algebric structures. Future research direction include integrating beals fuzzy subgroup models with machine learning, deep learning, and optimization techniques to improve intelligent system, cyber security, data classification methods under uncertain environment.

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