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INTUITIONISTIC FUZZY IDEAL OF PARTIALLY ORDERED NEAR-RING

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Abstract. In this paper, we introduce the concept of intuitionistic fuzzy ideals (IFI) within the framework of partially ordered near-rings ($PONR$). We explore the fundamental properties and characterizations of IFI in $PONR$, providing a comprehensive theoretical foundation for their study. Our work extends the existing literature on fuzzy algebraic structures, aligning with advancements in intuitionistic fuzzy sets and their applications in near-ring theory.

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1. Introduction

The introduction of fuzzy set theory by Zadeh in [15] marked a significant milestone in mathematics, enabling researchers to model uncertainty and vagueness across diverse fields. Rosenfeld [14] extended this groundbreaking idea to group theory, defining fuzzy groups and establishing their key properties. Subsequently, Kuroki[9, 10] generalized classical semigroups by introducing fuzzy semigroups and exploring various classes of semigroups through fuzzy ideals. Mordeson et al. [11] further enriched this area by systematically studying fuzzy semigroups and their applications in fuzzy coding, fuzzy finite state machines, and fuzzy languages.

The development of fuzzy algebraic structures continued with Abou-Zaid's introduction of fuzzy subnear-rings and ideals within near-rings in [1]. Atanassov's intuitionistic fuzzy sets (IFS) [2, 3], which incorporate both membership and non-membership degrees, broadened the scope of fuzzy set theory. Biswas [4] applied IFS to define intuitionistic

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fuzzy subgroups of groups, while Kim and Jun [7] investigated intuitionistic fuzzy ideals (IFI) of semigroups. Kim and Lee [8] further extended this study to intuitionistic fuzzy bi-ideals and introduced intuitionistic (T,S)-normed fuzzy ideals of rings [6].

Zhan Jianming and Ma Xueling [16] explored the properties of IFI in near-rings, and Cho et al. [5] introduced normal intuitionistic fuzzy N-subgroups in near-rings. Additionally, A. Radha Krishna and M. Bandari [10] defined partially ordered near-rings ($PONR$) and examined their properties. Building on these advancements, our study focuses on the exploration of IFI within $PONR$, contributing to the growing body of knowledge on intuitionistic fuzzy algebraic structures.

2. Preliminaries

For the sake of continuity we recall the following definitions.

Definition 2.1. [12] *An algebra $(\mathcal{N}, +, .)$ is said to be a near-ring(NR) if it satisfied the following conditions:*

(1) $(\mathcal{N}, +)$ is a (not necessarily abelian) group.
(2) $(\mathcal{N}, .)$ is a semigroup.
(3) for all $x, y, z \in \mathcal{N}, x(y+z) = xy + xz$.

Definition 2.2. [2] *Let X be a nonempty fixed set. An IFS L in X is an object having the form $L = \{\langle x, \wp_L(x), \ell_L(x)\rangle | x \in X\}$, where $\wp_L : X \to [0,1]$ and $\ell_L : X \to [0,1]$ of each element $x \in X$ to the set L, with $0 \leq \wp_L(x) + \ell_L(x) \leq 1$.*

Definition 2.3. [12] *An IFS $L = (\wp_L, \ell_L)$ of a NR $\mathcal{N}$ is called an intuitionistic fuzzy subnear-ring($IFSNR$) of $\mathcal{N}$ if:*

(1) $\wp_L(x-y) \geq min\{\wp_L(x), \wp_L(y)\}$ and $\ell_L(x-y) \leq max\{\ell_L(x), \ell_L(y)\}$.
(2) $\wp_L(xy) \geq min\{\wp_L(x), \wp_L(y)\}$ and $\ell_L(xy) \leq max\{\ell_L(x), \ell_L(y)\}$ for all $x, y \in N$.

Definition 2.4. [12] *An IFS $L = (\wp_L, \ell_L)$ in a NR $\mathcal{N}$ is called an IFI of $\mathcal{N}$ if:*

(1) $\wp_L(x-y) \geq \wp_L(x) \wedge \wp_L(y)$ and $\ell_L(x-y) \leq \ell_L(x) \vee \ell_L(y)$.
(2) $\wp_L(y+x-y) = \wp_L(x)$ and $\ell_L(y+x-y) = \ell_L(x)$.
(3) $\wp_L(rx) \geq \wp_L(x)$ and $\ell_L(rx) \leq \ell_L(x)$.
(4) $\wp((x+i)y - xy) \geq \wp_L(i)$ and $\ell((x+i)y - xy) \leq \ell_L(i)$ $\forall x, y, i, r \in \mathcal{N}$.

If $L = (\wp_L, \ell_L)$ satisfies (1),(2) and (3) then L is called an intuitionistic fuzzy left ideal($IFLI$) of $\mathcal{N}$ and it is satisfies (1), (2) and (4) then L is called an intuitionistic fuzzy right ideal($IFRI$) of $\mathcal{N}$.

Definition 2.5. [13] *Let $\mathcal{N}$ be a NR. A NR $\mathcal{N}$ is called a partially ordered near-ring $PONR$ if:*

(i) $a \leq b$ then $a + g \leq b + g$ $\forall$ $a, b, g \in \mathcal{N}$.
(ii) $a \leq b$ and $c \geq 0$ then $ac \leq bc$ and $ca \leq cb$ $\forall$ $a, b, c \in \mathcal{N}$.

3. Intuitionistic fuzzy ideals of partially ordered near-rings

Definition 3.1. *Let $\mathcal{N}$ be a $PONR$. An IFS $(\wp, \ell)$ of $\mathcal{N}$ is said to be an $IFSNR$ of $\mathcal{N}$ if:*

(i) $\wp(x-y) \geq \min\{\wp(x), \wp(\mathrm{y})\}$ and $\ell(x-y) \leq \max\{\ell(x), \ell(\mathrm{y})\}$.
(ii) $\wp(xy) \geq \min\{\wp(x),\ \wp(y)\}$ and $\ell(xy) \leq \max\{\ell(x),\ \ell(y)\}$.
(iii) $x \leq y \Rightarrow \wp(x) \geq \wp(y)$ and $\ell(x) \leq \ell(y)$ for all $x, y \in \mathcal{N}$.

Example 3.1. *Let us consider the Near-Ring $\mathcal{N} = \{0,1,2\}$ under Modulo 3 with the following operation table*

Subtraction Table *($x - y \mod 3$)*

x/y	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Multiplication Table *($x \cdot y \mod 3$)*

x/y	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Let $\wp, \ell$ be defined as:

x	$\wp(x)$	$\ell(x)$
0	1.0	0.0
1	0.7	0.2
2	0.4	0.5

We can verify that:

- $\wp(x-y) \geq \min(\wp(x), \wp(y))$
- $\ell(x-y) \leq \max(\ell(x), \ell(y))$
- $\wp(xy) \geq \min(\wp(x), \wp(y))$
- $\ell(xy) \leq \max(\ell(x), \ell(y))$
- $x \leq y \Rightarrow \wp(x) \geq \wp(y), \ell(x) \leq \ell(y)$

Thus, $\wp$ and ℓ define an $IFSNR$.

Definition 3.2. *Let $(\wp, \ell)$ be a non-empty IFS of a $PONR$ $\mathcal{N}$. Then $(\wp, \ell)$ is called an IFI of $\mathcal{N}$ if:*

(i) $\wp(x-y) \geq \min\{\wp(x), \{\wp(y)\}$ *and* $\ell(x-y) \leq \max\{\ell(x), \{\ell(y)\}$.
(ii) $\wp(xy) \geq \wp(y)$ *and* $\ell(xy) \leq \ell(y)$.
(iii) $\wp((x+z)y - xy) \geq \wp(z)$ *and* $\ell((x+z)y - xy) \leq \ell(z)$.
(iv) $x \leq y \Rightarrow \wp(x) \geq \wp(y)$ *and* $\ell(x) \leq \ell(y)$ *for all* $x, y, z \in \mathcal{N}$.

Example 3.2. *From Example 3.1 Let $\wp, \ell$ be defined as:*

x	$\wp(x)$	$\ell(x)$
0	0.9	0.1
1	0.6	0.3
2	0.6	0.3

These values satisfy:

- $\wp(x-y) \geq \min(\wp(x), \wp(y))$, $\ell(x-y) \leq \max(\ell(x), \ell(y))$
- $\wp(xy) \geq \wp(y)$, $\ell(xy) \leq \ell(y)$
- $\wp((x+z)y - xy) \geq \wp(z)$, $\ell((x+z)y - xy) \leq \ell(z)$
- $x \leq y \Rightarrow \wp(x) \geq \wp(y), \ell(x) \leq \ell(y)$

Thus, $\wp$ and ℓ define an IFI.

Definition 3.3. *An IFS $(\wp, \ell)$ of $PONR$ $\mathcal{N}$ is called $IFRI$ (resp. $IFLI$) if:*

(i) $\wp(x-y) \geq min(\wp(x),\ \wp(y))$ *and* $\ell(x-y) \leq max(\ell(x),\ \ell(y))$.
(ii) $\wp((x+z)y - xy) \geq \wp(z)(\wp(xy) \geq \wp(y))$ *and* $\ell((x+z)y - xy) \leq \ell(z)(\ell(xy) \leq \ell(y))$.
(iii) $x \leq y \Rightarrow \wp(x) \geq \wp(y)$ *and* $\ell(x) \leq \ell(y)$ *for all* $x, y, z \in \mathcal{N}$.

If $(\wp, \ell)$ is an $IFLI$ and $IFRI$ of a $PONR$ then $(\wp, \ell)$ is called an IFI of $\mathcal{N}$.

Example 3.3. *From Example 3.1 Let $\wp, \ell$ be defined as:*

x	$\wp(x)$	$\ell(x)$
0	1.0	0.0
1	0.8	0.1
2	0.6	0.2

These values meet all conditions for both left and right IFI:

- *Subtraction condition*
- *Left/right multiplication difference condition*
- *Order preservation*

Hence, $(\wp, \ell)$ defines both an $IFLI$ and $IFRI$, and is thus an IFI.

Theorem 3.1. *If $\{(\wp_i, \ell_i) : i \in \mathbb{I}\}$ is a family of IFI's of a $PONR\, \mathcal{N}$, then $(\bigvee_{i\in\mathbb{I}} \wp_i, \bigwedge_{i\in\mathbb{I}} \ell_i)$ is also an IFI of $\mathcal{N}$, where $\left(\bigvee_{i\in\mathbb{I}} \wp_i\right)$ and $\left(\bigwedge_{i\in\mathbb{I}} \ell_i\right)$ are defined by:*

$$\left(\bigvee_{i\in\mathbb{I}} \wp_i\right)(x) = \sup\{\wp_i(x) : i \in \mathbb{I}\}, \quad \left(\bigwedge_{i\in\mathbb{I}} \ell_i\right)(x) = \inf\{\ell_i(x) : i \in \mathbb{I}\}, \quad \textit{for all } x \in \mathcal{N}.$$

Proof. Let $\{(\wp_i, \ell_i) : i \in \mathbb{I}\}$ be a family of IFI's of a $PONR\, \mathcal{N}$. For any $x, y, z \in \mathcal{N}$, we have:

(i)

$$\begin{aligned}\left(\bigvee_{i\in I} \wp_i\right)(x-y) &= \sup\{\wp_i(x-y) : i \in I\} \\ &\geq \sup\{\min(\wp_i(x), \wp_i(y)) : i \in I\} \\ &= \min\{\sup \wp_i(x) : i \in I, \sup \wp_i(y) : i \in I\} \\ &= \min\left((\bigvee_{i\in I} \wp_i)(x), (\bigvee_{i\in I} \wp_i)(y)\right),\end{aligned}$$

and

$$\begin{aligned}\left(\bigwedge_{i\in I} \ell_i\right)(x-y) &= \inf\{\ell_i(x-y) : i \in I\} \\ &\leq \inf\{\max(\ell_i(x), \ell_i(y)) : i \in I\} \\ &= \max\{\inf \ell_i(x) : i \in I, \inf \ell_i(y) : i \in I\} \\ &= \max\left((\bigwedge_{i\in I} \ell_i)(x), (\bigwedge_{i\in I} \ell_i)(y)\right).\end{aligned}$$

(ii)

$$\begin{aligned}(\bigvee_{i\in I}\wp_i)(xy) &= \sup\{\wp_i(xy) : i\in I\}\\ &\geq \sup\{\wp_i(y) : i\in I\}\\ &= (\bigvee_{i\in I}\wp_i)(y),\\ (\bigvee_{i\in I}\wp_i)((x+z)y-xy) &= \sup\{\wp_i((x+z)y-xy) : i\in I\}\\ &\geq \sup\{\wp_i(z) : i\in I\}\\ &= (\bigvee_{i\in I}\wp_i)(z),\end{aligned}$$

and similarly,

$$\begin{aligned}(\bigwedge_{i\in I}\ell_i)(xy) &= \inf\{\ell_i(xy) : i\in I\}\\ &\leq \inf\{\ell_i(y) : i\in I\}\\ &= (\bigwedge_{i\in I}\ell_i)(y),\\ (\bigwedge_{i\in I}\ell_i)((x+z)y-xy) &= \inf\{\ell_i((x+z)y-xy) : i\in I\}\\ &\leq \inf\{\ell_i(z) : i\in I\}\\ &= (\bigwedge_{i\in I}\ell_i)(z).\end{aligned}$$

(iii) If $x\leq y$, then

$$\begin{aligned}(\bigvee_{i\in I}\wp_i)(x) &= \sup\{\wp_i(x) : i\in I\}\\ &\geq \sup\{\wp_i(y) : i\in I\}\\ &= (\bigvee_{i\in I}\wp_i)(y),\end{aligned}$$

and

$$\begin{aligned}(\bigwedge_{i\in I}\ell_i)(x) &= \inf\{\ell_i(x) : i\in I\}\\ &\leq \inf\{\ell_i(y) : i\in I\}\\ &= (\bigwedge_{i\in I}\ell_i)(y).\end{aligned}$$

Hence, $(\bigvee_{i\in I}\wp_i, \bigwedge_{i\in I}\ell_i)$ is an IFI of N. □

Theorem 3.2. *An epimorphic pre-image of an IFI of a $PONR$ N is an IFI.*

Proof. Let R_1 and S_1 be $PONR$'s, and let $\phi : R_1 \to S_1$ be an epimorphism. Let (ς, υ) be an IFI of S_1, and let $(\wp, \ell)$ be the pre-image of (ς, υ) under ϕ. We need to verify that $(\wp, \ell)$ satisfies the IFI conditions:

(i) For any $x, y \in R_1$,

$$\wp(x-y) = (\varsigma \circ \phi)(x-y) = \varsigma(\phi(x-y)).$$

Since ϕ is a homomorphism, $\phi(x-y) = \phi(x) - \phi(y)$. Thus:

$$\varsigma(\phi(x-y)) \geq \min(\varsigma(\phi(x)), \varsigma(\phi(y))).$$

Therefore:

$$\wp(x-y) \geq \min(\wp(x), \wp(y)).$$

A similar argument applies to ℓ.

(ii) For any $x, y \in R_1$,

$$\wp(xy) = (\varsigma \circ \phi)(xy) = \varsigma(\phi(xy)).$$

Since $\phi(xy) = \phi(x)\phi(y)$, and (ς, υ) is an IFI of S_1, we have:

$$\varsigma(\phi(xy)) \geq \varsigma(\phi(y)).$$

Therefore:

$$\wp(xy) \geq \wp(y).$$

A similar argument applies to ℓ.

(iii) If $x \leq y$, then $\phi(x) \leq \phi(y)$. Since (ς, υ) is an IFI of S_1, we have:

$$\varsigma(\phi(x)) \geq \varsigma(\phi(y)) \quad \text{and} \quad \upsilon(\phi(x)) \leq \upsilon(\phi(y)).$$

Thus:

$$\wp(x) \geq \wp(y) \quad \text{and} \quad \ell(x) \leq \ell(y).$$

Hence, $(\wp, \ell)$ is an IFI of R_1. □

Theorem 3.3. *Let $(\wp, \ell)$ be an IFI of $PONR\ \mathcal{N}$ and $(\wp^*, \ell^*)$ be an IFS in $\mathcal{N}$ defined by $(\wp^*(x), \ell^*(x)) = (\frac{\wp(x)}{\wp(1)}, \frac{\ell(x)}{\ell(0)})$ for all $x, y, z \in N$. Then $(\wp^*, \ell^*)$is normal IFI of $\mathcal{N}$ containing $(\wp, \ell)$.*

Proof. Let $(\wp, \ell)$ be an IFI of a $PONR\ \mathcal{N}$. For any $x, y, z \in \mathcal{N}$, then

$$\begin{aligned}(i)\wp^*(x-y) &= \frac{\wp(x-y)}{\wp(1)} \\ &\geq \frac{1}{\wp(1)} T((\wp(x), (\wp(y)) \\ &= T(\frac{1}{\wp(1)}\wp(x), \frac{1}{\wp(1)}\wp(y)) \\ &= T(\wp^*(x), \wp^*(y))\end{aligned}$$

and

$$\begin{aligned}\ell^*(x-y) &= \frac{\ell(x-y)}{\ell(0)} \\ &\leq \frac{1}{\ell(0)} S(\ell(x), (\ell(y))) \\ &= S(\frac{1}{\ell(0)}\ell(x), \frac{1}{\ell(0)}\ell(y)) \\ &= S(\ell^*(x), \ell^*(y))\end{aligned}$$

$$
\begin{aligned}
(ii)\ \wp^*(xy) &= \frac{\wp(xy)}{\wp(1)} \\
&\geq \frac{1}{\wp(1)}(\wp(y)) \\
&= \wp^*(y) \text{ and} \\
\wp^*((x+z)y - xy) &= \frac{\wp((x+z)y - xy)}{\wp(1)} \\
&\geq \frac{1}{\wp(1)}(\wp(z)) \\
&= \wp^*(z)\ .
\end{aligned}
$$

and

$$
\begin{aligned}
\ell^*(xy) &= \frac{\ell(xy)}{\ell(0)} \\
&\leq \frac{1}{\ell(0)}(\ell(y)) \\
&= \ell^*(y) \text{ and} \\
\ell^*((x+z)y - xy) &= \frac{\ell((x+z)y - xy)}{\ell(0)} \\
&\leq \frac{1}{\ell(0)}(\ell(z)) \\
&= \ell^*(z)\ .
\end{aligned}
$$

$$
\begin{aligned}
(iii)\ x \leq y = \wp^*(x) &= \frac{\wp(x)}{\wp(1)} \\
&\geq \frac{\wp(y)}{\wp(1)} \\
&= \wp^*(y).
\end{aligned}
$$

and

$$
\begin{aligned}
x \leq y = \ell^*(x) &= \frac{\ell(x)}{\ell(0)} \\
&\leq \frac{\ell(y)}{\ell(0)} \\
&= \ell^*(y).
\end{aligned}
$$

Hence $(\wp^*, \ell^*)$ is an IFI of $\mathcal{N}$. Clearly $\wp^*(1) = 1$ $\ell^*(0) = 0$ and $(\wp, \ell) \subset (\wp^*, \ell^*)$. □

Lemma 3.1. *Let* R_1 *and* S_1 *be a* $PONR's$ *and* $\varphi : R_1 \to S_1$ *is a homomorphism. Let* $(\wp, \ell)$ *be* φ*-invariant* IFI *of* R_1*. If* $x = \varphi(a)$ *, then* $\varphi(\wp, \ell)(x) = (\wp(a), \ell(a))$ $\forall a \in R$*.*

Theorem 3.4. *Let* $\phi : R_1 \to S_1$ *be an epimorphism of PONRs* R_1 *and* S_1*. If* $(\wp, \ell)$ *is a* ϕ*-invariant IFI of* R_1*, then* $\phi(\wp, \ell)$ *is an IFI of* S_1*.*

Proof. Let $(\wp, \ell)$ be a ϕ-invariant IFI of R_1. Define $\phi(\wp, \ell)$ on S_1 by:

$$\phi(\wp)(a) = \wp(\phi^{-1}(a)), \quad \phi(\ell)(a) = \ell(\phi^{-1}(a)), \quad \forall a \in S_1.$$

We need to verify that $\phi(\wp, \ell)$ satisfies the IFI conditions:

(i) For any $a, b \in S_1$,

$$\phi(\wp)(a - b) = \wp(\phi^{-1}(a - b)).$$

Since ϕ is a homomorphism, $\phi^{-1}(a - b) = \phi^{-1}(a) - \phi^{-1}(b)$. Thus:

$$\wp(\phi^{-1}(a - b)) \geq \min(\wp(\phi^{-1}(a)), \wp(\phi^{-1}(b))).$$

Therefore:

$$\phi(\wp)(a - b) \geq \min(\phi(\wp)(a), \phi(\wp)(b)).$$

A similar argument applies to ℓ.

(ii) For any $a, b \in S_1$,

$$\phi(\wp)(ab) = \wp(\phi^{-1}(ab)).$$

Since $\phi^{-1}(ab) = \phi^{-1}(a)\phi^{-1}(b)$, and $(\wp, \ell)$ is an IFI of R_1, we have:

$$\wp(\phi^{-1}(ab)) \geq \wp(\phi^{-1}(b)).$$

Therefore:

$$\phi(\wp)(ab) \geq \phi(\wp)(b).$$

A similar argument applies to ℓ.

(iii) If $a \leq b$, then $\phi^{-1}(a) \leq \phi^{-1}(b)$. Since $(\wp, \ell)$ is an IFI of R_1, we have:

$$\wp(\phi^{-1}(a)) \geq \wp(\phi^{-1}(b)) \quad \text{and} \quad \ell(\phi^{-1}(a)) \leq \ell(\phi^{-1}(b)).$$

Thus:

$$\phi(\wp)(a) \geq \phi(\wp)(b) \quad \text{and} \quad \phi(\ell)(a) \leq \phi(\ell)(b).$$

Hence, $\phi(\wp, \ell)$ is an IFI of S_1. □

Theorem 3.5. *Let $\varphi : R_1 \to S_1$ be an epimorphism of a $PONR's$ R_1 and S_1. If $(\wp, \ell)$ is φ- invariant IFI of R_1, then $\varphi(\wp, \ell)$ is an IFI of S_1.*

Proof. Let $a, b, c \in S_1$. Then $\exists x, y, z \in R_1 \ni \varphi(x) = a, \varphi(y) = b$ and $\varphi(z) = c$. suppose $(\wp, \ell)$ is φ-invariant IFI of R_1. Then we have

$$\begin{aligned} (i)\ \varphi(\wp)(a - b) &= \varphi(\wp)(\varphi(x) - \varphi(y)) \\ &= \varphi(\wp)\varphi(x - y)) \\ &= \wp(x - y) \\ &\geq min(\wp(x), \wp(y)) \\ &= min(\varphi(\wp)(a), \varphi(\wp)(b)). \end{aligned}$$

and

$$\begin{aligned} \varphi(\ell)(a - b) &= \varphi(\ell)(\varphi(x) - \varphi(y)) \\ &= \varphi(\ell)\varphi(x - y)) \\ &= \ell(x - y) \\ &\leq max(\ell(x), \ell(y)) \\ &= max(\varphi(\ell)(a), \varphi(\ell)(b)). \end{aligned}$$

$$\begin{aligned}
(ii)\ \varphi(\wp)(ab) &= \varphi(\wp)(\varphi(x)\varphi(y)) \\
&= \varphi(\wp)\varphi(xy)) \\
&= \wp(xy) \\
&\geq \wp(x) \\
&= \varphi(\wp)(b) \text{ and} \\
\varphi(\wp)((a+b)c-ab)) &= \varphi(\wp)(\varphi(x+y)z-\varphi(xy)) \\
&= \varphi(\wp)(\varphi(x)+\varphi(y))\varphi(z)-\varphi(x)\varphi(y)) \\
&= \varphi(\wp)(\varphi(x+y)\varphi(z)-\varphi(x)\varphi(y)) \\
&= \wp((x+y)z-xy) \\
&\geq \wp(z) \\
&= \varphi(\wp)(c)
\end{aligned}$$

and

$$\begin{aligned}
\varphi(\ell)(ab) &= \varphi(\ell)(\varphi(x)\varphi(y)) \\
&= \varphi(\ell)\varphi(xy)) \\
&= \ell(xy) \\
&\leq \ell(x) \\
&= \varphi(\ell)(b) \text{ and} \\
\varphi(\ell)((a+b)c-ab)) &= \varphi(\ell)(\varphi(x+y)z-\varphi(xy)) \\
&= \varphi(\ell)(\varphi(x)+\varphi(y))\varphi(z)-\varphi(x)\varphi(y)) \\
&= \varphi(\ell)(\varphi(x+y)\varphi(z)-\varphi(x)\varphi(y)) \\
&= \ell((x+y)z-xy) \\
&\leq \ell(z) \\
&= \varphi(\ell)(c)
\end{aligned}$$

$$\begin{aligned}
(iii)\ \text{Let } a \leq b &= \varphi(\wp)(a) \\
&= \varphi(\wp)(\varphi(x)) \\
&= \wp(x) \\
&\geq \wp(y) \\
&= \varphi(\wp)(b).
\end{aligned}$$

and

$$\begin{aligned}
\text{Let } a \leq b &= \varphi(\ell)(a) \\
&= \varphi(\ell)(\varphi(x)) \\
&= \ell(x) \\
&\leq \ell(y) \\
&= \varphi(\ell)(b).
\end{aligned}$$

Hence $\varphi(\wp, \ell)$ is an IFI of S. □

Theorem 3.6. *Let $(\wp, \ell)$ be an $IFLI$ of $PONR\ \mathcal{N}$ and $\wp^+(x) = \wp(x) + 1 - \wp(0)$ and $\ell^+(x) = \ell(x) - \ell(0)\ \forall x \in \mathcal{N}$. Then $(\wp^+, \ell^+)$ is a normal $IFLI$ of $\mathcal{N}$ containing $(\wp, \ell)$, provided t- norm holds for combined translation.*

Proof. Let $(\wp, \ell)$ be an $IFLI$ of $PONR\ \mathcal{N}$. We have $(\wp^+(x), \ell^+(x)) = (\wp(x) + 1 - \wp(0), \ell(x) - \ell(0))\ \forall x \in N$. Put $1 - \wp(0) = a,\ -\ell(0) = a$ then $(\wp^+(x), \ell^+(x)) = (\wp(x) + a, \ell(x) + a)$ and hence $(\wp^+(x), \ell^+(x)) = (\wp_a^T.\wp^+, \ell_a^T.\ell^+)$ is an $IFLI$ of $\mathcal{N}$. By definition of $(\wp^+, \ell^+), (\wp \leq \wp^+, \ell \geq \ell^+)$ and $(\wp^+(0), \ell^+(0)) = (\wp(0) + 1 - \wp(0), \ell(0) - \ell(0))$ and hence $\wp^+(0) = 1,\ \ell^+(0) = 0$. Therefore $(\wp^+,\ \ell^+)$ is a normal $IFLI$ of $\mathcal{N}$. □

Theorem 3.7. *Let ψ be an IFS of $PONR\ \mathcal{N}$. Then ψ is an $IFLI$ of a $PONR\ \mathcal{N}$ iff the strongest intuitionistic fuzzy relation $(\wp_\psi, \ell_\psi)$ on $\mathcal{N}$ is an $IFLI$ of $PONR\ \mathcal{N} \times \mathcal{N}$.*

Proof. Suppose that ψ is an $IFLI$ of $PONR\ \mathcal{N}$ then obviously $(\wp_\psi, \ell_\psi)$ is an $IFLI$ of a $PONR\ \mathcal{N} \times \mathcal{N}$, for any $(x_1, x_2), (y_1, y_2) \in \mathcal{N} \times \mathcal{N}$. Then

$$\begin{aligned}
\wp_\psi(x_1, x_2) &= T(\psi(x_1), \psi(x_2)) \\
&\geq T(T(\psi(x_1 - y_1), \psi(y_1)), T(T(\psi(x_2 - y_2), \psi(y_2))) \\
&= T(\wp_\psi(x_1 - y_1,\ x_2 - y_2),\ \wp_\psi(y_1 - y_2)) \\
T(\wp_\psi(x_1,\ x_2),\ \wp_\psi(x_1,\ x_2) &= T(T(\psi(x_1),\ \psi(x_2),\ T(\psi(x_1), \psi(x_2)) \\
&= T(T(\psi(x_1), \psi(x_1)), T(\psi(x_2), \psi(x_2))) \\
&= T(\psi(x_1), \psi(x_2)) = \wp_\psi(x_1, x_2)
\end{aligned}$$

$$\begin{aligned}
\ell_\psi(x_1, x_2) &= S(\psi(x_1), \psi(x_2)) \\
&\leq S(S(\psi(x_1 - y_1), \psi(y_1)), S(S(\psi(x_2 - y_2), \psi(y_2))) \\
&= S(\ell_\psi(x_1 - y_1,\ x_2 - y_2),\ \ell_\psi(y_1 - y_2)) \\
S(\ell_\psi(x_1,\ x_2),\ \ell_\psi(x_1,\ x_2) &= S(S(\psi(x_1),\ \psi(x_2),\ S(\psi(x_1), \psi(x_2)) \\
&= S(S(\psi(x_1), \psi(x_1)), S(\psi(x_2), \psi(x_2))) \\
&= S(\psi(x_1), \psi(x_2)) = \ell_\psi(x_1, x_2)
\end{aligned}$$

Suppose $(x_1, x_2), (y_1, y_2) \in \mathcal{N} \times \mathcal{N}$ and $(x_1, x_2) \leq (y_1, y_2)$ then $x_1 \leq y_1$ and $x_2 \leq y_2$. Therefore $T(\psi(x_1), \psi(x_2)) \geq T(\psi(y_1), \psi(y_2)),\ S(\psi(x_1), \psi(x_2)) \leq S(\psi(y_1), \psi(y_2))$. Hence $\wp_\psi(x_1,\ x_2) \geq \wp_\psi(y_1,\ y_2),\ \ell_\psi(x_1,\ x_2) \leq \ell_\psi(y_1,\ y_2)$. Thus $(\wp_\psi, \ell_\psi)$ is an $IFLI$ of a $PONR$. Let $x, y \in \mathcal{N}$. Then

$$\begin{aligned}
(i)\ \psi(x - y) &= T(\psi(x - y), \psi(x - y)) = \wp_\psi(x - y, x - y) = \wp_\psi((x, x) - (y, y)) \\
&\geq T(\wp_\psi(x, x), (y, y)) = T(T(\psi(x), \psi(x)), T(\psi(y), \psi(y))) \\
&= T(\wp_\psi((x, y), \wp_\psi(x, y)) = T(\wp_\psi(x, y)) = T(\psi(x), \psi(y))
\end{aligned}$$

$$\begin{aligned}
\psi(x - y) &= S(\psi(x - y), \psi(x - y)) = \ell_\psi(x - y, x - y) = \ell_\psi((x, x) - (y, y)) \\
&\leq S(\ell_\psi(x, x), (y, y)) = S(S(\psi(x), \psi(x)), S(\psi(y), \psi(y))) \\
&= S(\ell_\psi((x, y), \ell_\psi(x, y)) = S(\ell_\psi(x, y)) = S(\psi(x), \psi(y))
\end{aligned}$$

$$
\begin{aligned}
(ii)\ \psi(xy) &= T(\psi(xy),\psi(xy)) = \wp_\psi(xy,\ xy)\\
&= \wp_\psi((x,x)(y,y)) \geq T(\wp_\psi(y,y)) = T(\psi(y),\psi(y)) = \psi(y) \text{ and}\\
\psi(x) &= T(\psi(x),\psi(x)) = \wp_\psi(x)\\
&\geq T(\wp_\psi(x-y,\ x-y),\ \wp_\psi(y,\ y))\\
&= T(T(\psi(x-y,\psi(x-y)),T(\psi(y),\psi(y))) = T(\psi(x-y),\psi(y))
\end{aligned}
$$

$$
\begin{aligned}
\psi(xy) &= S(\psi(xy),\psi(xy)) = \ell_\psi(xy,\ xy)\\
&= \ell_\psi((x,x)(y,y)) \leq S(\ell_\psi(y,y)) = S(\psi(y),\psi(y)) = \psi(y) \text{ and}\\
\psi(x) &= S(\psi(x),\psi(x)) = \ell_\psi(x)\\
&\leq S(\ell_\psi(x-y,\ x-y),\ \ell_\psi(y,\ y))\\
&= S(S(\psi(x-y,\psi(x-y)),S(\psi(y),\psi(y))) = S(\psi(x-y),\psi(y))
\end{aligned}
$$

$$
\begin{aligned}
(iii) \text{ Let } x,y\in\mathcal{N} \text{ and } x\leq y, \text{then}(x,x) &\leq (y,y)\\
\wp_\psi(x,\ x) &\geq \wp_\psi(y,\ y)\\
T(\psi(x),\psi(x)) &\geq T(\psi(y),\psi(y))\\
\text{Therefore } \psi(x) &\geq \psi(y)
\end{aligned}
$$

$$
\begin{aligned}
\text{Let } x,y\in\mathcal{N} \text{ and } x\leq y, \text{then}(x,x) &\leq (y,y)\\
\ell_\psi(x,\ x) &\leq \ell_\psi(y,\ y)\\
S(\psi(x),\psi(x)) &\leq S(\psi(y),\psi(y))\\
\text{Therefore } \psi(x) &\leq \psi(y)
\end{aligned}
$$

.

Hence ψ is a $IFLI$ of a $PONR\,\mathcal{N}$. □

4. Conclusion

In this paper, we introduced and explored the concept of intuitionistic fuzzy ideals (IFI) within the framework of partially ordered near-rings ($PONR$), systematically investigating their properties and characterizations to establish a comprehensive theoretical foundation for their application in algebraic structures. Key results were established, including the preservation of the IFI structure under union and intersection operations, demonstrating its robustness. Additionally, it was proven that the pre-image and image of IFI under epimorphisms retain their ideal properties, highlighting compatibility with homomorphic mappings. Normalization techniques for IFI were developed, ensuring adaptability to broader contexts while maintaining defining characteristics. Furthermore, the relationship between IFI and the strongest intuitionistic fuzzy relations was analyzed, reinforcing their role in extending classical algebraic concepts to fuzzy settings. These findings contribute significantly to the growing body of knowledge on intuitionistic fuzzy algebraic structures, particularly in the context of near-rings and partially ordered systems, generalizing existing theories and paving the way for further research into applications such as decision-making, coding theory, and fuzzy automata.

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