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DISTANCE-DEGREE BASED ASSESSMENT OF (k, ℓ)-ANONYMITY IN EVEN REGULAR CIRCULANT NETWORKS AND ITS APPLICATIONS TO TOPOLOGICAL INDICES

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Abstract. This paper presents a graph-theoretic framework to evaluate (k, ℓ)-anonymity in circulant-like social networks using the concept of distance-degree. The (k, ℓ)-anonymity model protects against active attacks by ensuring that at least k nodes remain indistinguishable based on their distance profiles. We identify k-antimetric generators and compute k-metric antidimensions using τ-distance neighborhoods and their degrees, forming equivalence classes that define privacy levels. Exact conditions for optimal anonymity in even regular circulant networks $C_n(1, 2, \ldots, q)$ are established. Moreover, the framework extends naturally to compute key topological indices: the Wiener index and the distance eccentric connectivity index. Our findings reveal that even a single well-chosen node can provide strong anonymity guarantees in these networks.

Keywords: (k, ℓ)-anonymity; k-antimetric generator (k-AMG); k-metric antidimension; distance-degree; circulant networks; structural privacy; Wiener index; distance eccentric connectivity index.

AMS Subject Classification: 05C12, 68R10, 05C90, 05C92.

1. Introduction

The rise of large-scale social networks has introduced significant concerns regarding user privacy, especially in the face of *active attacks*, where adversaries can infiltrate networks by inserting or manipulating nodes. One of the most effective structural measures for evaluating and safeguarding privacy under such threats is the (k, ℓ)-anonymity model. Introduced by Trujillo-Rasua and Yero [15], this model guarantees that any attacker with up to ℓ controlled nodes cannot uniquely identify any individual node among fewer than k candidates.

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The theoretical foundation for this privacy measure lies in two graph-theoretic concepts: the *k-antimetric generator* (k-AMG) and the *k-metric antidimension* of a graph. A k-AMG is a subset of nodes whose distance-based representations (codes) leave at least k nodes indistinguishable, while the k-metric antidimension refers to the minimum cardinality of such a set. Together, these parameters yield a privacy profile (k, ℓ), where k is the privacy threshold and ℓ bounds the size of the smallest identifying subset.

This topic has seen growing interest across a variety of graph classes, such as complete bipartite graphs, cycles, trees, wheel-related graphs, generalized Petersen graphs, and Hamming graphs [4, 6, 9, 14, 15, 16]. These studies have demonstrated that in many real-world and synthetic networks, strong anonymity guarantees (i.e., large k with small ℓ are rare). For instance, most social networks achieve only minimal privacy such as $(1, 1)$-anonymity [10, 11, 16], which is considered insufficient in practice. Therefore, it remains a crucial goal to identify graph classes that can sustain higher levels of anonymity, especially using new structural methods.

In this work, we introduce a novel and systematic approach for evaluating the (k, ℓ)-anonymity of social networks using the concept of *distance-degree*, a local structural metric that captures the distribution of distances from a node. Specifically, we investigate (k, ℓ)-anonymity in the context of *even regular circulant networks*, a class of highly symmetric and scalable topologies that appear in distributed systems, wireless networks, and memory architectures. Instead of identifying k-AMGs directly through brute-force or combinatorial methods, we propose an analytical method that uses τ-distance neighborhoods and their cardinalities (distance-degrees) to determine equivalence classes and privacy guarantees.

Beyond privacy, we also demonstrate in the Appendix that the concept of distance-degree extends naturally to evaluating two fundamental topological indices from chemical graph theory and network science, the *Wiener index* and the *distance eccentric connectivity index*. These indices quantify network compactness and eccentricity-weighted connectivity, and can be computed efficiently using our proposed framework.

Our contributions can be summarized as follows:

- We develop a distance-degree based framework for identifying k-AMGs and computing k-metric antidimensions.
- We apply this method to even regular circulant networks ($C_n(1, 2, \ldots, q)$) and derive exact conditions for their (k, ℓ)-anonymity.
- We show that for certain values of n, a singleton node suffices to protect the entire network.
- We extend the utility of distance-degree to compute the Wiener index and distance eccentric connectivity index of circulant networks.

2. Preliminaries

Let $\mathcal{N} = (V(\mathcal{N}), E(\mathcal{N}))$ be a simple, undirected, and connected network of order $n = |\mathcal{N}|$. Two nodes in $\mathcal{N}$ are said to be neighbors if they share an edge. The number of neighbors of a node u is called its degree, denoted by $d(u)$. If every node in $\mathcal{N}$ has the same degree, then $\mathcal{N}$ is said to be a *regular network*.

A *path* is a walk with no repeated nodes, and a *shortest path* between two nodes is referred to as a *geodesic*. The number of edges in a path is called its *length*, and the length of a geodesic between nodes u and v is the *distance* between them, denoted $d(u, v)$. The *eccentricity* of a node u, denoted $\mathrm{ecc}(u)$, is the maximum distance from u to any other node in $\mathcal{N}$. If a node v lies at a distance equal to $\mathrm{ecc}(u)$, then v is called an *eccentric vertex* of u [3].

Let $S = \{x_1, x_2, \ldots, x_t\} \subseteq V(\mathcal{N})$ be an ordered set. The *code* of a node $x \in V(\mathcal{N})$ with respect to S is the vector

$$c_S(x) = (d(x, x_1), d(x, x_2), \ldots, d(x, x_t))$$

as introduced in [12, 13]. The following definitions, introduced by Trujillo-Rasua and Yero [15], formalize key concepts of structural anonymity.

- A set $S \subseteq V(\mathcal{N})$ is called a *k-antimetric generator* (k-AMG) of $\mathcal{N}$ if every node $y \in V(\mathcal{N}) \setminus S$ shares its code $c_S(y)$ with at least $k-1$ other distinct nodes in $V(\mathcal{N}) \setminus S$. That is, for each such y, there exist distinct nodes $y_1, y_2, \ldots, y_{k-1} \in V(\mathcal{N}) \setminus S$ such that

$$c_S(y) = c_S(y_1) = \cdots = c_S(y_{k-1}).$$

- The *k-metric antidimension* of $\mathcal{N}$, denoted $\mathrm{adim}_k(\mathcal{N})$, is the minimum cardinality of a k-AMG in $\mathcal{N}$.
- The largest integer k for which $\mathcal{N}$ admits a k-AMG is called the *metric antidimension* of $\mathcal{N}$, and such a network is said to be *k-metric antidimensional*.
- A network $\mathcal{N}$ satisfies *(k,ℓ)-anonymity* with respect to active attacks if ℓ is an upper bound on the size of the smallest k-AMG. Equivalently, $\mathcal{N}$ satisfies (k,ℓ)-anonymity if $\mathrm{adim}_k(\mathcal{N}) \leq \ell$.
- For a given $S \subseteq V(\mathcal{N})$, define a relation ρ_S on $V(\mathcal{N}) \setminus S$ by,

$$x \ \rho_S \ y \iff c_S(x) = c_S(y), \quad \forall x, y \in V(\mathcal{N}) \setminus S. \tag{1}$$

  This relation is an equivalence relation and partitions $V(\mathcal{N}) \setminus S$ into equivalence classes $S_1, S_2, \ldots, S_m$.

Remark 2.1. *A set $S \subseteq V(\mathcal{N})$ is a k-AMG if and only if $\min_{1 \leq i \leq m} |S_i| = k$, where each S_i is an equivalence class under the relation ρ_S* [9].

3. Methodology: Distance-Degree Approach for (k,ℓ)-Anonymity

To assess the (k,ℓ)-anonymity of circulant-like social networks, we adopt a two-fold strategy grounded in the graph-theoretic notions of anti-metric generators and distance-degree. Our approach departs from direct construction of k-AMGs and instead utilizes τ-distance neighborhoods and their corresponding degrees to derive privacy-relevant node partitions.

3.1. Step 1: Interpreting (k,ℓ)-Anonymity via k-AMG and k-Metric Antidimension.

- A k-anti-metric generator (or k-AMG) of a network $\mathcal{N}$ may exist for multiple values of $k \in \mathbb{Z}^+$. Each such k-AMG induces a corresponding k-metric antidimension, denoted $\mathrm{adim}_k(\mathcal{N}) = \ell$, which counts the number of equivalence classes in the node code partition induced by the set.
- The pair (k,ℓ) thus encapsulates the privacy configuration of the network. For example, if $\mathcal{N}$ satisfies $(4,2)$-anonymity, then there exists a 4-AMG of $\mathcal{N}$ for which $\mathrm{adim}_4(\mathcal{N}) = 2$, and no such k-AMG exists for any $k > 4$.
- Our primary objective is to identify the largest integer k for which a k-AMG exists, and compute the corresponding ℓ. This ensures the strongest anonymity configuration supported by the network structure.

3.2. **Step 2: Constructing k-AMGs Using Distance-Degree Concepts.** We propose a novel mechanism to identify k-AMGs by leveraging the concepts of τ-distance neighborhood and τ-distance-degree.

(1) Let u be a node in a connected network $\mathcal{N}$ with eccentricity $\varepsilon = \text{ecc}(u)$. For each integer τ satisfying $1 \leq \tau \leq \varepsilon$, define the τ-distance neighborhood of u as

$$N_\tau(u) = \{v \in V(\mathcal{N}) \mid d(u, v) = \tau\}.$$

The cardinality of this neighborhood is the τ-distance-degree of u, denoted by

$$d_\tau(u) = |N_\tau(u)|.$$

(2) For a non-empty set $S = \{s_1, s_2, \ldots, s_q\} \subseteq V(\mathcal{N})$, the *code* of a node $v \in V(\mathcal{N})$ with respect to S is defined as

$$c_S(v) = (d(v, s_1), d(v, s_2), \ldots, d(v, s_q)).$$

Then v lies in the δ_i-distance neighborhood of each s_i, where $\delta_i = d(v, s_i)$ for $1 \leq i \leq q$. The equivalence classes induced by the relation ρ_S are formed by grouping nodes with identical codes.

(3) To illustrate the process, consider a singleton set $S = \{x\}$ in $\mathcal{N}$. Let $\varepsilon = \text{ecc}(x)$, and define the τ-distance neighborhoods as $N_1(x), N_2(x), \ldots, N_\varepsilon(x)$. Each node $y \in N_\tau(x)$ satisfies $c_S(y) = (\tau)$, and thus belongs to the same equivalence class. Therefore, $d_\tau(x)$ gives the size of this class. The smallest such class among all τ-distance neighborhoods determines the k-value of the k-AMG, where

$$k = \min_\tau \{d_\tau(x)\}.$$

Hence, S is a k-AMG by Remark 2.1.

This methodology enables us to systematically compute the k-metric antidimension and assess the (k, ℓ)-anonymity of networks, particularly the even regular circulant networks considered in this study.

4. Main Results on Circulant Networks

4.1. **Circulant Networks.** A *circulant-like social network* is a network in which each node is connected to its neighbors in a symmetric, circular manner. Due to their structural symmetry, *circulant networks* have proven to be highly effective topologies in various domains. These networks offer desirable properties such as *scalability*, *fault tolerance*, and *efficient data transmission*, making them well-suited for applications in *Mobile Ad Hoc Networks (MANETs)*, *Wireless Sensor Networks (WSNs)*, and *cellular networks*. Furthermore, circulant structures serve as the basis for the design of efficient *data alignment networks* in complex memory systems [18].

Formally, a circulant network is defined as follows. For an integer $n \geq 3$, a circulant network is a *Cayley graph* (see [1]) over the additive group $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ under modulo n addition. The vertex set is given by $\{v_i : i \in \mathbb{Z}_n\}$, and the edge set is defined by

$$E = \left\{v_i v_{(i+s) \bmod n} : 1 \leq i \leq n,\ s \in S\right\},$$

where $S \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$ is known as the *generating set*. If S is closed under additive inverses, i.e., $s \in S \Rightarrow -s \in S$, then the resulting graph is *undirected*; otherwise, it is *directed*.

An undirected circulant network is denoted by $C_n(a_1, a_2, \ldots, a_m)$, where each generator satisfies $1 \leq a_i \leq \lfloor n/2 \rfloor$. The corresponding edge set is

$$E = \left\{v_i v_{i+a_j \bmod n} : 1 \leq i \leq n,\ 1 \leq j \leq m\right\}.$$

A circulant network is called *even regular* if $a_i \neq n/2$ for all generators a_i; otherwise, it is referred to as *odd regular* [7] (a 4-regular circulant network is depicted in Figure 1). In this paper, we focus on even regular circulant networks of the form $C_n(1, 2, \ldots, q)$, where $1 \leq q < \lfloor \frac{n}{2} \rfloor$.

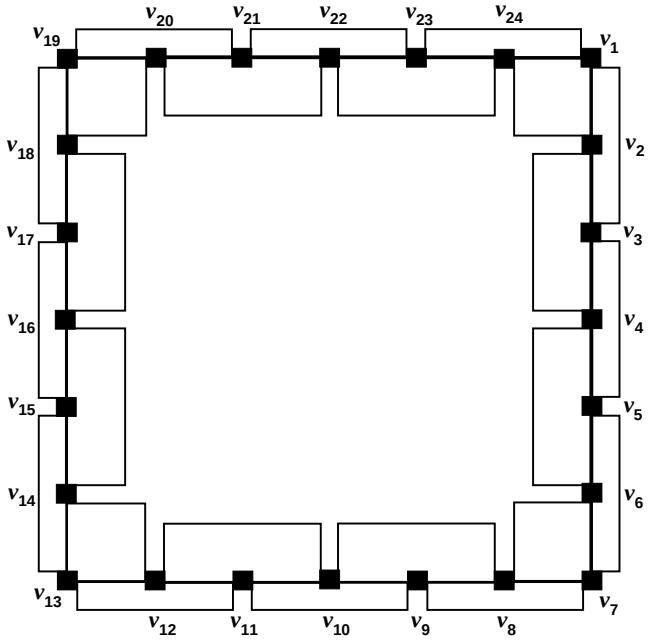


FIGURE 1. A 4-regular circulant network $C_{24}(1, 2)$

4.2. **Results for Even Regular Circulant Networks.** This section presents the main theoretical results for circulant-like social networks with even regularity. We analyze the distance-degree distribution and its implications for determining k-antimetric generators, which leads to the evaluation of (k, ℓ)-anonymity.

Theorem 4.1. *Let $n \geq 3$ and $\mathcal{N} = C_n(1, 2, 3, \ldots, q)$ be an even regular circulant network, where $1 \leq q < \lfloor \frac{n}{2} \rfloor$. For any fixed node v_i $(1 \leq i \leq n)$, the τ-distance-degree $d_\tau(v_i)$ is given by*

$$d_\tau(v_i) = \begin{cases} 2q, & \text{if } 1 \leq \tau < \operatorname{ecc}(v_i), \\ 2q-1, & \text{if } n \equiv 0 \pmod{2q} \text{ and } \tau = \operatorname{ecc}(v_i), \\ 2q, & \text{if } n \equiv 1 \pmod{2q} \text{ and } \tau = \operatorname{ecc}(v_i), \\ d-1, & \text{if } n \equiv d \pmod{2q},\ 2 \leq d \leq 2q-1,\ \tau = \operatorname{ecc}(v_i). \end{cases}$$

Proof. We first determine the eccentricity $\operatorname{ecc}(v_i)$ of a node v_i. Due to the symmetry of $\mathcal{N}$, geodesics from v_i to its eccentric vertex can be traced using the q-th generator:

- If $n \equiv d = 0, 1 \pmod{2q}$, then
$$\operatorname{ecc}(v_i) = \frac{n-d}{2q}.$$
- If $n \equiv d$ for $2 \leq d \leq 2q-1$, then
$$\operatorname{ecc}(v_i) = \frac{n-d}{2q} + 1.$$

In each case, v_i has $2q$ neighbors: q in the clockwise and q in the counter-clockwise direction.

Case 1: $1 \leq \tau < \text{ecc}(v_i)$. Geodesics of length τ reach unique nodes, so $|N_\tau(v_i)| = 2q$.

Case 2: $\tau = \text{ecc}(v_i)$. The number of distinct end nodes depends on n modulo $2q$.

- If $n \equiv 0$, then $d_\tau(v_i) = 2q - 1$.
- If $n \equiv 1$, then $d_\tau(v_i) = 2q$.
- If $n \equiv d$, $2 \leq d \leq 2q - 1$, then $d_\tau(v_i) = d - 1$.

□

Proposition 4.1. *No subset of nodes in $\mathcal{N} = C_n(1, 2, \ldots, q)$ can be a k-AMG for $k > 2q$.*

Proof. From Theorem 4.1, the maximum number of nodes at any τ-distance is $2q$. Hence, no equivalence class can have cardinality $\geq k$ for $k > 2q$, and thus no k-AMG can exist. □

Proposition 4.2. *Every subset of nodes in $\mathcal{N} = C_n(1, 2, \ldots, q)$ with $m \geq 3$ is a 1-AMG.*

Proof. For any $m \geq 3$, there exists at least one node v in $\mathcal{N}$ whose code with respect to the set S is unique. Hence, the smallest equivalence class has size 1, so S is a 1-AMG. □

Theorem 4.2. *Let $\mathcal{N} = C_n(1, 2, \ldots, q)$ be a $2q$-regular circulant network. Then any singleton set $S = \{v_i\}$ is a k-AMG, where*

$$k = \begin{cases} 2q - 1, & \text{if } n \equiv 0 \pmod{2q}, \\ 2q, & \text{if } n \equiv 1 \pmod{2q}, \\ d - 1, & \text{if } n \equiv d \pmod{2q},\ 2 \leq d \leq 2q - 1. \end{cases}$$

Proof. This follows from Theorem 4.1 and the definition of k-AMG in terms of the smallest equivalence class size derived from τ-distance neighborhoods. □

Theorem 4.3. *If $n \equiv 0 \pmod{2q}$, then $\mathcal{N} = C_n(1, 2, \ldots, q)$ satisfies $(2q-1, 1)$-anonymity.*

Proof. By Theorem 4.2, every singleton set is a $(2q - 1)$-AMG, and its size is 1, the minimum possible. Hence, $\text{adim}_{2q-1}(\mathcal{N}) = 1$. □

Theorem 4.4. *If $n \equiv 1 \pmod{2q}$, then $\mathcal{N} = C_n(1, 2, \ldots, q)$ satisfies $(2q, 1)$-anonymity.*

Proof. From Theorem 4.2, each singleton set is a $2q$-AMG. As no smaller subset can achieve this property, $\text{adim}_{2q}(\mathcal{N}) = 1$. □

Theorem 4.5. *If $n \equiv 2 \pmod{2q}$, then $\mathcal{N} = C_n(1, 2, \ldots, q)$ satisfies $(2q, 2)$-anonymity.*

Proof. Let $S = \{v_i, v_{i+n/2}\}$ be a two-node set. From Theorem 4.1, $\text{ecc}(v_i) = \frac{n-2}{2q} + 1$. The pairwise τ-distance neighborhoods overlap symmetrically, yielding $2q$-sized equivalence classes. Hence, S is a $2q$-AMG, and since singleton sets provide only 1-AMG, we conclude $\text{adim}_{2q}(\mathcal{N}) = 2$. □

Theorem 4.6. *Let $n \equiv d \pmod{2q}$ for $3 \leq d \leq 2q-1$. Then $\mathcal{N} = C_n(1, 2, \ldots, q)$ satisfies $(d - 1, 1)$-anonymity.*

Proof. From Theorem 4.2, singleton sets provide $(d - 1)$-AMGs. All sets of size ≥ 3 are 1-AMGs (Proposition 4.2). A pairwise analysis shows that no set of size two consistently yields larger equivalence classes than size one. Thus, $\text{adim}_{d-1}(\mathcal{N}) = 1$. □

Summary Table of (k, ℓ)-Anonymity.

$n \bmod 2q$	k (Maximum k for k-AMG)	$\ell = \text{adim}_k(\mathcal{N})$
0	$2q - 1$	1
1	$2q$	1
2	$2q$	2
$3 \leq d \leq 2q - 1$	$d - 1$	1

5. Conclusion

In the context of safeguarding social networks against active attacks, the (k, ℓ)-anonymity model serves as a critical structural privacy measure. This paper proposed a novel analytical framework based on the concept of *distance-degree* to evaluate the (k, ℓ)-anonymity of even regular circulant networks. Rather than relying on combinatorial identification of k-antimetric generators (k-AMGs), we introduced a more structured and theoretically grounded approach that uses the τ-distance neighborhoods and their cardinalities to derive equivalence classes and determine anonymity parameters.

Our findings demonstrate that

- The maximum τ-distance-degree of any node in the considered circulant networks equals the network's regularity.
- The largest possible value of k for which a k-AMG exists in such networks is bounded above by the degree $2q$, where q is the number of generators.
- All node subsets of cardinality greater than two are inherently 1-AMGs, reflecting limited privacy gains from increasing set size.
- Singleton sets offer a range of k-values depending on the parity of the network size, and in many configurations, a single well-chosen node suffices to provide optimal (k, ℓ)-anonymity.
- We characterized precise (k, ℓ)-anonymity configurations for circulant networks based on the congruence of n modulo $2q$, leading to results such as:

$n \bmod 2q$	0	1	2	$3 \leq d \leq 2q-1$
k	$2q-1$	$2q$	$2q$	$d-1$
$\ell = \operatorname{adim}_k(\mathcal{N})$	1	1	2	1

Finally, we highlighted the versatility of the distance-degree concept by applying it to compute two significant topological indices in the Appendix, the Wiener index and the distance eccentric connectivity index, both of which play key roles in network topology and molecular graph analysis.

This study not only strengthens the theoretical underpinnings of privacy in circulant-like networks but also introduces scalable tools for structural analysis. Future work may extend these techniques to broader classes of graphs, including directed circulants, Cayley graphs on non-cyclic groups, and dynamically evolving networks under adversarial influence.

Appendix: Application of Distance-Degree to Topological Indices

In addition to its utility for assessing (k, ℓ)-anonymity, the concept of *distance-degree* plays a key role in computing two important connectivity-based topological indices: the **Wiener index** and the **distance eccentric connectivity index**. These indices are widely used in chemical graph theory, social network analysis, and complex systems.

1. Wiener Index. The Wiener index, originally introduced by Wiener in 1947 [17], is defined as the sum of shortest-path distances between all pairs of vertices in a graph. Mathematically, the formula is

$$W(\mathcal{N}) = \sum_{\{u,v\} \subseteq V(\mathcal{N})} d(u, v).$$

This index reflects the compactness and average distance in a network and is useful for predicting physical properties of chemical compounds as well as analyzing communication and social networks. In 1947, Wiener introduced a graph-theoretic concept *Wiener index* to predict the boiling points of alkanes. Since then, the Wiener index has become one of the

most prominent and widely studied *topological indices* in mathematical chemistry. Beyond its original application, the Wiener index serves as a key measure of a graph's structural compactness, as it directly relates to the *average shortest-path distance* between pairs of nodes.

The average distance, alongside the *clustering coefficient* and *degree distribution*, is considered one of the three most robust descriptors of network topology. Today, the Wiener index finds extensive use in fields such as *sociometry* and the *theory of social networks* [5], where it helps quantify the cohesion and reachability within a network. Additional applications of this invariant extend to *crystallography*, *communication theory*, and *facility location problems* [8], making it a versatile and powerful tool in both theoretical and applied network analysis.

The Wiener index can also be expressed using the **transmission** (or total distance) of each node $v \in V(\mathcal{N})$, denoted $D(v|\mathcal{N})$, as follows:

$$W(\mathcal{N}) = \frac{1}{2} \sum_{v \in V(\mathcal{N})} D(v|\mathcal{N}), \quad \text{where} \quad D(v|\mathcal{N}) = \sum_{u \in V(\mathcal{N})} d(v,u).$$

Using the concept of distance-degree, this transmission can be reformulated as

$$D(v|\mathcal{N}) = \sum_{\tau=1}^{\mathrm{ecc}(v)} \tau \cdot d_\tau(v),$$

where $d_\tau(v)$ is the number of nodes at distance τ from v.

For the circulant network $\mathcal{N} = C_n(1,2,\ldots,q)$, the values of $d_\tau(v)$ from Theorem 4.1 yield the following transmission formulas

$$D(v|\mathcal{N}) = \begin{cases} \frac{1}{4q} \cdot n(n+2q-2), & \text{if } n \equiv 0 \pmod{2q}, \\ \frac{1}{4q} \cdot (n-1)(n+2q-1), & \text{if } n \equiv 1 \pmod{2q}, \\ \frac{1}{4q} \cdot (n+2q-d)(n+4q(d-1)-d), & \text{if } n \equiv d,\ 2 \le d \le 2q-1. \end{cases}$$

Thus, the Wiener index becomes

$$W(\mathcal{N}) = \begin{cases} \frac{n}{8q} \cdot n(n+2q-2), & \text{if } n \equiv 0 \pmod{2q}, \\ \frac{n}{8q} \cdot (n-1)(n+2q-1), & \text{if } n \equiv 1 \pmod{2q}, \\ \frac{n}{8q} \cdot (n+2q-d)(n+4q(d-1)-d), & \text{if } n \equiv d,\ 2 \le d \le 2q-1. \end{cases}$$

2. Distance Eccentric Connectivity Index. The *distance eccentric connectivity index*, introduced by Alkqesmah et al. [2], combines local and global graph properties. It is defined by the formula

$$\xi_{De}(\mathcal{N}) = \sum_{u \in V(\mathcal{N})} \deg_{De}(u) \cdot \mathrm{ecc}(u),$$

where $\deg_{De}(u)$ is the *distance eccentricity degree* of u, i.e., the number of nodes at distanceecc(u) from u

$$\deg_{De}(u) = |\mathcal{N}_{\mathrm{ecc}(u)}(u)| = d_{\mathrm{ecc}(u)}(u).$$

By applying Theorem 4.1 to nodes in $\mathcal{N} = C_n(1,2,\ldots,q)$, we obtain

$$\deg_{De}(u) = \begin{cases} 2q-1, & \text{if } n \equiv 0 \pmod{2q}, \\ 2q, & \text{if } n \equiv 1 \pmod{2q}, \\ d-1, & \text{if } n \equiv d,\ 2 \le d \le 2q-1. \end{cases}$$

Combining this with known values of $\mathrm{ecc}(u)$, the index is computed as

$$\xi_{De}(\mathcal{N}) = \begin{cases} \frac{n}{2q} \cdot n(2q-1), & \text{if } n \equiv 0 \pmod{2q}, \\ \frac{n}{2q} \cdot 2q(n-1), & \text{if } n \equiv 1 \pmod{2q}, \\ \frac{n}{2q} \cdot (d-1)(n+2q-d), & \text{if } n \equiv d,\ 2 \leq d \leq 2q-1. \end{cases}$$

These results illustrate how the distance-degree framework enables efficient and exact computation of structurally significant network parameters.

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