TWMS J. App. and Eng. Math. V.16, N.11, 2026, pp. 1373-1384

$\mathbb{P}$-ORDER ANALYSIS OF e-OPEN SETS IN CUBIC PICTURE FUZZY TOPOLOGICAL SPACES

G. SARAVANAKUMAR[1*], K. SUGANTHI[1,2], §

Abstract. In this paper, we introduce and investigate new classes of generalized open sets-namely, e-open, e^*-open, a-open, and β-open sets—within the framework of Cubic Picture Fuzzy Topological Spaces ($CPFTS$'s) under $\mathbb{P}$-order. Our study extends the theory of generalized open sets to the richer structure of $CPFTS$'s, which combine interval-valued and tri-component fuzzy membership functions. We provide formal definitions for these open sets in $CPFTS$'s and explore their fundamental topological properties. We establish a series of characterizations and inclusion relationships among the defined sets, supported by illustrative examples and counterexamples. These results highlight the distinct nature of each open set type and their non-reversible implications under set-theoretic operations. Our approach emphasizes both algebraic and topological aspects, revealing a refined hierarchy of openness in the context of $CPFTS$'s. Furthermore, we analyze the behavior of these sets under $\mathbb{P}$-order, offering insights into how order structures influence topological properties in fuzzy environments. The dual-order framework enables a deeper understanding of neighborhood systems and proximity relations in $CPFTS$. This study fills an important gap in the literature by assimilating generalized open set theory with $CPFTS$'s for the first time. The findings pave the way for future research in fuzzy topology and its applications in decision-making systems, intelligent environments, and uncertainty modeling.

Keywords: Cubic Picture Fuzzy Topological Spaces, e-Open Sets, $\mathbb{P}$-Order.

AMS Subject Classification: 54A40, 03E72, 54A05, 54D99.

1. Introduction

The growing complexity and uncertainty inherent in real-world systems have led to the emergence of advanced mathematical tools capable of handling vagueness, indeterminacy, and imprecision. Among these tools, the theory of fuzzy sets, first introduced by Zadeh [18], has undergone substantial extensions through the development of intuitionistic fuzzy

[1] Department of Mathematics, Vel Tech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology (Deemed to be University), Avadi, Chennai-600062, India.
e-mail: saravananguru2612@gmail.com; ORCID: https://orcid.org/0009-0003-1309-7609.
[2] Department of Mathematics, Asan Memorial college of Engineering and Technology, Keerapakkam, Chengalpattu, India.
e-mail: suganthiselvam0980@gmail.com; ORCID: https://orcid.org/0009-0009-6318-6256.
* Corresponding author.
§ Manuscript received: July 07, 2025; accepted: December 08, 2025.
TWMS Journal of Applied and Engineering Mathematics, Vol.16, No.11; © Işık University, Department of Mathematics, 2026; all rights reserved.

sets[3], picture fuzzy sets [4], and more recently, cubic picture fuzzy sets ($CPFS$'s) which combine the interval-valued structure of cubic sets with the tri-component membership of picture fuzzy sets. Ashraf et al. [2] first proposed $CPFS$'s as a robust extension of fuzzy models, providing enhanced representational capabilities using three degrees of evaluation: positive, neutral, and negative.

Building upon this foundation, Jun et al. [17] and others have developed algebraic operations and structural properties of cubic sets, which were later applied to $CPFS$'s. $CPFS$'s have since found wide relevance in practical domains including decision-making, medical diagnosis, and more recently, intelligent supply chain systems and blockchain-integrated environments [13]. In particular, $CPTS$'s under $\mathbb{P}$-order and $\mathbb{R}$-order frameworks have been introduced to model spatial uncertainty with greater logical expressiveness and flexibility [13]. These topological structures accommodate dual-ordering operations and provide a foundation for investigating open set structures under uncertainty.

Parallel to these developments, the concept of generalized open sets—notably e-open sets-introduced by Ekici [5], has played a significant role in refining continuity, compactness, and related topological properties in classical and fuzzy topological spaces. Ekici's subsequent works introduced further generalizations such as e^*-open sets [9], a-open sets [10], and $(D, S)^*$-sets, along with decompositions of continuity [5, 6, 7]. These generalized open sets form a hierarchy of openness that bridges classical and semi-open sets, and have been studied extensively in fuzzy and intuitionistic fuzzy contexts.Recently, several authors have advanced the study of cubic and picture fuzzy topological structures through the development of cubic intuitionistic fuzzy topologies [14] and cubic m-polar fuzzy topological spaces [16].

Furthermore, topological investigations under cubic hesitant fuzzy environments have been explored in [15], extending the applicability of cubic multi-valued logic.

Studies on picture fuzzy soft topological spaces and their practical implications in medical decision-making appear in [1], highlighting the growing importance of generalized picture fuzzy topologies. Additionally, new forms of generalized open sets in picture fuzzy topological spaces have been introduced in [11], enriching the hierarchy of openness in fuzzy settings. However, despite this extensive exploration, no significant research has extended these generalized open sets-especially e-open sets and their nearby variants-into the richer structure of $CPFTS$'s.

Research Gap. Although $CPFS$'s have seen meaningful application in modeling uncertainty in real-world systems, the integration of generalized open sets such as e-open sets into the $CPFTS$'s setting remains largely unexplored. Previous works have either addressed open set structures in intuitionistic fuzzy or basic fuzzy spaces [5, 10], or investigated topological properties of $CPFS$'s without introducing refined notions of openness. This absence of foundational work on e-open sets and their characterizations in $CPFTS$'s presents a significant gap in the literature.

Motivation and Novelty. Motivated by this gap, the present study is the first to explore e-open sets and their nearby generalizations including e^*-open, a-open, and β-open sets within the framework of $CPFTS$'s under $\mathbb{P}$-order. We introduce new characterizations of these sets and investigate their mutual relationships, topological behaviors, and algebraic properties through a series of theorems and illustrative examples. This $\mathbb{P}$-order exploration enables a richer understanding of proximity and neighborhood-based structures in $CPFTS$'s.

Objective. This study aims to:

- Introduce and formally define e-open sets and their nearby open variants in CPFTSs.
- Characterize these sets under $\mathbb{P}$-order frameworks.

- Explore the relationships between these generalized open sets through set-theoretic operations.
- Illustrate the topological structure and non-reversible implications among the generalized open sets.

Significance. This work advances the theoretical landscape of fuzzy topological spaces by unifying multi-valued logic with interval-valued membership and order-based topologies. It opens new pathways for further research in uncertainty modeling, artificial intelligence, and systems analysis where fuzzy, granular, and non-binary data representations are essential. The insights obtained can also aid in refining decision support systems, smart transport frameworks, and metaverse-based digital environments where neighborhood sensitivity and flexible openness criteria are critical.

2. Preliminaries

Definition 2.1 ([18]). *A fuzzy set λ in a set X is defined to be a function $\lambda : \mathrm{X} \to \mathrm{I}$ where $\mathrm{I} = [0, 1]$.*

Definition 2.2 ([19]). *A closed sub interval $\tilde{a} = [a^-,\ a^+]$ where $0 \leq a^- \leq a^+ \leq 1$ of $\mathrm{I} = [0,\ 1]$ is called Interval number. Where, $[I]$ denote the set of all interval numbers.*

Definition 2.3 ([19]). *Let X be a nonempty set. A function $\mu : X \to [I]$ is called an interval-valued fuzzy set(briefly, $IVFS$) in X. Let $[I]$ stand for the set of all $IVFS$'s in X. For every $\mu \in [I]^X$ and $x \in \mathrm{X}, \mu(x) = [\mu^-(x), \mu^+(x)]$ is called the degree of membership of an element x to μ, where $\mu^- : X \to I$ and $\mu^+ : X \to I$ are fuzzy sets in X which are called lower fuzzy set and an upper fuzzy set in X, respectively.*

Definition 2.4 ([17]). *Let X be a non-empty set. Then a structure $A = \{\langle x, \mu(x), \lambda(x)\rangle \mid x \in X\}$ is a cubic set in X in which μ is an $IVFS$ in X and λ is a fuzzy set in X.*

Definition 2.5 ([4]). *Consider a non-empty set X. We define a picture fuzzy set(PFS) as follows $J(x) = \{\langle(s, \alpha(x), \beta(x), \gamma(x)) \mid x \in X\rangle\}$, where $\alpha : X \to [0, 1], \beta : X \to [0, 1], \gamma : X \to [0, 1]$ represent the positive membership degree, neutral membership degree and negative membership degree of the element x with respect to the set X, respectively. Additionally, α, β and γ must satisfy the condition $0 \leq \alpha(x) + \beta(x) + \gamma(x) \leq 1$ for all $x \in X$.*

Definition 2.6 ([4]). *Consider a non-empty set X. Let $S(x)$ denote an interval valued picture fuzzy set($IVPFS$) defined in X as follows $S(x) = \{\langle x, \alpha(x), \beta(x), \gamma(x) \mid x \in X\}$, where $\alpha : X \to [0, 1], \beta : X \to [0, 1], \gamma : X \to [0, 1]$ represent the positive membership degree, neutral membership degree and negative membership degree of the element x with respect to the set X. Moreover $\alpha(x), \beta(x)$ and $\gamma(x)$ must satisfy the condition $0 \leq sup(\alpha(x)) + sup(\beta(x)) + sup(\gamma(x)) \leq 1$, for all $x \in X$.*

Definition 2.7 ([2]). *Consider a non-empty set X. The concept of a cubic picture fuzzy set($CPFS$) can be represented as follows $A = \{\langle[\alpha_i, \alpha_j], [\beta_i, \beta_j], [\gamma_i, \gamma_j]\rangle, \langle\alpha, \beta, \gamma\rangle\}$, where $\alpha : X \to [0, 1], \beta : X \to [0, 1], \gamma : X \to [0, 1]$ represent the positive membership degree, neutral membership degree and negative membership degree of every element with respect to the set X.*

Definition 2.8 ([2]). *Let us consider a universal set X. Then the $CPFS$ $A = \{\langle[\alpha_i, \alpha_j]$, $[\beta_i, \beta_j]$, $[\gamma_i, \gamma_j]\rangle, \langle\alpha, \beta, \gamma\rangle\}$ is classified as follows*

(i) Positive-internal if $\alpha_i(x) \leq \alpha(x) \leq \alpha_j(x), \forall x \in X$.
(ii) Neutral-internal if $\beta_i(x) \leq \beta(x) \leq \beta_j(x), \forall x \in X$.

(iii) Negative-internal if $\gamma_i(x) \leq \gamma(x) \leq \gamma_j(x), \forall x \in X$.

If the $CPFS$ $A = \{\langle[\alpha_i, \alpha_j], [\beta_i, \beta_j], [\gamma_i, \gamma_j]\rangle, \langle\alpha, \beta, \gamma\rangle\}$ *in* X *is satisfied all the above properties then the* $CPFS$ *is said to be an internal* $CPFS$ *in* X.

Definition 2.9 ([2]). *Let us consider a universal set* X. *Then the* $CPFS$ $A = \{\langle[\alpha_i, \alpha_j], [\beta_i, \beta_j], [\gamma_i, \gamma_j]\rangle, \langle\alpha, \beta, \gamma\rangle\}$ *is classified as follows*

(i) Positive-external if $\alpha(x) \notin [\alpha_i(x), \alpha_j(x)], \forall x \in X$
(ii) Neutral-external if $\beta(x) \notin [\beta_i(x), \beta_j(x)], \forall x \in X$
(iii) Negative-external if $\gamma(x) \notin [\gamma_i(x), \gamma_j(x)], \forall x \in X$

Definition 2.10 ([2]). *Let* X *be a set. Consider a* $CPFS$ $A = \{\langle[\alpha_i, \alpha_j], [\beta_i, \beta_j], [\gamma_i, \gamma_j]\rangle, \langle\alpha, \beta, \gamma\rangle\}$ *in* X. *Then its complement is defined as* $A^c = \{\langle[\gamma_i, \gamma_j], [\beta_i, \beta_j], [\alpha_i, \alpha_j]\rangle, \langle\gamma, \beta, \alpha\rangle\}$

Definition 2.11 ([13]). *Let us consider a universal set* X. *Then the* $CPFS$, $\ddot{0} = \{\langle[0, 0], [0, 0], [1, 1]\rangle, \langle0, 0, 1\rangle\}$ *is called a null P-CPFS.*

Definition 2.12 ([13]). *Let us consider a universal set* X. *Then the* $CPFS$, $\ddot{1} = \{\langle[1, 1], [0, 0], [0, 0]\rangle, \langle1, 0, 0\rangle\}$ *is called a absolute P-CPFS.*

Definition 2.13 ([2]). *Let* $A = \{\langle[\alpha_{i_1}, \alpha_{j_1}], [\beta_{i_1}, \beta_{j_1}], [\gamma_{i_1}, \gamma_{j_1}]\rangle, \langle\alpha_1, \beta_1, \gamma_1\rangle\}$ *and* $B = \{\langle[\alpha_{i_2}, \alpha_{j_2}], [\beta_{i_2}, \beta_{j_2}], [\gamma_{i_2}, \gamma_{j_2}]\rangle, \langle\alpha_2, \beta_2, \gamma_2\rangle\}$ *be two* $CPFS$*'s. Then the operations on* $CPFS$*'s under the* P*-order are defined as follows.*

(i) $A \subseteq_{\mathbb{P}} B$ *iff* $\forall x \in X$, $[\alpha_{i_1}, \alpha_{j_1}] \leq [\alpha_{i_2}, \alpha_{j_2}], [\beta_{i_1}, \beta_{j_1}] \geq [\beta_{i_2}, \beta_{j_2}], [\gamma_{i_1}, \gamma_{j_1}] \geq [\gamma_{i_2}, \gamma_{j_2}]$, $\alpha_1 \leq \alpha_2, \beta_1 \geq \beta_2, \gamma_1 \geq \gamma_2$
(ii) Union of two $CPFS$*'s*

$$A \cup_{\mathbb{P}} B = \left\{ \begin{array}{l} \{max\{\alpha_{i_1}, \alpha_{i_2}\}, min\{\beta_{i_1}, \beta_{i_2}\}, min\{\gamma_{i_1}, \gamma_{i_2}\}\} \\ \{max\{\alpha_{j_1}, \alpha_{j_2}\}, min\{\beta_{j_1}, \beta_{j_2}\}, min\{\gamma_{j_1}, \gamma_{j_2}\}\} \\ \{max\{\alpha_1, \alpha_2\}, min\{\beta_1, \beta_2\}, min\{\gamma_1, \gamma_2\}\} \end{array} \right\}$$

(iii) Intersection of two $CPFS$*'s*

$$A \cap_{\mathbb{P}} B = \left\{ \begin{array}{l} \{min\{\alpha_{i_1}, \alpha_{i_2}\}, min\{\beta_{i_1}, \beta_{i_2}\}, max\{\gamma_{i_1}, \gamma_{i_2}\}\} \\ \{min\{\alpha_{j_1}, \alpha_{j_2}\}, min\{\beta_{j_1}, \beta_{j_2}\}, max\{\gamma_{j_1}, \gamma_{j_2}\}\} \\ \{min\{\alpha_1, \alpha_2\}, min\{\beta_1, \beta_2\}, max\{\gamma_1, \gamma_2\}\} \end{array} \right\}$$

Definition 2.14 ([13]). *A* $\mathbb{P}$*-CPFT is the family* $\mathcal{F}_{\mathbb{P}}$ *of* $CPFS$*'s in* X *which satisfies the following conditions:*

(i) $\ddot{0}, \ddot{1} \in \mathcal{F}_{\mathbb{P}}$
(ii) If $A_{\mathrm{i}} \in \mathcal{F}_{\mathbb{P}}$, *then* $\cup_{\mathbb{P}}^{i \in N} A_i \in \mathcal{F}_{\mathbb{P}}$
(iii) If $\mathrm{A}, \mathrm{B} \in \mathcal{F}_{\mathbb{P}}$, *then* $A \cap_{\mathbb{P}} \mathrm{B} \in \mathcal{F}_{\mathbb{P}}$

The pair $(X, \mathcal{F}_{\mathbb{P}})$ *is called the* $\mathbb{P}$*-CPFTS and any* $CPFS$ *in* $\mathcal{F}_{\mathbb{P}}$ *is known as* $\mathbb{P}$*-cubic picture fuzzy open set(*$CPF_{\mathbb{P}}os$*) in* X. *The complement* A^{c} *of a* $CPF_{\mathbb{P}}os$ A *in* $\mathbb{P}$*-CPFTS* $(X, \mathcal{F}_{\mathbb{P}})$ *is called a* $\mathbb{P}$*-cubic picture fuzzy closed set(*$CPF_{\mathbb{P}}cs$*) in* X.

Definition 2.15 ([13]). *Those cubic sets which are both* $CPF_{\mathbb{P}}os$ *and* $CPF_{\mathbb{P}}cs$ *are called* $\mathbb{P}$*-cubic picture fuzzy clopen sets(*$CPF_{\mathbb{P}}clop.s$*) in* $(X, \mathcal{F}_{\mathbb{P}})$.

Definition 2.16 ([13]). *Let* $(X, \mathcal{F}_{\mathbb{P}})$ *be a* $\mathbb{P}$*-CPFTS and* A *be a* $CPFS$ *in* $(X, \mathcal{F}_{\mathbb{P}})$. *Then the* $\mathbb{P}$ *-CPF interior and* $\mathbb{P}$ *-CPF closure are defined by*

$CPF_{\mathbb{P}}int(A) = \cup_{\mathbb{P}}\{A_i \mid A_i$ *is a* $\mathbb{P}$*-* $CPF_{\mathbb{P}}os$ *in* X *and* $A_i \subseteq_{\mathbb{P}} A\}$
$CPF_{\mathbb{P}}cl(A) = \cap_{\mathbb{P}}\{A_i \mid A_i$ *is a* $\mathbb{P}$*-* $CPF_{\mathbb{P}}cs$ *in* X *and*$A \subseteq_{\mathbb{P}} A_i\}$.

3. ℙ-order e-open sets in Cubic Picture Fuzzy Topological Spaces

Definition 3.1. *A $CPFS$ L in a $\mathbb{P}$ -$CPFTS$ $(X, F_{\mathbb{P}})$ is said to be*

(i) $\mathbb{P}$ *- CPF regular open set(briefly,$CPF_{\mathbb{P}}ros$) if $L = CPF_{\mathbb{P}}int(CPF_{\mathbb{P}}cl(L))$*
(ii) $\mathbb{P}$ *- CPF regular closed set (briefly,$CPF_{\mathbb{P}}rcs$) if $L = CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}int(L))$*

Definition 3.2. *A $CPFS$ L is said to be a $\mathbb{P}$ -CPF*

(i) δ interior of L (briefly, $CPF_{\mathbb{P}}\delta int$) is defined by $CPF_{\mathbb{P}}\delta int = \bigcup\{G : G \subseteq L$ and G is a $CPF_{\mathbb{P}}\delta ros$ in $X\}$.
(ii) δ closure of L (briefly, $CPF_{\mathbb{P}}\delta cl$) is defined by $CPF_{\mathbb{P}}\delta cl = \bigcap\{G : G \supseteq L$ and G is a $CPF_{\mathbb{P}}\delta rcs$ in $X\}$.

Definition 3.3. *A $CPFS$ L is said to be a $\mathbb{P}$ - CPF*

(i) δ-pre open set(briefly, $CPF_{\mathbb{P}}\delta\mathcal{P}os$) if $L \subseteq_{\mathbb{P}} CPF_{\mathbb{P}}int(CPF_{\mathbb{P}}\delta clL)$.
(ii) δ-pre closed set(briefly, $CPF_{\mathbb{P}}\delta\mathcal{P}cs$) if $L \supseteq_{\mathbb{P}} CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}\delta intL)$.
(iii) δ-semi open set(briefly, $CPF_{\mathbb{P}}\delta\mathcal{S}os$) if $L \subseteq_{\mathbb{P}} CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}\delta intL)$.
(iv) δ-semi closed set(briefly, $CPF_{\mathbb{P}}\delta\mathcal{S}cs$) if $L \supseteq_{\mathbb{P}} CPF_{\mathbb{P}}int(CPF_{\mathbb{P}}\delta clL)$.
(v) e-open set (briefly, $CPF_{\mathbb{P}}eos$) if $L \subseteq_{\mathbb{P}} CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}\delta intL) \cup CPF_{\mathbb{P}}int\ (CPF_{\mathbb{P}}\delta clL)$.
(vi) e-closed set (briefly, $CPF_{\mathbb{P}}ecs$)$L \supseteq_{\mathbb{P}} CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}\delta intL) \cap CPF_{\mathbb{P}}int\ (CPF_{\mathbb{P}}\delta clL)$.
(vii) e^*-open set(briefly, $CPF_{\mathbb{P}}e^*os$) if $L \subseteq_{\mathbb{P}} CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}int(CPF_{\mathbb{P}}\delta clL))$.
(viii) e^*-closed set(briefly, $CPF_{\mathbb{P}}e^*cs$) if $L \supseteq_{\mathbb{P}} CPF_{\mathbb{P}}int(CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}\delta intL))$.
(ix) a-open set (briefly, $CPF_{\mathbb{P}}aos$) if $L \subseteq_{\mathbb{P}} CPF_{\mathbb{P}}int(CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}\delta intL))$.
(x) a-closed set (briefly, $CPF_{\mathbb{P}}acs$) if $L \supseteq_{\mathbb{P}} CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}int(CPF_{\mathbb{P}}\delta clL))$.
(xi) β open set (briefly, $CPF_{\mathbb{P}}\beta os$) if $L \subseteq_{\mathbb{P}} CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}int(CPF_{\mathbb{P}}cl(L)))$
(xii) β closed set (briefly, $CPF_{\mathbb{P}}\beta cs$) if $L \supseteq_{\mathbb{P}} CPF_{\mathbb{P}}int(CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}int(L)))$

The family of all $CPF_{\mathbb{P}}\delta\mathcal{P}os$ (resp. $CPF_{\mathbb{P}}\delta\mathcal{P}cs$, $CPF_{\mathbb{P}}\delta\mathcal{S}os$, $CPF_{\mathbb{P}}\delta\mathcal{S}cs$, $CPF_{\mathbb{P}}eos$, $CPF_{\mathbb{P}}ecs$, $CPF_{\mathbb{P}}aos$,$CPF_{\mathbb{P}}acs$, $CPF_{\mathbb{P}}\beta os$,$CPF_{\mathbb{P}}\beta cs$, $CPF_{\mathbb{P}}e^*os$, & $CPF_{\mathbb{P}}e^*cs$) of X is denoted by $CPF_{\mathbb{P}}\delta\mathcal{P}\ OS(X)$ (resp. $CPF_{\mathbb{P}}\delta\mathcal{P}CS(X)$, $CPF_{\mathbb{P}}\delta\mathcal{S}OS(X)$, $CPF_{\mathbb{P}}\delta\mathcal{S}CS(X)$, $CPF_{\mathbb{P}}eOS(X)$, $CPF_{\mathbb{P}}eCS(X)$, $CPF_{\mathbb{P}}aOS(X)$, $CPF_{\mathbb{P}}aCS(X)$,$CPF_{\mathbb{P}}\beta OS(X)$, $CPF_{\mathbb{P}}\beta$ $CS(X)$, $CPF_{\mathbb{P}}e^*OS(X)$ & $CPF_{\mathbb{P}}e^*CS(X)$).

Definition 3.4. *A $CPFS$ L is said to be a $\mathbb{P}$-CPF*

(i) e-interior(resp. e^*-interior) of L is defined by $CPF_{\mathbb{P}}eintL$(resp. $CPF_{\mathbb{P}}e^*intL$) = $\bigcup_{\mathbb{P}}\{G : G \subseteq L$ & G is a $CPF_{\mathbb{P}}eos$(resp. $CPF_{\mathbb{P}}e^*os$)in$X\}$.
(ii) e-closure (resp. e^*-closure) of L is defined by $CPF_{\mathbb{P}}eclL$(resp. $CPF_{\mathbb{P}}e^*clL$) = $\bigcap_{\mathbb{P}}\{G : L \subseteq G$ & G is a $CPF_{\mathbb{P}}ecs$(resp. $CPF_{\mathbb{P}}e^*cs$)in$X\}$.
(iii) δ pre interior(resp. δ semi interior) of L (briefly, $CPF_{\mathbb{P}}\delta\mathcal{P}int$) (resp. $CPF_{\mathbb{P}}\delta\mathcal{S}int$) is defined by $CPF_{\mathbb{P}}\delta\mathcal{P}int$(resp. $CPF_{\mathbb{P}}\delta\mathcal{S}int$) = $\bigcup_{\mathbb{P}}\{G : G \subseteq L$ & G is a $CPF_{\mathbb{P}}\delta\mathcal{P}os$ (resp.$CPF_{\mathbb{P}}\delta\mathcal{S}os$) in $X\}$.
(iv) δ pre closure (resp. δ semi closure) of L (briefly, $CPF_{\mathbb{P}}\delta\mathcal{P}cl$ (resp.$CPF_{\mathbb{P}}\delta\mathcal{S}cl$)) is defined by $CPF_{\mathbb{P}}\delta\mathcal{P}cl$(resp. $CPF_{\mathbb{P}}\delta\mathcal{S}cl$) = $\bigcap_{\mathbb{P}}\{G : L \subseteq G$ & G is a $CPF_{\mathbb{P}}ecs$ (resp.$CPF_{\mathbb{P}}\delta\mathcal{P}cs$ & $CPF_{\mathbb{P}}\delta\mathcal{S}cs$) in $X\}$.
(v) a-interior(resp. β-interior) of L is defined by $CPF_{\mathbb{P}}aintL$(resp. $CPF_{\mathbb{P}}\beta intL$) = $\bigcup_{\mathbb{P}}\{G : G \subseteq L$ & G is a $CPF_{\mathbb{P}}aos$(resp. $CPF_{\mathbb{P}}\beta os$)in$X\}$.
(vi) a-closure (resp. β-closure) of L is defined by $CPF_{\mathbb{P}}aclL$(resp. $CPF_{\mathbb{P}}\beta clL$) = $\bigcap_{\mathbb{P}}\{G : L \subseteq G$ & G is a $CPF_{\mathbb{P}}acs$(resp. $CPF_{\mathbb{P}}\beta cs$)in$X\}$.

Proposition 3.1. *Let A, B, C, D be$CPFSs$ on a nonempty set X. Then the following statements hold under $\mathbb{P}$–order:*

(i) If $A \subseteq_{\mathbb{P}} B$ and $C \subseteq_{\mathbb{P}} D$, then

$$A \cup_{\mathbb{P}} C \subseteq_{\mathbb{P}} B \cup_{\mathbb{P}} D.$$

(ii) If $A \subseteq_{\mathbb{P}} B$ and $C \subseteq_{\mathbb{P}} D$, then

$$A \cap_{\mathbb{P}} C \subseteq_{\mathbb{P}} B \cap_{\mathbb{P}} D.$$

(iii) If $A \subseteq_{\mathbb{P}} B$, then the complements reverse the order:

$$B^c \subseteq_{\mathbb{P}} A^c.$$

Proof. Let $x \in X$ be arbitrary. Write each cubic picture fuzzy value as

$$A(x) = \big([\alpha_i^A, \alpha_j^A], [\beta_i^A, \beta_j^A], [\gamma_i^A, \gamma_j^A], \langle \alpha^A, \beta^A, \gamma^A \rangle\big),$$

and similarly for B, C, D.

(i) Preservation of $\mathbb{P}$–order under $\mathbb{P}$–union.

Assume $A \subseteq_{\mathbb{P}} B$ and $C \subseteq_{\mathbb{P}} D$. Thus,

$$\alpha_i^A \le \alpha_i^B, \quad \alpha_i^C \le \alpha_i^D,$$

and similarly for all other positive, neutral and negative interval bounds and crisp membership components in the directions dictated by $\mathbb{P}$–order.

For the positive–interval lower bound we compute:

$$\alpha_i^{A \cup_{\mathbb{P}} C} = \max\{\alpha_i^A, \alpha_i^C\} \le \max\{\alpha_i^B, \alpha_i^D\} = \alpha_i^{B \cup_{\mathbb{P}} D},$$

since max is monotone in each argument.

Similarly,

$$\alpha_j^{A \cup_{\mathbb{P}} C} = \max\{\alpha_j^A, \alpha_j^C\} \le \max\{\alpha_j^B, \alpha_j^D\} = \alpha_j^{B \cup_{\mathbb{P}} D}.$$

For the neutral lower bound (min is used in union):

$$\beta_i^{A \cup_{\mathbb{P}} C} = \min\{\beta_i^A, \beta_i^C\} \ge \min\{\beta_i^B, \beta_i^D\} = \beta_i^{B \cup_{\mathbb{P}} D},$$

because $A \subseteq_{\mathbb{P}} B$ and $C \subseteq_{\mathbb{P}} D$ imply $\beta_i^A \ge \beta_i^B$ and $\beta_i^C \ge \beta_i^D$.

The same arguments hold for (β_j) and (γ_i, γ_j) using the fact that min and max preserve the inequality directions required by $\mathbb{P}$–order.

For the crisp components,

$$\alpha^{A \cup_{\mathbb{P}} C} = \max\{\alpha^A, \alpha^C\} \le \max\{\alpha^B, \alpha^D\} = \alpha^{B \cup_{\mathbb{P}} D},$$

and

$$\beta^{A \cup_{\mathbb{P}} C} = \min\{\beta^A, \beta^C\} \ge \min\{\beta^B, \beta^D\}, \qquad \gamma^{A \cup_{\mathbb{P}} C} = \min\{\gamma^A, \gamma^C\} \ge \min\{\gamma^B, \gamma^D\}.$$

Thus every coordinate satisfies the inequalities required by $A \cup_{\mathbb{P}} C \subseteq_{\mathbb{P}} B \cup_{\mathbb{P}} D$.

(ii) Preservation of $\mathbb{P}$–order under $\mathbb{P}$–intersection.

Now intersection uses the dual min/max structure:

$$\alpha_i^{A \cap_{\mathbb{P}} C} = \min\{\alpha_i^A, \alpha_i^C\} \le \min\{\alpha_i^B, \alpha_i^D\} = \alpha_i^{B \cap_{\mathbb{P}} D},$$

because $\alpha_i^A \le \alpha_i^B$ and $\alpha_i^C \le \alpha_i^D$.

For neutral components (intersection uses max):

$$\beta_i^{A \cap_{\mathbb{P}} C} = \max\{\beta_i^A, \beta_i^C\} \ge \max\{\beta_i^B, \beta_i^D\},$$

using $\beta_i^A \ge \beta_i^B$ and $\beta_i^C \ge \beta_i^D$.

Negative intervals and crisp components follow identically by monotonicity of min and max.

Thus,

$$A \cap_{\mathbb{P}} C \subseteq_{\mathbb{P}} B \cap_{\mathbb{P}} D.$$

(iii) Complement reverses the ℙ–order.

Using the definition:

$$A^c(x) = ([\gamma_i^A, \gamma_j^A], [\beta_i^A, \beta_j^A], [\alpha_i^A, \alpha_j^A], \langle \gamma^A, \beta^A, \alpha^A \rangle),$$

and similarly for B.

Assume $A \subseteq_{\mathbb{P}} B$. Thus,

$$\gamma_i^A \geq \gamma_i^B, \quad \gamma_j^A \geq \gamma_j^B,$$

implying

$$\alpha_i^{B^c} = \gamma_i^B \leq \gamma_i^A = \alpha_i^{A^c},$$

and similarly for every positive coordinate, showing

$$\alpha^{B^c} \leq \alpha^{A^c}.$$

Neutral components satisfy $\beta^{B^c} = \beta^B \leq \beta^A = \beta^{A^c}$ in the sense of reversed inclusion, and the negative components satisfy the dual relation.

Therefore each coordinate of B^c satisfies the ℙ–order inequality with respect to A^c, hence

$$B^c \subseteq_{\mathbb{P}} A^c.$$

This proves all three parts. □

Proposition 3.2. *The statements are hold but the converse does not true.*

(i) Every $CPF_{\mathbb{P}}\delta os$ (resp. $CPF_{\mathbb{P}}\delta cs$) is a $CPF_{\mathbb{P}}os$ (resp. $CPF_{\mathbb{P}}cs$).
(ii) Every $CPF_{\mathbb{P}}os$ (resp. $CPF_{\mathbb{P}}cs$) is a $CPF_{\mathbb{P}}\delta\mathcal{S}os$ (resp. $CPF_{\mathbb{P}}\delta\mathcal{S}cs$).
(iii) Every $CPF_{\mathbb{P}}os$ (resp. $CPF_{\mathbb{P}}cs$) is a $CPF_{\mathbb{P}}\delta\mathcal{P}os$ (resp. $CPF_{\mathbb{P}}\delta\mathcal{P}cs$).
(iv) Every $CPF_{\mathbb{P}}\delta\mathcal{S}os$ (resp. $CPF_{\mathbb{P}}\delta\mathcal{S}cs$) is a $CPF_{\mathbb{P}}eos$ (resp. $CPF_{\mathbb{P}}ecs$).
(v) Every $CPF_{\mathbb{P}}\delta\mathcal{P}os$ (resp. $CPF_{\mathbb{P}}\delta\mathcal{P}cs$) is a $CPF_{\mathbb{P}}eos$ (resp. $CPF_{\mathbb{P}}ecs$).
(vi) Every $CPF_{\mathbb{P}}eos$ (resp. $CPF_{\mathbb{P}}ecs$) is a $CPF_{\mathbb{P}}e^*os$ (resp. $CPF_{\mathbb{P}}e^*cs$).

Proof. (i) If L is a $CPF_{\mathbb{P}}\delta os$, then $L = CPF_{\mathbb{P}}\delta intL \subseteq CPF_{\mathbb{P}}intL$. $\therefore$ L is a $CPF_{\mathbb{P}}os$.

(ii) If L is a $CPF_{\mathbb{P}}os$, then $L = CPF_{\mathbb{P}}intL$. So, $L = CPF_{\mathbb{P}}intL \subseteq CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}\delta intL)$. $\therefore$ L is a $CPF_{\mathbb{P}}\delta\mathcal{S}os$.

(iii) If L is a $CPF_{\mathbb{P}}os$, then $L = CPF_{\mathbb{P}}intL$. So, $L = CPF_{\mathbb{P}}intL \subseteq CPF_{\mathbb{P}}int(CPF_{\mathbb{P}}\delta clL)$. $\therefore$ L is a $CPF_{\mathbb{P}}\delta\mathcal{P}os$.

(iv) If L is a $CPF_{\mathbb{P}}\delta\mathcal{S}os$, then $L \subseteq CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}\delta intL) \subseteq CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}\delta intL) \cup CPF_{\mathbb{P}}int\ (CPF_{\mathbb{P}}\delta clL)$. $\therefore$ L is a $CPF_{\mathbb{P}}eos$.

(v) If L is a $CPF_{\mathbb{P}}\delta\mathcal{P}os$, then $L \subseteq CPF_{\mathbb{P}}int(CPF_{\mathbb{P}}\delta clL) \subseteq CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}\delta intL) \cup CPF_{\mathbb{P}}int\ (CPF_{\mathbb{P}}\delta clL)$. $\therefore$ L is a $CPF_{\mathbb{P}}eos$.

(vi) If L is a $CPF_{\mathbb{P}}eos$ then $L \subseteq CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}\delta intL) \cup CPF_{\mathbb{P}}int(CPF_{\mathbb{P}}\delta clL)$. So $L \subseteq CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}\delta intL) \cup CPF_{\mathbb{P}}int(CPF_{\mathbb{P}}\delta clL) \subseteq CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}int(CPF_{\mathbb{P}}\delta clL))$. $\therefore$ L is a $CPF_{\mathbb{P}}e^*os$.

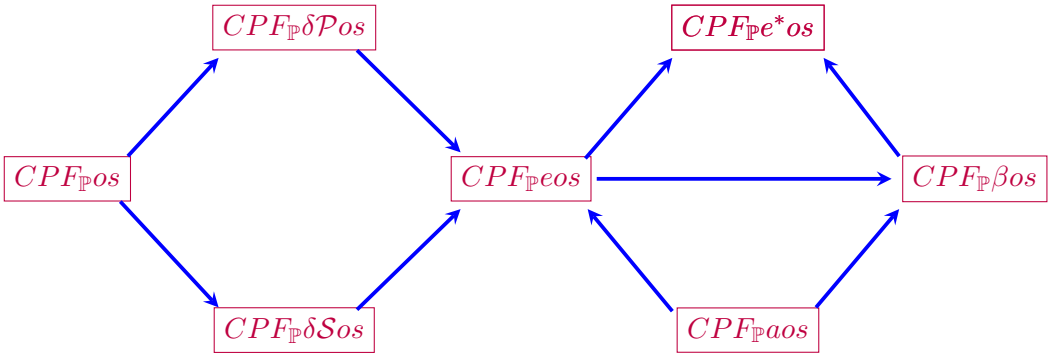
It is also true for their respective closed sets. □

Example 3.1. *Let X be a non-empty set and $\mathcal{F}_{\mathbb{P}}$ be the collection of ℙ cubic picture fuzzy sets in X then $(X, \mathcal{F}_{\mathbb{P}})$ be $CPF_{\mathbb{P}}ts$. Let $\mathcal{F}_{\mathbb{P}}$=$\{\ddot{0}, \ddot{1}, P_1, P_2, P_3, P_4\}$ where $P_1 = \{\langle[0,0],[0.05, 0.10],[0.85,0.90]\rangle,\ \langle 0, 0.10, 0.90\rangle\}$, $P_2 = \{\langle[0.85,0.90],[0.05,0.10],[0,0]\rangle, \langle 0.90, 0.10, 0\rangle\}$, $P_3 = \{\langle[0.15,0.25],[0.05,0.10],\ [0.40,0.42]\rangle, \langle 0.20, 0.10, 0.40\rangle\}$, $P_4 = \{\langle[0.30,0.37],[0.05,0.10],[0.38,0.41]\rangle, \langle 0.32, 0.10, 0.40\rangle\}$, $P_5 = \{\langle[0.29,0.30],[0.05,0.10],[0.40,0.41]\rangle, \langle 0.30, 0.10, 0.36\rangle\}$, $P_6 = \{\langle[0.32, 0.35], [0.05,0.10],[0.36,0.37]\rangle, \langle 0.33, 0.10, 0.36\rangle\}$, $P_7 = \{\langle[0.10,0.20],[0.05,0.10],$*

$[0.30, 0.37]\rangle, \langle 0.15, 0.10, 0.32\rangle\}$, $P_8 = \{\langle [0.40, 0.42], [0.05, 0.10], [0.15, 0.25]\rangle, \langle 0.41, 0.10, 0.20\rangle\}$ *then*

(i) P_5 *is* $CPF_{\mathbb{P}}\delta\mathcal{P}os$ *but not* $CPF_{\mathbb{P}}os$
(ii) P_6 *is* $CPF_{\mathbb{P}}\delta\mathcal{S}os$ *but not* $CPF_{\mathbb{P}}os$
(iii) P_6 *is* $CPF_{\mathbb{P}}eos$ *but not* $CPF_{\mathbb{P}}\delta\mathcal{P}os$
(iv) P_5 *is* $CPF_{\mathbb{P}}eos$ *but not* $CPF_{\mathbb{P}}\delta\mathcal{S}os$
(v) P_7 *is* $CPF_{\mathbb{P}}e^*os$ *but not* $CPF_{\mathbb{P}}eos$
(vi) P_8 *is* $CPF_{\mathbb{P}}e^*os$ *but not* $CPF_{\mathbb{P}}\beta os$
(vii) P_7 *is* $CPF_{\mathbb{P}}\beta os$ *but not* $CPF_{\mathbb{P}}eos$
(viii) P_7 *is* $CPF_{\mathbb{P}}\beta os$ *but not* $CPF_{\mathbb{P}}aos$
(ix) P_6 *is* $CPF_{\mathbb{P}}eos$ *but not* $CPF_{\mathbb{P}}aos$

Remark 3.1. *The diagram illustrates that the converse of the above proposition 3.2 does not hold.*



Theorem 3.1. *The statements are true.*

(i) $CPF_{\mathbb{P}}\delta\mathcal{P}clL \supseteq L \cup CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}\delta intL)$.
(ii) $CPF_{\mathbb{P}}\delta\mathcal{P}intL \subseteq L \cap CPF_{\mathbb{P}}int(CPF_{\mathbb{P}}\delta clL)$.
(iii) $CPF_{\mathbb{P}}\delta\mathcal{S}clL \supseteq L \cup CPF_{\mathbb{P}}int(CPF_{\mathbb{P}}\delta clL)$.
(iv) $CPF_{\mathbb{P}}\delta\mathcal{S}intL \subseteq L \cap CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}\delta intL)$.

Proof. (i) Since $CPF_{\mathbb{P}}\delta\mathcal{P}clL$ is $CPF_{\mathbb{P}}\delta\mathcal{P}cs$, we have

$$CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}\delta intL) \subseteq C_{\mathbb{P}}cl(C_{\mathbb{P}}\delta int(C_{\mathbb{P}}\delta\mathcal{P}clL)) \subseteq CPF_{\mathbb{P}}\delta\mathcal{P}clL.$$

Thus $L \cup CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}\delta intL) \subseteq CPF_{\mathbb{P}}\delta\mathcal{P}clL$.
The other cases are similar. □

Theorem 3.2. L *is a* $CPF_{\mathbb{P}}eos$ *iff* $L = CPF_{\mathbb{P}}\delta\mathcal{P}intL \cup CPF_{\mathbb{P}}\delta\mathcal{S}intL$.

Proof. Let L be a $C_{\mathbb{P}}eos$. Then $L \subseteq CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}\delta intL) \cup CPF_{\mathbb{P}}int(CPF_{\mathbb{P}}\delta clL)$.
By Theorem 3.1, we have

$$\begin{aligned} CPF_{\mathbb{P}}\delta\mathcal{P}intL \cup CPF_{\mathbb{P}}\delta\mathcal{S}intL &\subseteq L \cap (CPF_{\mathbb{P}}int(CPF_{\mathbb{P}}\delta clL)) \cup (L \cap CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}\delta intL)) \\ &= L \cap (CPF_{\mathbb{P}}int(CPF_{\mathbb{P}}\delta clL)) \cap CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}\delta intL) \\ &= L. \end{aligned}$$

Conversely, if $L = CPF_{\mathbb{P}}\delta\mathcal{P}intL \cup CPF_{\mathbb{P}}\delta\mathcal{S}intL$ then, by Theorem 3.1

$$\begin{aligned} L &= CPF_{\mathbb{P}}\delta\mathcal{P}intL \cup CPF_{\mathbb{P}}\delta\mathcal{S}intL \\ &\subseteq \big(L \cap CPF_{\mathbb{P}}int(CPF_{\mathbb{P}}\delta clL)\big) \cup \big(L \cap CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}\delta intL)\big) \\ &= L \cap \big(CPF_{\mathbb{P}}int(CPF_{\mathbb{P}}\delta clL) \cup CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}\delta intL)\big) \\ &\subseteq CPF_{\mathbb{P}}int(CPF_{\mathbb{P}}\delta clL) \cup CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}\delta intL) \end{aligned}$$

and hence L is a $CPF_{\mathbb{P}}eos$. □

Theorem 3.3. *The union (resp. intersection) of any family of $CPF_{\mathbb{P}}eOS(X)$ (resp. $CPF_{\mathbb{P}}eCS(X)$) is a $CPF_{\mathbb{P}}eOS(X)$ (resp. $CPF_{\mathbb{P}}eCS(X)$).*

Proof. Let $\{L_a : a \in \mathcal{F}_{\mathbb{P}}\}$ be a family of $CPF_{\mathbb{P}}eos$.
For each $a \in \mathcal{F}_{\mathbb{P}}$, $L_a \subseteq CPF_{\mathbb{P}}cl\big(CPF_{\mathbb{P}}\delta int(L_a)\big) \cup CPF_{\mathbb{P}}int\big(CPF_{\mathbb{P}}\delta cl(L_a)\big)$.

$$\begin{aligned} \bigcup_{a\in\mathcal{F}_{\mathbb{P}}} L_a &\subseteq \bigcup_{a\in\mathcal{F}_{\mathbb{P}}} CPF_{\mathbb{P}}cl\big(CPF_{\mathbb{P}}\delta int(R_a)\big) \cup CPF_{\mathbb{P}}int\big(CPF_{\mathbb{P}}\delta cl(L_a)\big) \\ &\subseteq CPF_{\mathbb{P}}cl\big(CPF_{\mathbb{P}}\delta int(\cup L_a)\big) \cup CPF_{\mathbb{P}}int\big(CPF_{\mathbb{P}}\delta cl(\cup L_a)\big) \end{aligned}$$

The other case is similar. □

Proposition 3.3. *If L is a*

(i) $CPF_{\mathbb{P}}eos$ and $CPF_{\mathbb{P}}\delta intL = \ddot{0}$, then L is a $CPF_{\mathbb{P}}\delta\mathcal{P}os$.
(ii) $CPF_{\mathbb{P}}eos$ and $CPF_{\mathbb{P}}\delta clL = \ddot{0}$, then L is a $CPF_{\mathbb{P}}\delta\mathcal{S}os$.
(iii) $CPF_{\mathbb{P}}eos$ and $CPF_{\mathbb{P}}\delta cs$, then L is a $CPF_{\mathbb{P}}\delta\mathcal{S}os$.
(iv) $CPF_{\mathbb{P}}\delta\mathcal{S}os$ and $CPF_{\mathbb{P}}\delta cs$, then L is a $CPF_{\mathbb{P}}eos$.

Proof. (i) Let L be a $CPF_{\mathbb{P}}eos$, that is

$$\begin{aligned} L \subseteq CPF_{\mathbb{P}}cl\big(CPF_{\mathbb{P}}\delta intL\big) \cup CPF_{\mathbb{P}}int\big(CPF_{\mathbb{P}}\delta clL\big) &= \ddot{0} \cup CPF_{\mathbb{P}}int\big(CPF_{\mathbb{P}}\delta clL\big) \\ &= CPF_{\mathbb{P}}int\big(CPF_{\mathbb{P}}\delta clL\big) \end{aligned}$$

Hence L is a $CPF_{\mathbb{P}}\delta\mathcal{P}os$.
(ii) Let L be a $CPF_{\mathbb{P}}eos$, that is

$$\begin{aligned} L \subseteq CPF_{\mathbb{P}}cl\big(CPF_{\mathbb{P}}\delta intL\big) \cup CPF_{\mathbb{P}}int\big(CPF_{\mathbb{P}}\delta clL\big) &= CPF_{\mathbb{P}}cl\big(CPF_{\mathbb{P}}\delta intL\big) \cup \ddot{0} \\ &= CPF_{\mathbb{P}}cl\big(CPF_{\mathbb{P}}\delta intL\big) \end{aligned}$$

Hence L is a $CPF_{\mathbb{P}}\delta\mathcal{S}os$.
(iii) Let L be a $CPF_{\mathbb{P}}eos$ and $CPF_{\mathbb{P}}\delta cs$, that is

$$L \subseteq CPF_{\mathbb{P}}cl\big(CPF_{\mathbb{P}}\delta intL\big) \cup CPF_{\mathbb{P}}int\big(CPF_{\mathbb{P}}\delta clL\big) = CPF_{\mathbb{P}}cl\big(CPF_{\mathbb{P}}\delta intL\big).$$

Hence L is a $CPF_{\mathbb{P}}\delta\mathcal{S}os$.
(iv) Let L be a $CPF_{\mathbb{P}}\delta\mathcal{S}os$ and $CPF_{\mathbb{P}}\delta cs$, that is

$$L \subseteq CPF_{\mathbb{P}}cl\big(CPF_{\mathbb{P}}\delta intL\big) \subseteq CPF_{\mathbb{P}}cl\big(CPF_{\mathbb{P}}\delta intL\big) \cup CPF_{\mathbb{P}}int\big(CPF_{\mathbb{P}}\delta clL\big).$$

Hence L is a $CPF_{\mathbb{P}}eos$. □

Theorem 3.4. *L is a $CPF_{\mathbb{P}}ecs$ (resp.$CPF_{\mathbb{P}}eos$) iff $L = CPF_{\mathbb{P}}eclL$ (resp.$L = CPF_{\mathbb{P}}eintL$.*

Proof. Suppose $L = CPF_{\mathbb{P}}eclL = \bigcap\{G : L \subseteq G \ \&\ G \text{ is a } CPF_{\mathbb{P}}ecs\}$.
This means $L \in \bigcap\{G : L \subseteq G \ \&\ G \text{ is a } CPF_{\mathbb{P}}ecs\}$ and hence L is $CPF_{\mathbb{P}}ecs$.Conversely, suppose L be a $CPF_{\mathbb{P}}ecs$ in X. Then, we have $L \in \bigcap\{G : L \subseteq G \ \&\ G \text{ is a } CPF_{\mathbb{P}}ecs\}$.
Hence, $L \subseteq G$ implies $L = \bigcap\{G : L \subseteq G \ \&\ G \text{ is a } CPF_{\mathbb{P}}ecs\} = CPF_{\mathbb{P}}eclL$.
Similarly $L = CPF_{\mathbb{P}}eintL$. □

Theorem 3.5. *Let L and G in X, then the $CPF_{\mathbb{P}}ecl$ sets have*

(i) $CPF_{\mathbb{P}}ecl(\ddot{0}) = \ddot{0}$, $CPF_{\mathbb{P}}ecl(\ddot{1}) = \ddot{1}$.
(ii) $CPF_{\mathbb{P}}eclL$ *is a* $CPF_{\mathbb{P}}ecs$ *in* X.
(iii) $CPF_{\mathbb{P}}eclL \subseteq CPF_{\mathbb{P}}eclG$ *if* $L \subseteq G$.
(iv) $CPF_{\mathbb{P}}ecl\big(CPF_{\mathbb{P}}eclL\big) = CPF_{\mathbb{P}}eclL$.

Proof. The proofs are directly from definitions of $CPF_{\mathbb{P}}ecl$ set. □

Theorem 3.6. *Let L and G in X, then the $CPF_{\mathbb{P}}eint$ sets have*

(i) $CPF_{\mathbb{P}}eint(\ddot{0}) = \ddot{0}$, $CPF_{\mathbb{P}}eint(\ddot{1}) = \ddot{1}$.
(ii) $CPF_{\mathbb{P}}eintL$ *is a* $CPF_{\mathbb{P}}eos$ *in* X.
(iii) $CPF_{\mathbb{P}}eintL \subseteq CPF_{\mathbb{P}}eintG$ *if* $L \subseteq G$.
(iv) $CPF_{\mathbb{P}}eint\big(CPF_{\mathbb{P}}eintL\big) = CPF_{\mathbb{P}}eintL$.

Proof. The proofs are directly from definitions of $CPF_{\mathbb{P}}eint$ set. □

Proposition 3.4. *Let L and G are in X, then*

(i) $CPF_{\mathbb{P}}eclL^c = [C_{\mathbb{P}}eintL]^c$, $C_{\mathbb{P}}eintL^c = [C_{\mathbb{P}}eclL]^c$.
(ii) $CPF_{\mathbb{P}}ecl\big(L \cup G\big) \supseteq CPF_{\mathbb{P}}eclL \cup CPF_{\mathbb{P}}eclG$, $CPF_{\mathbb{P}}ecl\big(L \cap G\big) \subseteq CPF_{\mathbb{P}}eclL \cap CPF_{\mathbb{P}}eclG$.
(iii) $CPF_{\mathbb{P}}eint\big(L \cup G\big) \supseteq CPF_{\mathbb{P}}eintL \cup CPF_{\mathbb{P}}eintG$, $CPF_{\mathbb{P}}eint\big(L \cap G\big) \subseteq CPF_{\mathbb{P}}eintL \cap CPF_{\mathbb{P}}eintG$.

Proof. (i) The proof is directly from definition.
(ii) $L \subseteq L \cup G$ or $G \subseteq L \cup G$. Hence $CPF_{\mathbb{P}}eclL \subseteq CPF_{\mathbb{P}}ecl\big(L \cup G\big)$ or $CPF_{\mathbb{P}}eclG \subseteq CPF_{\mathbb{P}}ecl\big(L \cup G\big)$. Therefore, $CPF_{\mathbb{P}}ecl\big(L \cup G\big) \supseteq CPF_{\mathbb{P}}eclL \cup CPF_{\mathbb{P}}eclG$.
The other one is similar.
(iii) $L \subseteq L \cup G$ or $G \subseteq L \cup G$. Hence $CPF_{\mathbb{P}}eintL \subseteq CPF_{\mathbb{P}}eint\big(L \cup G\big)$ or $CPF_{\mathbb{P}}eintG \subseteq CPF_{\mathbb{P}}eint\big(L \cup G\big)$. Therefore, $CPF_{\mathbb{P}}eint\big(L \cup G\big) \supseteq CPF_{\mathbb{P}}eintL \cup CPF_{\mathbb{P}}eintG$.
The other one is similar.

□

Proposition 3.5. *If L is in X, then*

(i) $CPF_{\mathbb{P}}eclL \supseteq CPF_{\mathbb{P}}cl\big(CPF_{\mathbb{P}}\delta intL\big) \cap CPF_{\mathbb{P}}int\big(CPF_{\mathbb{P}}\delta clL\big)$.
(ii) $CPF_{\mathbb{P}}eintL \subseteq CPF_{\mathbb{P}}cl\big(CPF_{\mathbb{P}}\delta intL\big) \cup CPF_{\mathbb{P}}int\big(CPF_{\mathbb{P}}\delta clL\big)$.

Proof. (i) $CPF_{\mathbb{P}}eclL$ is a $CPF_{\mathbb{P}}ecs$ and $L \subseteq CPF_{\mathbb{P}}eclL$, then

$$\begin{aligned} CPF_{\mathbb{P}}eclL &\supseteq CPF_{\mathbb{P}}cl\big(CPF_{\mathbb{P}}\delta int(CPF_{\mathbb{P}}eclL)\big) \cap CPF_{\mathbb{P}}int\big(CPF_{\mathbb{P}}\delta cl(CPF_{\mathbb{P}}eclL)\big) \\ &\supseteq CPF_{\mathbb{P}}cl\big(CPF_{\mathbb{P}}\delta intL\big) \cap CPF_{\mathbb{P}}int\big(CPF_{\mathbb{P}}\delta clL\big). \end{aligned}$$

(ii) $CPF_{\mathbb{P}}eintL$ is a $CPF_{\mathbb{P}}eos$ and $L \supseteq CPF_{\mathbb{P}}eintL$, then

$$\begin{aligned} CPF_{\mathbb{P}}eintL &\subseteq CPF_{\mathbb{P}}cl\big(CPF_{\mathbb{P}}\delta int(CPF_{\mathbb{P}}eintL)\big) \cup CPF_{\mathbb{P}}int\big(CPF_{\mathbb{P}}\delta cl(CPF_{\mathbb{P}}eintL)\big) \\ &\subseteq CPF_{\mathbb{P}}cl\big(CPF_{\mathbb{P}}\delta intL\big) \cup CPF_{\mathbb{P}}int\big(CPF_{\mathbb{P}}\delta clL\big). \end{aligned}$$

□

Theorem 3.7. *Let L be in X, then*

(i) $CPF_{\mathbb{P}}eclL = CPF_{\mathbb{P}}\delta\mathcal{P}clL \cap CPF_{\mathbb{P}}\delta\mathcal{S}clL$.
(ii) $CPF_{\mathbb{P}}eintL = CPF_{\mathbb{P}}\delta\mathcal{P}intL \cap CPF_{\mathbb{P}}\delta\mathcal{S}intL$.

Proof. (i) It is obvious that, $CPF_{\mathbb{P}}eclL \subseteq CPF_{\mathbb{P}}\delta\mathcal{P}clL \cap CPF_{\mathbb{P}}\delta\mathcal{S}clL$. Conversely, from Definition 3.4, we have

$$\begin{aligned} CPF_{\mathbb{P}}eclL &\supseteq CPF_{\mathbb{P}}cl\big(CPF_{\mathbb{P}}\delta int(CPF_{\mathbb{P}}eclL)\big) \cap CPF_{\mathbb{P}}int\big(CPF_{\mathbb{P}}\delta cl(CPF_{\mathbb{P}}eclL)\big) \\ &\supseteq CPF_{\mathbb{P}}cl\big(CPF_{\mathbb{P}}\delta intL\big) \cap CPF_{\mathbb{P}}int\big(CPF_{\mathbb{P}}\delta clL\big). \end{aligned}$$

Since $CPF_{\mathbb{P}}eclL$ is $CPF_{\mathbb{P}}ecs$, by Theorem 3.1, we have

$$\begin{aligned} CPF_{\mathbb{P}}\delta\mathcal{P}clL \cap CPF_{\mathbb{P}}\delta\mathcal{S}clL &= L \cup CPF_{\mathbb{P}}cl\big(CPF_{\mathbb{P}}\delta intL\big) \cap \big(L \cup CPF_{\mathbb{P}}int(CPF_{\mathbb{P}}\delta clL)\big) \\ &= L \cap \big(CPF_{\mathbb{P}}cl(CPF_{\mathbb{P}}\delta intL) \cap CPF_{\mathbb{P}}int(CPF_{\mathbb{P}}\delta clL)\big) \\ &= L \subseteq CPF_{\mathbb{P}}eclL. \end{aligned}$$

Therefore, $CPF_{\mathbb{P}}eclL = CPF_{\mathbb{P}}\delta\mathcal{P}clL \cap CPF_{\mathbb{P}}\delta\mathcal{S}clL$.

(ii) is similar from (i).

□

Theorem 3.8. *Let L be in X. Then*

(i) $CPF_{\mathbb{P}}ecl(\ddot{1} - L) = \ddot{1} - CPF_{\mathbb{P}}eintL$.

(ii) $CPF_{\mathbb{P}}eint(\ddot{1} - L) = \ddot{1} - CPF_{\mathbb{P}}eclL$.

Proof. (i) Let G be $CPF_{\mathbb{P}}ecs$ in X and L be any set in X.

$$\begin{aligned} \text{Then } CPF_{\mathbb{P}}eintL &= \bigcup\{\ddot{1} - G : \ddot{1} - G \subseteq L, \ddot{1} - G \text{ is a } CPF_{\mathbb{P}}eos \text{ in } X\} \\ &= \ddot{1} - \bigcap\{G : G \supseteq \ddot{1} - L, G \text{ is a } CPF_{\mathbb{P}}ecs \text{ in } X\} \\ &= \ddot{1} - CPF_{\mathbb{P}}eclL. \end{aligned}$$

Thus, $CPF_{\mathbb{P}}ecl(\ddot{1} - L) = \ddot{1} - CPF_{\mathbb{P}}eintL$

(ii) Let G be $CPF_{\mathbb{P}}eos$ in X and L be any set in X.

$$\begin{aligned} \text{Then } CPF_{\mathbb{P}}eclL &= \bigcap\{\ddot{1} - G : \ddot{1} - G \supseteq L, \ddot{1} - G \text{ is a } CPF_{\mathbb{P}}ecs \text{ in } X\} \\ &= \ddot{1} - \bigcup\{G : G \subseteq \ddot{1} - L, G \text{ is a } CPF_{\mathbb{P}}eos \text{ in } X\} \\ &= \ddot{1} - CPF_{\mathbb{P}}eintL. \end{aligned}$$

Thus, $CPF_{\mathbb{R}}eint(\ddot{1} - L) = \ddot{1} - CPF_{\mathbb{R}}eclL$.

□

4. Conclusion

In this paper, we have introduced and analyzed novel classes of generalized open sets-namely, e-open, e^*-open, a-open, and β-open sets—within the framework of Cubic Picture Fuzzy Topological Spaces ($CPFTS$'s) under ℙ-order. Through rigorous definitions, illustrative examples, and counterexamples, we have demonstrated how these new open sets exhibit distinct structural behaviors and contribute to a refined hierarchy of openness in $CPFTS$'s. Our exploration of inclusion relationships and non-reversible implications under set-theoretic operations establishes a deeper algebraic and topological foundation for generalized openness.

The incorporation of ℙ-order has proven essential in understanding the interplay between order structures and fuzzy topological properties, enriching the conceptual landscape of neighborhood systems in such environments. This research bridges a critical gap by extending generalized open set theory to $CPFTS$'s for the first time, laying the groundwork for future studies in fuzzy topology, particularly in applications involving intelligent systems, multi-criteria decision-making, and uncertainty modeling.

Acknowledgement. The authors thank the anonymous reviewer and editor for their helpful comments and guidance.

References

[1] Ali, A., Karataş, S., and Qiyas, M., (2024), Picture fuzzy soft topological spaces and their applications in medical diagnosis, AIP Conference Proceedings, 3122(1), 040031, pp. 1–8.
[2] Ashraf, S., Abdullah, S., and Qadir, A., (2018), Novel concept of cubic picture fuzzy sets, Journal of New Theory, 24, pp. 59–72.
[3] Atanassov, K. T., (1986), Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20, pp. 87–96.
[4] Cuoung, B. C., and Phan, V. H., (2015), Some fuzzy logic operators for picture fuzzy sets, Proceedings of the 2015 Seventh International Conference on Knowledge and Systems Engineering (KSE), IEEE, pp. 132–137.
[5] Ekici, E., (2008), On e-open sets, DP^*-sets and $DP\epsilon^*$-sets and decompositions of continuity, Arabian Journal for Science and Engineering, 33(2A), pp. 269–282.
[6] Ekici, E., (2007), Some generalizations of almost contra-super-continuity, Filomat, 21(2), pp. 31–44.
[7] Ekici, E., (2008), New forms of contra-continuity, Carpathian Journal of Mathematics, 24(1), pp. 37–45.
[8] 1. Wu, X. X., Zhu, Z., Wang, T., and Zhang, X., (2025), Picture fuzzy interactional aggregation operators via strict triangular norms and applications to multi-criteria decision making, Applied and Computational Mathematics, 24(1), pp.62-100
[9] Ekici, E., (2009), On e^*-open sets and $(D, S)^*$-sets, Mathematica Moravica, 13(1), pp. 29–36.
[10] Ekici, E., (2008), A note on a-open sets and e^*-open sets, Filomat, 22(1), pp. 89–96.
[11] Narayanamoorthy, S., and Vijayalakshmi, R., (2023),Generalized open sets in picture fuzzy topological spaces, International Journal of Fuzzy Systems, 25(3), pp. 678–691.
[12] Gunduz C., Bayramov S ., Zeynalov J., (2024), Separation axioms in bipolar soft topological spaces, TWMS J. Pure Appl. Math., 15(1), pp.15-25.
[13] Riaz, M., Kausar, R., Jameel, T., and Pamucar, D., (2024), Cubic picture fuzzy topological data analysis with integrating blockchain and the metaverse for uncertain supply chain management, Engineering Applications of Artificial Intelligence, 131, Article ID: 107827.
[14] Riaz, M., (2022), Cubic intuitionistic fuzzy topology with application to approximate reasoning, Journal of Mathematics, Article ID 9631579, pp. 1–15.
[15] Riaz, M., (2022), Topological data analysis with cubic hesitant fuzzy sets, Symmetry, 14(5), Article 865, pp. 1–22.
[16] Riaz, M., Akmal, K., Almalki, Y., and Alblowi, S. A., (2022), Cubic m-polar fuzzy topology and its application to multi-criteria group decision making, AIMS Mathematics, 7(5), pp. 7843–7865.
[17] Jun, Y. B., Kim, C. S., and Yang, K. O., (2012), Cubic sets and operations on cubic sets, Inform., 4(1), pp. 83–98.
[18] Zadeh, L. A., (1965), Fuzzy sets, Information and Control, 8, pp. 338–353.
[19] Zadeh, L. A., (1975), The concept of a linguistic variable and its application to approximate reasoning. Part 1, Information Sciences, 8, pp. 199–249.



Dr. G. Saravanakumar holds a Ph.D. in Mathematics from Annamalai University, along with an M.Phil., M.Sc., and B.Sc. in Mathematics from reputable institutions in Tamil Nadu, India. He has over 15 years of teaching experience and is currently serving as an Associate Professor at Vel Tech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology. His primary research interests include fuzzy topology, intuitionistic fuzzy sets, and their applications.



Ms. K. Suganthi is an Assistant Professor and Head of Science and Humanities at Asan Memorial College of Engineering and Technology, Chengalpattu. She holds M.Sc. and M.Phil. degrees in Mathematics, along with B.Ed. and M.Ed. qualifications. With rich teaching and administrative experience across institutions, she has been serving at AMET since 2022. She is currently pursuing her Ph.D. in Mathematics at Vel Tech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology under the supervision of Dr. G. Saravanakumar, with her research focused on Cubic Picture Fuzzy Topological Spaces.