TWMS J. App. and Eng. Math. V.16, N.6, 2026, pp. 696-712

# MODULI OF CONTINUITY FOR FUNCTIONS IN SOBOLEV SPACES AND HAAR WAVELET SOLUTIONS TO FRACTIONAL BASSET EQUATION

S. LAL[1], ABHILASHA[2*], §

ABSTRACT. In this paper, the method for solving fractional differential equations have been proposed using the Haar wavelet operational matrix of fractional integration. An operational matrix of fractional integration using the Haar wavelet is designed to solve a linear multi-term fractional differential equation as well as a system of fractional differential equations. The Basset equation for different fractional orders and a system of fractional differential equations both have been solved in order to validate and show the viability of the proposed method. Furthermore, it has also been demonstrated to approximate functions in Sobolev space via the Haar wavelet approach with the help of moduli of continuity. By treating fractional differential equations as a set of algebraic equations, this study significantly advances both the moduli of continuity and numerical solutions of fractional differential equations.

Keywords: Haar wavelet, moduli of continuity, wavelet approximations, Haar wavelet operational matrix of fractional integration, Sobolev space.

AMS Subject Classification: 26A33, 42C40, 46E35, 65T60, 65L10, 65H10.

## 1. INTRODUCTION

Fractional calculus [28] has become an exciting field of mathematics for solving diverse problems arising in nature. Unlike the classical calculus, it involves not only instant time but also previous time details. As a consequence, this newly emerging field has wide applications [25] in viscoelastic systems, plasma physics, fluid mechanics, electrochemistry, signal processing, mathematical biology, thermodynamics, aerodynamics, and many more, see [22, 29, 30, 2, 20, 17, 11, 27, 21]. Wavelet theory, with its inherent multiresolution analysis and time-frequency localization, has proven to be a versatile framework for tackling such problems. Unlike traditional Fourier methods, wavelets provide sparse representations of functions, making them ideal for approximating solutions to FDEs. The

[1] Department of Mathematics, Faculty of Science, Banaras Hindu University, Varanasi, India.
e-mail: shyam_lal@rediffmail.com; https://orcid.org//0000-0001-8598-2207.

[2] Department of Mathematics, Institute of Integrated & Honors Studies, Kurukshetra University, Kurukshetra, India.
e-mail: yadavabhilasha1942@kuk.ac.in; https://orcid.org/0000-0012-3456-789X.

[*] Corresponding author.

§ Manuscript received: April 09, 2025; accepted: September 09, 2025.
TWMS Journal of Applied and Engineering Mathematics, Vol.16, No.6; 

Haar wavelet, in particular, stands out due to its simplicity, orthogonality and compact support, enabling efficient numerical implementations.

In the recent years, researchers have begun to depict a variety of physical phenomena using fractional differential equations. After framing the mathematical model of a problem as a fractional differential equation, the main task is to find out the solution of the equation and study its properties. Despite their broad applicability, solving FDEs analytically remains a significant challenge, as many fractional-order problems lack exact solutions. Over the years, various methods like finite difference [24], differential transform [1], Adomian decomposition [12], operational matrices [5, 31] and Laplace transforms [26] have been proposed. Also, due to the complexity of fractional problems over integer-ordered problems, solutions often do not exist in classical spaces of functions. While traditional function spaces like Lipschitz [11] and Hölder spaces Hölder [6] are restrictive, Sobolev spaces provide a more flexible framework for analyzing fractional differential equations.

The objectives of this paper are to:

(i) establish an estimate of functions in Sobolev space $H_h^s(\mathbb{R})$ using family of Haar wavelets as function bases.

(ii) construct operational matrix of fractional integration.

(iii) solve fractional Basset equation [4] by converting the differential equation into algebraic one.

(iv) extend the operational matrix method to solve a system of fractional differential equations.

This paper is organised as follows. Section 2. contains some definitions and preliminaries used in this paper. The convergence analysis and the error bounds using the moduli of continuity of functions in homogeneous Sobolev space $H_h^s(\mathbb{R})$ are provided in Section 3. In Section 4, the basis functions and operational matrix of fractional integration are calculated using Haar wavelet. Section 5 discusses the algorithm to solve Basset equation using the Haar wavelet operational matrix of fractional integration. The numerical examples of the Basset problem and a system of fractional differential equations are solved in Section 6. Discussions about the consequences of the proposed are given in Section 7. Section 8 contains the main results and concluding remarks and lastly the references have been written which are used in framing this paper.

## 2. Definitions and Fundamentals

In this section, an overview of Haar wavelet, Sobolev space, fractional calculus and fractional operational matrix have been given.

**Definition 2.1.** *Haar wavelet on the interval* $[0,1)$ *is a square wave function* $\psi(t) \in L^2[0,1)$ *defined as,*

$$\psi(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2}, \\ -1, & \frac{1}{2} \le t < 1, \\ 0, & \text{otherwise,} \end{cases}$$

*and*

$$\phi(t) = \begin{cases} 1, & 0 \le t < 1, \\ 0, & \text{otherwise.} \end{cases}$$

*Here,* $\phi(t)$ *and* $\psi(t)$ *are called Haar scaling function and Haar wavelet mother function respectively* [17].

*The family* $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ *of Haar wavelets is generated by using translation and dilation of mother wavelet function as,*

$$\psi_{j,k} = 2^{\frac{j}{2}}\psi(2^{\frac{j}{2}}t - k),$$

*where* $j = 0, 1, 2, ...$ *and* $k = 0, 1, 2, ..., 2^j - 1$.

**Definition 2.2.** *The Riemann-Lioville fractional integral operator* [10] *of a function* $f(t)$ *is defined as,*

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-x)^{\alpha-1}f(x)dx, \qquad \alpha > 0, t > 0,$$

*where* $\Gamma(\alpha) = \int_0^\infty z^{\alpha-1}e^{-z}dz$, *is the Gamma function.*

**Definition 2.3.** *The fractional derivative of a function* $f(t)$ *in Caputo sense* [3] *is defined as,*

$$D^{\alpha}f(t) = I^{n-\alpha}D^n f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-x)^{n-\alpha-1}f^{(n)}(t)dt, \quad n-1 < \alpha \leq n, n \in \mathbb{Z}.$$

**Definition 2.4.** *The Mechee fractional* $\alpha$*-derivative* [23] *of a real function* $f(t) : [a, \infty) \to \mathbb{R}$ *is defined as,*

$$f^{\alpha}(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t - \epsilon t^{1-\alpha})}{2\epsilon}.$$

***Remark:*** *(i)* $f^{\alpha}(t) = t^{1-\alpha}f'(t)$.
*(ii)* $f^1(t) = f'(t)$.
*(iii) Comparison of Caputo and Mechee fractional derivative:*
*Taking* $f(t) = e^{ct}$,

$$\begin{aligned} \text{Caputo derivative, } D^{\alpha}f(t) &= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-x)^{n-\alpha-1}\frac{d^n}{dt}e^{ct}dt \\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{1}{(t-x)^{\alpha}}\frac{d}{dt}e^{ct}dt, \quad n = 1, \\ &= c^{\alpha}e^{ct}. \\ \text{Mechee derivative, } f^{\alpha}(t) &= \lim_{\epsilon \to 0}\frac{e^{c(t+\epsilon t^{1-\alpha})} - e^{c(t-\epsilon t^{1-\alpha})}}{2\epsilon} \\ &= ct^{1-\alpha}e^{ct}. \end{aligned}$$

**Definition 2.5.** *A function* $f(t) \in L^2(\mathbb{R})$ *is said to be in Sobolev space* $H^s(\mathbb{R}), s > 0$, [14] *if*

$$\int_{\mathbb{R}} |\hat{f}(\omega)|^2(1 + |\omega|^2)^s d\omega < \infty,$$

*and a function* $f(t) \in L^2(\mathbb{R})$ *is said to be in homogeneous Sobolev space* $H_h^s(\mathbb{R})$, $s > 0$, *if*

$$\int_{\mathbb{R}} |\hat{f}(\omega)|^2|\omega|^{2s}d\omega < \infty,$$

*where* $\hat{f}(\omega) = \int_{\mathbb{R}} f(t)e^{-i\omega t}dt$, *is the Fourier transform of function* $f(t)$.

**Definition 2.6.** *The modulus of continuity* [8] *of a function* $f \in L^2[0, 1)$ *is defined as,*

$$\begin{aligned} W(f, \delta) &= \sup_{0<h\leq\delta} ||f(t+h) - f(t)||_2, \qquad \forall t \in [0, 1), \\ &= \sup_{0<h\leq\delta} \left(\int_0^1 |f(t+h) - f(t)|^2 dt\right)^{1/2}. \end{aligned}$$

**Definition 2.7.** *A function $f(t) \in L^2[0,1)$ can be expanded as Haar wavelet series as*

$$f(t) = \sum_{k_1=0}^{2^N-1} d_{N,k_1}\phi_{N,k_1}(t) + \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} c_{j,k}\psi_{j,k}(t), \ N \text{ is a non-negative integer } [8].$$

*Here,*

$$\phi_{N,k_1}(t) = \frac{1}{\sqrt{n}}\begin{cases} 2^{\frac{N}{2}}, & \frac{k_1}{n} \le t < \frac{k_1+1}{n}, \\ 0, & \text{otherwise,} \end{cases}$$

*where $k_1 = 0,1,2,3,...,2^N-1$, $n = 2^N$ and*

$$\psi_{j,k}(t) = \frac{1}{\sqrt{m}}\begin{cases} 2^{\frac{j}{2}}, & \frac{k}{m} \le t < \frac{k+\frac{1}{2}}{m}, \\ -2^{\frac{j}{2}}, & \frac{k+\frac{1}{2}}{m} \le t < \frac{k+1}{m}, \\ 0, & \text{otherwise,} \end{cases}$$

*where $k = 0,1,2,...,2^j-1$, $m = 2^j$ and $j = 0,1,2,...J$, $J$ is the maximum level of resolution* [16].
*Since, $f(t) = (P_N f)(t) + \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} c_{j,k}\psi_{j,k}(t)$, $N$ is a non-negative integer, $f(t) - (P_N f)(t) = \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} c_{j,k}\psi_{j,k}(t)$ gives the Haar wavelet approximation of the function $f$ by $P_N(f)$, where $c_{j,k} = \langle f, \psi_{j,k}\rangle$.*

## 3. Convergence Analysis

The following convergence theorem has been proved in this section:

**Theorem 3.1.** *If a function $f \in H_h^s(\mathbb{R})$, $s > 0$, a homogeneous Sobolev space, is expanded as Haar wavelet series as,*

$$f(t) = \sum_{k_1=0}^{2^N-1} d_{N,k_1}\phi_{N,k_1}(t) + \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} c_{j,k}\psi_{j,k}(t),$$

*then the series converges uniformly to $f(t)$ in $H_h^s(\mathbb{R})$.*

*Proof.* From previous section, $f(t) - (P_N f)(t) = \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} c_{j,k}\psi_{j,k}(t)$.

$$\begin{aligned} ||f - P_N f||_2^2 &= \left\| \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} \langle f, \psi_{j,k}\rangle \psi_{j,k}(t) \right\|_2^2 \\ &= \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} |c_{j,k}|^2 \le ||f||_2^2, \qquad \text{(By Bessel's inequality)} \\ &\Longrightarrow \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} |c_{j,k}|^2 \to 0 \ \text{ as } \ N \to 0. \end{aligned}$$

Therefore, $||f - P_N f||_2^2 \to 0$ as $N \to 0$.
$\Longrightarrow f - P_N f \to 0$ as $N \to 0$.
Hence, the series $\sum_{k_1=0}^{2^N-1} d_{N,k_1}\phi_{N,k_1}(t) + \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} c_{j,k}\psi_{j,k}(t)$ converges uniformly to $f(t)$. □

**Theorem 3.2.** *If a function $f \in H_h^s(\mathbb{R})$, $s > 0$, a homogeneous Sobolev space, is expanded as Haar wavelet series as described in section (2.6), then the moduli of continuity $W\left(f - P_N f, \frac{1}{2^j}\right)$ of $f$ by $P_N f$, under norm $||.||_2$, is given by*

$$W\left(f - P_N f, \frac{1}{2^j}\right) = \begin{cases} O(1), & s \le \frac{1}{2}, \\ O\left(\frac{1}{2^{N(s-\frac{1}{2})}}\right), & \frac{1}{2} < s < \frac{3}{2}, \\ O\left(\frac{1}{2^N}\right) & s \ge \frac{3}{2}. \end{cases}$$

*Proof.* For $j, k \in \mathbb{Z}$,

$$c_{j,k} = \langle f, \psi_{j,k}\rangle = \frac{1}{2\pi}\langle \hat{f}, \hat{\psi}_{j,k}\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\overline{\hat{\psi}_{j,k}(\omega)}d\omega.$$

Now,

$$\begin{aligned} \hat{\psi}_{j,k}(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega t}\psi_{j,k}(t)dt \\ &= \left(\frac{2^{\frac{j}{2}}}{-i\omega}\right)\left[2e^{\frac{-i\omega(k+\frac{1}{2})}{2^j}} - e^{\frac{-i\omega k}{2^j}} - e^{\frac{-i\omega(k+1)}{2^j}}\right] \\ &= \frac{4i}{\omega}2^{\frac{j}{2}}e^{\frac{-i\omega(2k+1)}{2^{j+1}}}sin^2(\frac{\omega}{2^{j+2}}). \end{aligned}$$

$$\begin{aligned} \text{This gives, } |c_{j,k}| &= \left|\frac{2^{\frac{j}{2}+1}}{\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)\frac{e^{\frac{i\omega(2k+1)}{2^{j+1}}}}{\omega}sin^2(\frac{\omega}{2^{j+2}})d\omega\right| \\ &\le \frac{2^{\frac{j}{2}+1}}{\pi}\left(\int_{-\infty}^{\infty}|\hat{f}(\omega)|^2|\omega|^{2s}d\omega\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}\left|\frac{sin^4(\frac{\omega}{2^{j+2}})}{\omega^{2(s+1)}}\right|d\omega\right)^{\frac{1}{2}} \\ &\le \frac{C.2^{\frac{j}{2}+1}}{\pi}\left(2\int_0^{\infty}\frac{sin^4(\frac{\omega}{2^{j+2}})}{|\omega|^{2(s+1)}}d\omega\right)^{\frac{1}{2}}, \\ |c_{j,k}|^2 &\le \frac{C^2 2^{j+3}}{\pi^2}\int_0^{\infty}\frac{sin^4(\frac{\omega}{2^{j+2}})}{|\omega|^{2(s+1)}}d\omega \\ &= \frac{C^2 2^{j+3}}{\pi^2}\left[\int_0^{2^{j+2}}\frac{sin^4(\frac{\omega}{2^{j+2}})}{|\omega|^{2(s+1)}}d\omega + \int_{2^{j+2}}^{\infty}\frac{sin^4(\frac{\omega}{2^{j+2}})}{|\omega|^{2(s+1)}}d\omega\right] \\ &= I_1 + I_2. \end{aligned}$$

Now,

$$\begin{aligned} I_1 &= \frac{C^2 2^{j+3}}{\pi^2}\int_0^{2^{j+2}}\frac{sin^4(\frac{\omega}{2^{j+2}})}{|\omega|^{2(s+1)}}d\omega \\ &\le \frac{C^2 2^{j+3}}{\pi^2}\int_0^{2^{j+2}}\frac{(\frac{\omega}{2^{j+2}})^4}{|\omega|^{2(s+1)}}d\omega \\ &= \frac{C^2}{\pi^2}\frac{1}{2^{2sj}}\frac{1}{2^{4s-1}}\frac{1}{3-2s}, \quad \text{for } s < \frac{3}{2}, \end{aligned}$$

$$\begin{aligned}
\text{and } I_2 &= \frac{C^2 2^{j+3}}{\pi^2}\int_{2^{j+2}}^{\infty}\frac{sin^4(\frac{\omega}{2^{j+2}})}{|\omega|^{2(s+1)}}d\omega\\
&\le \frac{C^2 2^{j+3}}{\pi^2}\int_{2^{j+2}}^{\infty}\frac{d\omega}{|\omega|^{2(s+1)}}\\
&= \frac{C^2}{\pi^2}\frac{1}{2^{2sj}}\frac{1}{2^{4s-1}}\frac{1}{2s+1}, \quad \text{for } s>\frac{1}{2}.
\end{aligned}$$

$$|c_{j,k}|^2 \le \frac{C^2}{\pi^2}\frac{1}{2^{2sj}}\frac{1}{2^{4s-1}}\left[\frac{1}{3-2s}+\frac{1}{2s+1}\right], \quad \text{for } \frac{1}{2}<s<\frac{3}{2}.$$

$$\begin{aligned}
\text{Therefore, } ||f-P_Nf||_2^2 &= \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1}|c_{j,k}|^2\\
&\le \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1}\frac{C^2}{\pi^2}\frac{1}{2^{2sj}}\frac{1}{2^{4s-1}}\left(\frac{1}{3-2s}+\frac{1}{2s+1}\right)\\
&= O\left(\frac{1}{2^{N(2s-1)}}\right), \quad \text{for } \frac{1}{2}<s<\frac{3}{2}.\\
\text{Hence, } ||f-P_Nf||_2 &= O\left(\frac{1}{2^{N(s-\frac{1}{2})}}\right), \quad \text{for } \frac{1}{2}<s<\frac{3}{2}.
\end{aligned}$$

Now,

$$\begin{aligned}
W\left(f-P_Nf,\frac{1}{2^j}\right) &= \sup_{0<h\le\frac{1}{2^j}}||(f-P_Nf)(t+h)-(f-P_Nf)(t)||_2\\
&\le \sup_{0<h\le\frac{1}{2^j}}[||(f-P_Nf)(t+h)||_2+||(f-P_Nf)(t)||_2]\\
&= O\left(\frac{1}{2^{N(s-\frac{1}{2})}}\right), \quad \text{for } \frac{1}{2}<s<\frac{3}{2}.
\end{aligned}$$

(ii) When $s\le\frac{1}{2}$.

For $s=\frac{1}{2}$,

$$\begin{aligned}
|c_{j,k}|^2 &\le \frac{C^2 2^{j+3}}{\pi^2}\int_0^{\infty}\frac{sin^4(\frac{\omega}{2^{j+2}})}{|\omega|^3}d\omega\\
&= \frac{C^2 2^{j+3}}{\pi^2}\left[\int_0^{2^{j+2}}\frac{sin^4(\frac{\omega}{2^{j+2}})}{|\omega|^3}d\omega+\int_{2^{j+2}}^{\infty}\frac{sin^4(\frac{\omega}{2^{j+2}})}{|\omega|^3}d\omega\right]\\
&= \frac{C^2 2^{j+3}}{\pi^2}(J_1+J_2),\\
\text{where } J_1 &= \int_0^{2^{j+2}}\frac{sin^4(\frac{\omega}{2^{j+2}})}{|\omega|^3}d\omega \le \int_0^{2^{j+2}}\frac{(\frac{\omega}{2^{j+2}})^4}{|\omega|^3}d\omega=\frac{2^{2j+4}}{2(2^{4j+8})},\\
\text{and } J_2 &= \int_{2^{j+2}}^{\infty}\frac{sin^4(\frac{\omega}{2^{j+2}})}{|\omega|^3}d\omega \le \int_{2^{j+2}}^{\infty}\frac{1}{|\omega|^3}d\omega=\frac{2}{2^{2j+4}}.\\
\text{Therefore, } |c_{j,k}|^2 &\le \frac{5C^2}{2\pi^2 2^j}.
\end{aligned}$$

$$\begin{aligned}
\text{Now,}\ \ ||f-P_Nf||_2^2 &= \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1}|c_{j,k}|^2 \le \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1}\frac{5C^2}{2\pi^2 2^j}\\
&= \sum_{j=N}^{\infty}\frac{5C^2 2^j}{2\pi^2 2^j} = O(1).
\end{aligned}$$

$$\begin{aligned}
\text{Hence,}\ \ W\left(f-P_Nf,\frac{1}{2^j}\right) &= \sup_{0<h\le\frac{1}{2^j}} ||(f-P_Nf)(t+h)-(f-P_Nf)(t)||_2\\
&\le \sup_{0<h\le\frac{1}{2^j}} [||(f-P_Nf)(t+h)||_2 + ||(f-P_Nf)(t)||_2]\\
&= O(1), \quad \text{for } s\le\frac{1}{2}.
\end{aligned}$$

(iii) When $s\ge\frac{3}{2}$.
For $s=\frac{3}{2}$,

$$\begin{aligned}
|c_{j,k}|^2 &\le \frac{C^2 2^{j+3}}{\pi^2}\int_0^{\infty}\frac{sin^4(\frac{\omega}{2^{j+2}})}{|\omega|^5}d\omega\\
&= \frac{C^2 2^{j+3}}{\pi^2}\left[\int_0^{2^{j+2}}\frac{sin^4(\frac{\omega}{2^{j+2}})}{|\omega|^5}d\omega + \int_{2^{j+2}}^{\infty}\frac{sin^4(\frac{\omega}{2^{j+2}})}{|\omega|^5}d\omega\right]\\
&= \frac{C^2 2^{j+3}}{\pi^2}(S_1+S_2),\\
\text{where}\ \ S_1 &= \int_0^{2^{j+2}}\frac{sin^4(\frac{\omega}{2^{j+2}})}{|\omega|^5}d\omega \le \int_{\epsilon>0}^{2^{j+2}}\frac{(\frac{\omega}{2^{j+2}})^4}{|\omega|^5}d\omega, \quad \omega>0,\\
&= O\left(\frac{1}{2^{4j}}\right),\\
\text{and}\ \ S_2 &= \int_{2^{j+2}}^{\infty}\frac{sin^4(\frac{\omega}{2^{j+2}})}{|\omega|^5}d\omega \le \int_{2^{j+2}}^{\infty}\frac{1}{|\omega|^5}d\omega = \frac{4}{2^{4j+8}}.\\
\text{Therefore,}\ \ |c_{j,k}|^2 &\le \frac{3C^2}{\pi^2 2^{3j+6}}.\\
\text{Now,}\ \ ||f-P_Nf||_2^2 &= \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1}|c_{j,k}|^2 \le \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1}\frac{3C^2}{\pi^2 2^{3j+6}}\\
&= \sum_{j=N}^{\infty}\frac{3C^2 2^j}{\pi^2 2^{3j+6}} \le \frac{3C^2}{\pi^2 2^{2N+6}} = O\left(\frac{1}{2^{2N}}\right).
\end{aligned}$$

Similarly, for $s>\frac{3}{2}$, applying the same process $||f-P_nf||_2^2 = O\left(\frac{1}{2^{2N}}\right)$ is obtained.

$$\begin{aligned}
W\left(f-P_Nf,\frac{1}{2^j}\right) &= \sup_{0<h\le\frac{1}{2^j}} ||(f-P_Nf)(t+h)-(f-P_Nf)(t)||_2\\
&\le \sup_{0<h\le\frac{1}{2^j}} [||(f-P_Nf)(t+h)||_2 + ||(f-P_Nf)(t)||_2]\\
&= O\left(\frac{1}{2^N}\right), \quad s\ge\frac{3}{2}.
\end{aligned}$$

□

Following corollary is derived from theorem (3.1):

**Corollary 3.1.** *If a function* $f \in H_h^s(\mathbb{R})$, $s > 0$, *a homogeneous Sobolev space, then the Haar wavelet approximation* $||f - P_N f||_2$ *of* $f$ *by* $P_N f$ *satisfies*

$$||f - P_N f||_2 = \begin{cases} O(1), & s \leq \frac{1}{2}, \\ O\left(\frac{1}{2^{N(s-\frac{1}{2})}}\right), & \frac{1}{2} < s < \frac{3}{2}, \\ O\left(\frac{1}{2^N}\right) & s \geq \frac{3}{2}. \end{cases}$$

## 4. Haar operational matrix of fractional integration

In this section, considering $N = 0$ and $J = 2$, the eight basis functions are obtained as

$$\phi_{0,0}(t) = \begin{cases} 1, & 0 \leq t < 1, \\ 0, & \text{otherwise}, \end{cases} \qquad \psi_{0,0}(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{2}, \\ -1, & \frac{1}{2} \leq t < 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\psi_{1,0}(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{4}, \\ -1, & \frac{1}{4} \leq t < \frac{1}{2}, \\ 0, & \text{otherwise}, \end{cases} \qquad \psi_{1,1}(t) = \begin{cases} 1, & \frac{1}{2} \leq t < \frac{3}{4}, \\ -1, & \frac{3}{4} \leq t < 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\psi_{2,0}(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{8}, \\ -1, & \frac{1}{8} \leq t < \frac{1}{4}, \\ 0, & \text{otherwise}, \end{cases} \qquad \psi_{2,1}(t) = \begin{cases} 1, & \frac{1}{4} \leq t < \frac{3}{8}, \\ -1, & \frac{3}{8} \leq t < \frac{1}{2}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\psi_{2,2}(t) = \begin{cases} 1, & \frac{1}{2} \leq t < \frac{5}{8}, \\ -1, & \frac{5}{8} \leq t < \frac{3}{4}, \\ 0, & \text{otherwise}, \end{cases} \qquad \psi_{2,3}(t) = \begin{cases} 1, & \frac{3}{4} \leq t < \frac{7}{8}, \\ -1, & \frac{7}{8} \leq t < 1, \\ 0, & \text{otherwise}, \end{cases}$$

Taking the collocation points as $t_l = \dfrac{l - \frac{1}{2}}{8}$, $l = 1, 2, ..., 2^{J+1}$, the Haar coefficient matrix is given by $H_{8\times 8} = [\psi(t_1) \;\; \psi(t_2) \;\; \psi(t_3) \;\; \psi(t_4) \;\; \psi(t_5) \;\; \psi(t_6) \;\; \psi(t_7) \;\; \psi(t_8)]$,
where $\psi(t)$ is given by $[\phi_{0,0}(t) \;\; \psi_{0,0}(t) \;\; \psi_{1,0}(t) \;\; \psi_{1,1}(t) \;\; \psi_{2,0}(t) \;\; \psi_{2,1}(t) \;\; \psi_{2,2}(t) \;\; \psi_{2,3}(t)]^T$.
Hence, the Haar coefficient matrix [7] of order $2^{J+1} \times 2^{J+1}$ for $J = 2$ is

$$H_{8\times 8} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

If $I^\alpha$ is the fractional integral operator then $(I^\alpha \psi)(t) = P^\alpha_{8\times 8}\psi(t)$, where $P^\alpha_{8\times 8}$ is the Haar operational matrix of fractional integration of order $\alpha$.
Also, $P^\alpha_{8\times 8} = H_{8\times 8} F^\alpha_{8\times 8} H^{-1}_{8\times 8}$, where $F^\alpha_{8\times 8}$ is the block-pulse operational matrix of fractional integration [19] and is given by

$$F^\alpha_{8\times 8} = \frac{1}{8^\alpha} \frac{1}{\Gamma(\alpha + 2)} \begin{pmatrix} 1 & \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 & \xi_7 \\ 0 & 1 & \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 \\ 0 & 0 & 1 & \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 \\ 0 & 0 & 0 & 1 & \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ 0 & 0 & 0 & 0 & 1 & \xi_1 & \xi_2 & \xi_3 \\ 0 & 0 & 0 & 0 & 0 & 1 & \xi_1 & \xi_1 \end{pmatrix}.$$

Here, $\xi_i = (i+1)^{\alpha+1} - 2i^{\alpha+1} + (i-1)^{\alpha+1}$.
Therefore, using the matrix $F^{\alpha}_{8\times 8}$, Haar operational matrix of fractional integration $P^{\alpha}_{8\times 8}$ for different values of $\alpha$ is obtained.

## 5. Implementation of Haar Wavelet in Physical Problems

Interaction of fluid and a solid body is a classical phenomenon in nature. Basset [4] in 1888 studied a sphere under gravity and introduced the significant force called 'Basset force'. The significance of this force is that it considers the history of relative acceleration of the body and hence give better results regarding the motion of body through the fluid. This problem can be written as a fractional differential equation [26] :

$$Dy(t) + \left(\frac{9}{1+2\lambda}\right)^{\frac{1}{2}} D^{\alpha}y(t) + y(t) = 1, \quad y(0) = 0, \quad 0 < t < T. \tag{1}$$

In this section, the Basset equation has been solved using Haar operational matrix of fractional integration.
For solving the eq.(1), the differential coefficients are approximated as:
$Dy(t) \approx C^T\psi(t)$, where $C$ and $\psi(t)$ are $8\times 1$ vectors of the form

$$C = [d_{0,0}\ c_{0,0}\ c_{1,0}\ c_{1,1}\ c_{2,0}\ c_{2,1}\ c_{2,2}\ c_{2,3}]^T,$$

and

$$\psi(t) = [\phi_{0,0}(t)\ \psi_{0,0}(t)\ \psi_{1,0}(t)\ \psi_{1,1}(t)\ \psi_{2,0}(t)\ \psi_{2,1}(t)\ \psi_{2,2}(t)\ \psi_{2,3}(t)]^T.$$

Then integrating it gives $y(t) \approx C^T P_{8\times 8}\psi(t)$,
and $D^{\alpha}y(t) \approx C^T P^{1-\alpha}_{8\times 8}\psi(t)$.
Putting all these values in the fractional diff. eqn. (1), the system of algebraic equations is obtained,

$$C^T\psi(t) + \left(\frac{9}{1+2\lambda}\right)^{\frac{1}{2}} C^T P^{1-\alpha}_{8\times 8}\psi(t) + C^T P_{8\times 8}\psi(t) = U^T\psi(t), \tag{2}$$

where $U$ is the known $8\times 1$ column vector.
Solving the above system of algebraic equations (2) gives the numerical solution of Basset problem.

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**Algorithm 1** Solving a System of Equations Using Operational Matrices in MATLAB

---

**Formatting the values:** Set MATLAB to use long decimal format for numerical accuracy using *format long;*
**Define the Matrices:** Assembling the matrices like operational matrix, identity matrix and their transposes as *P=[...];*
**Evaluate the Coefficient Matrix 'C':** Using matrix inversion in MATLAB,
*A=eye(8)+sqrt(9+(1+2*λ)).**$P^{1-\alpha}_{8\times 8}$*+*$P_{8\times 8}$*;*
C=inv(A)*U;
**Define Haar basis vector:** Define Haar basis vectors in terms of Haar coefficient matrix, *H=[...];*
**Evaluating the** $y$**:** Solution can be calculated for different basis vectors,
*Y=trans(H)*trans(P)*C;*

---

## 6. Numerical Examples

**Example 6.1.** *(i)For $\lambda = 0$, the Basset equation becomes,*

$$Dy(t) + 3D^{\alpha}y(t) + y(t) = 1, \quad y(0) = 0, \quad 0 < t < T. \tag{3}$$

*For $\lambda = 0$ and $\alpha = 0, 1/3, 1/2, 3/4, 1$, the following numerical values of $y(t)$ are obtained using Haar wavelet operational matrix of fractional integration.*

Table 1. Haar solution of Basset equation (1) for $\lambda = 0; \alpha = 0, 1/3, 1/2$

| | *For* $\alpha = 0$ | | | *For* $\alpha = 1/3$ | *For* $\alpha = 1/2$ |
|---|---|---|---|---|---|
| ***t*** | ***Exact solution*** | ***Haar solution*** | ***Abs.Error*** | ***Haar solution*** | ***Haar solution*** |
| 1/16 | 0.05529 | 0.05514 | $1.5\times10^{-4}$ | 0.05588 | 0.04672 |
| 3/16 | 0.131908 | 0.131963 | $5.5\times10^{-5}$ | 0.11723 | 0.10219 |
| 5/16 | 0.17837 | 0.17834 | $3\times10^{-5}$ | 0.14366 | 0.13363 |
| 7/16 | 0.20655 | 0.20683 | $2.8\times10^{-4}$ | 0.17003 | 0.16587 |
| 9/16 | 0.22365 | 0.22361 | $4\times10^{-5}$ | 0.19035 | 0.18461 |
| 11/16 | 0.23401 | 0.23403 | $2\times10^{-5}$ | 0.20929 | 0.20357 |
| 13/16 | 0.240306 | 0.240303 | $3\times10^{-6}$ | 0.221804 | 0.21549 |
| 15/16 | 0.24412 | 0.24419 | $7\times10^{-5}$ | 0.238619 | 0.22407 |

Table 2. Haar solution of Basset equation (1) for $\lambda = 0; \alpha = 1/2, 3/4, 1$

| | *For* $\alpha = 1/2$ | *For* $\alpha = 3/4$ | *For* $\alpha = 1$ | | |
|---|---|---|---|---|---|
| ***t*** | ***Haar solution*** | ***Haar solution*** | ***Exact solution*** | ***Haar solution*** | ***Abs. Error*** |
| 1/16 | 0.04672 | 0.03566 | 0.015503 | 0.01548 | $2.3\times10^{-5}$ |
| 3/16 | 0.10219 | 0.05215 | 0.04579 | 0.04578 | $1\times10^{-5}$ |
| 5/16 | 0.13363 | 0.08557 | 0.07515 | 0.07514 | $1\times10^{-5}$ |
| 7/16 | 0.16587 | 0.11564 | 0.103605 | 0.103607 | $2\times10^{-6}$ |
| 9/16 | 0.18461 | 0.13593 | 0.131184 | 0.131164 | $2\times10^{-5}$ |
| 11/16 | 0.20357 | 0.16965 | 0.15791 | 0.15793 | $2\times10^{-5}$ |
| 13/16 | 0.21549 | 0.19402 | 0.18382 | 0.18389 | $7\times10^{-5}$ |
| 15/16 | 0.22407 | 0.21627 | 0.20893 | 0.20894 | $1\times10^{-5}$ |

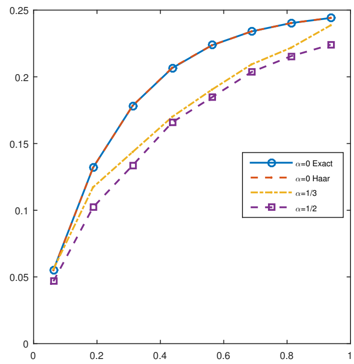


FIGURE 1. For $\alpha = 0, 1/3, 1/2$

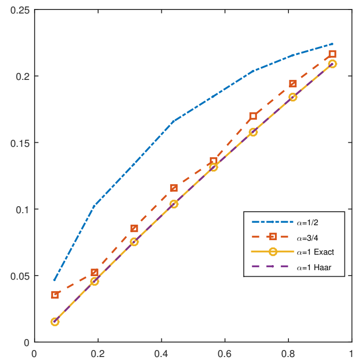


FIGURE 2. For $\alpha = 1/2, 3/4, 1$

***Comparison of solutions of Basset equation for*** $\lambda = 0$

**Example 6.2.** *For $\lambda = 4$, the Basset equation becomes,*

$$Dy(t) + D^{\alpha}y(t) + y(t) = 1, \quad y(0) = 0, \quad 0 < t < T. \tag{4}$$

*Using Haar wavelet operational matrix of fractional integration for $\lambda = 4$ and $\alpha = 0, 1/3, 1/2, 3/4, 1$, the following numerical values of $y(t)$ are obtained.*

TABLE 3. Haar solution of Basset equation (1) for $\lambda = 4$

| | ***For*** $\alpha = 0$ | | | ***For*** $\alpha = 1/3$ | ***For*** $\alpha = 1/2$ |
|---|---|---|---|---|---|
| ***t*** | ***Exact solution*** | ***Haar solution*** | ***Abs. Error*** | ***Haar solution*** | ***Haar solution*** |
| *1/16* | *0.05875* | *0.05872* | $3\times10^{-5}$ | *0.07525* | *0.068164* |
| *3/16* | *0.15635* | *0.15628* | $7\times10^{-5}$ | *0.097911* | *0.095091* |
| *5/16* | *0.23236* | *0.23239* | $3\times10^{-5}$ | *0.18031* | *0.166428* |
| *7/16* | *0.29156* | *0.29136* | $2\times10^{-4}$ | *0.23648* | *0.209066* |
| *9/16* | *0.33767* | *0.33782* | $1.5\times10^{-4}$ | *0.27152* | *0.256169* |
| *11/16* | *0.37358* | *0.37331* | $2.7\times10^{-4}$ | *0.33579* | *0.31758* |
| *13/16* | *0.40154* | *0.40158* | $4\times10^{-5}$ | *0.37149* | *0.36036* |
| *15/16* | *0.42332* | *0.42374* | $4.2\times10^{-4}$ | *0.39325* | *0.38583* |

TABLE 4. Haar solution of Basset equation (1) for $\lambda = 4$

| | For $\alpha = 1/2$ | For $\alpha = 3/4$ | For $\alpha = 1$ | | |
|---|---|---|---|---|---|
| ***t*** | ***Haar solution*** | ***Haar solution*** | ***Exact solution*** | ***Haar solution*** | ***Abs. Error*** |
| *1/16* | *0.068164* | *0.060339* | *0.03076* | *0.03079* | $3\times10^{-5}$ |
| *3/16* | *0.095091* | *0.08754* | *0.08948* | *0.08946* | $2\times10^{-5}$ |
| *5/16* | *0.166428* | *0.15636* | *0.14465* | *0.14447* | $1.8\times10^{-4}$ |
| *7/16* | *0.209066* | *0.20639* | *0.19647* | *0.19649* | $2\times10^{-5}$ |
| *9/16* | *0.256169* | *0.250676* | *0.24516* | *0.24521* | $5\times10^{-5}$ |
| *11/16* | *0.31758* | *0.30821* | *0.29089* | *0.29072* | $1.7\times10^{-4}$ |
| *13/16* | *0.36036* | *0.35139* | *0.33385* | *0.33383* | $2\times10^{-5}$ |
| *15/16* | *0.38583* | *0.37951* | *0.37421* | *0.37426* | $5\times10^{-5}$ |

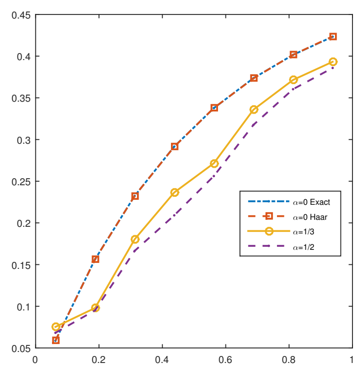


FIGURE 3. For $\alpha = 0, 1/3, 1/2$

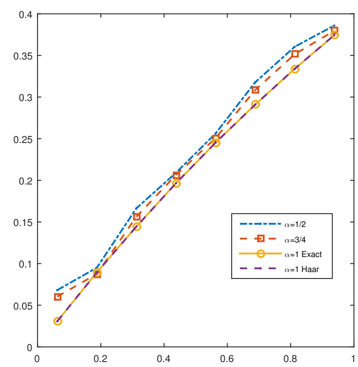


FIGURE 4. For $\alpha = 1/2, 3/4, 1$

***Comparison of solutions of Basset equation for*** $\lambda = 4$

**Example 6.3.** *Many physical problems in nature can be described as the fractional differential equations. Considering this fact, the system of fractional differential equations comes into account whose numerical solution can be obtained using the Haar wavelet operational matrix of fractional integration.*
*Consider the following system of fractional differential equations* [13]*:*

$$D^{\alpha}x(t) = x(t) + y(t),$$
$$D^{\beta}y(t) = -x(t) + y(t), \tag{5}$$

*with initial conditions* $x(0) = 0$ *and* $y(0) = 1$*. Here, the fractional derivatives are taken in Caputo sense.*
*For* $\alpha, \beta = 1$ *the exact solution of above system is given by* $x(t) = e^t \sin t$ *and* $y(t) = e^t \cos t$*.*
*For calculating numerical solution of above system (5), approximate the fractional derivatives as:*

$$D^{\alpha}x(t) \approx C^T\psi(t),$$
$$D^{\beta}y(t) \approx E^T\psi(t).$$

*Then, upon integration,* $x(t) \approx C^T P^{\alpha}_{8\times 8}\psi(t)$ *and* $y(t) \approx E^T P^{\beta}_{8\times 8}\psi(t) + V^T\psi(t)$, *where* $V$ *is the known* $8 \times 1$ *column vector.*
*The numerical solutions of above system of fractional differential equations (5) for* $\alpha, \beta = 1$ *and* $\alpha, \beta = 1/2$ *are given in table 5.*

TABLE 5. Solutions of system of fractional differential equations (5)

| | For $\alpha,\beta = 1$ | | | | | | For $\alpha,\beta = 1/2$ | |
|---|---|---|---|---|---|---|---|---|
| | Exact solution | | Haar solution | | Absolute Error | | Haar solution | |
| t | x(t) | y(t) | x(t) | y(t) | x(t) | y(t) | x(t) | y(t) |
| 1/16 | 0.06648 | 1.06241 | 0.06643 | 1.06283 | $5\times10^{-5}$ | $4.2\times10^{-4}$ | 0.06462 | 0.98486 |
| 3/16 | 0.22484 | 1.18508 | 0.22409 | 1.18504 | $7.5\times10^{-4}$ | $4\times10^{-5}$ | 0.46026 | 1.24477 |
| 5/16 | 0.42021 | 1.30063 | 0.42038 | 1.30060 | $1.7\times10^{-4}$ | $3\times10^{-5}$ | 0.87125 | 1.31942 |
| 7/16 | 0.65620 | 1.40295 | 0.65670 | 1.39998 | $5\times10^{-4}$ | $2.97\times10^{-3}$ | 1.08710 | 1.41158 |
| 9/16 | 0.93597 | 1.48464 | 0.93518 | 1.48461 | $7.9\times10^{-4}$ | $3\times10^{-5}$ | 1.26481 | 1.42113 |
| 11/16 | 1.26206 | 1.53696 | 1.26207 | 1.53682 | $1\times10^{-5}$ | $1.4\times10^{-4}$ | 1.65546 | 1.46391 |
| 13/16 | 1.63608 | 1.54972 | 1.63620 | 1.54973 | $1.2\times10^{-4}$ | $1\times10^{-5}$ | 2.002451 | 1.43986 |
| 15/16 | 2.05840 | 1.51122 | 2.05863 | 1.51183 | $2.3\times10^{-4}$ | $6.1\times10^{-4}$ | 2.589031 | 1.38193 |

*The comparison between graphs of exact and Haar solutions of system of fractional differential equations (5) for* $\alpha, \beta = 1$ *is shown in figure 5.*
*The graphs of solution of system of fractional differential equations (5) for* $\alpha, \beta = 1/2$, *using Haar wavelet operational matrix of fractional integration are shown in figure 6.*

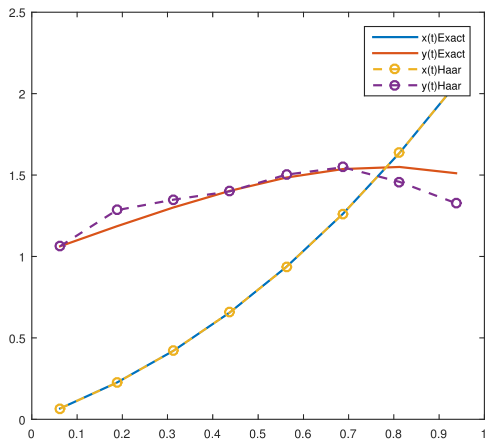


FIGURE 5. Comparison of exact and Haar solution of system (5) for $\alpha, \beta = 1$

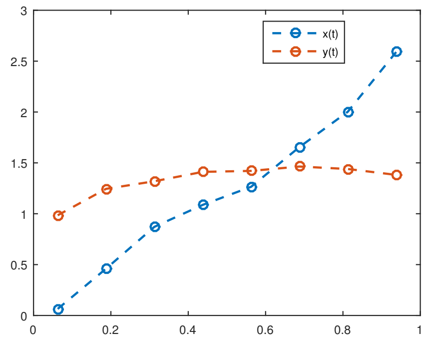


FIGURE 6. Haar solution of system (5) for $\alpha, \beta = 1/2$

## 7. Conclusions and Discussion

In this study, we investigated the applicability and efficiency of the Haar wavelet operational matrix method for solving fractional differential equations, particularly the Basset equation and a system of fractional differential equations. The exact solutions for integer values of equation (3) $y(t) = \frac{1}{4}(1 - e^{-4t})$ and $y(t) = 1 - e^{\frac{-t}{4}}$ respectively. The numerical solutions obtained via the Haar wavelet method are compared with the exact solutions. The results demonstrated excellent agreement between the exact and Haar wavelet solutions, as evidenced by the numerical comparisons in Tables 1 & 2 and the graphical representations in Figures 1 & 2. Similarly, the exact solution of above equation (4) for the values $\alpha = 0$ and $\alpha = 1$ are $y(t) = \frac{1}{2}(1 - e^{-2t})$ and $y(t) = 1 - e^{\frac{-t}{2}}$ respectively. The Tables 3 and 4 show that the exact solution and the Haar solution obtained is in high agreement with each other. The graphs shown in Figures 1-4 verify the shape and numerical values of Basset equation as given in [26] and hence give the applicability of the proposed method. A key observation from this study is that the Haar wavelet method not only provides accurate solutions for fractional differential equations but also converges to the exact solutions when the fractional order approaches integer values. This consistency reinforces the reliability of the proposed method.
Additionally, the computational efficiency, which gives the high precision and low compuational cost, of the Haar wavelet method compared to other methods, such as Chebyshev collocation, LDG and numerical inverse Laplace transform [18][26], is highlighted in the Table (6).

TABLE 6. Comparison of Haar solution with other known methods

| **Method** | $\lambda$ | **N** | **Max. Absolute error** |
|---|---|---|---|
| Haar wavelet | $\frac{1}{2}$ | 8 | $10^{-4}$ |
| Chebyshev collocation | $\frac{1}{2}$ | 10 | $10^{-3}$ |
| Local Discontinuous Galerkin(LDG) | $\frac{1}{2}$ | 10 | $10^{-3}$ |
| Numerical inverse Laplace transform | $\frac{1}{2}$ | 15 | $10^{-3}$ |

The absolute errors among different methods for solving Basset equation using different values of no. of iterations $N$ is shown in figure 7.

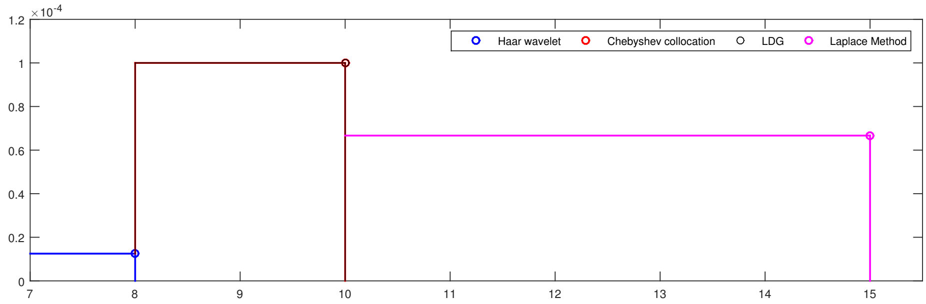


FIGURE 7. Comparison of Haar solution with other numerical methods

The method's superiority was further demonstrated in solving systems of fractional differential equations, where the solution patterns remained consistent with integer-order systems, albeit with a shift in the intersection point towards the origin as shown in Figure 6. The comparative analysis in Table 7 revealed that the Haar wavelet method outperforms collocation and differential transform methods in terms of both accuracy and computational cost, requiring fewer iterations.

TABLE 7. Comparison of Haar solution of system of fractional differential eqautions with other known methods

| **Method** | $\lambda$ | **N** | **Max. Absolute error** |
|---|---|---|---|
| Haar wavelet | $\frac{1}{2}$ | 8 | $10^{-4}$ |
| Collocation method | $\frac{1}{2}$ | 8 | $10^{-2}$ |
| Differential transform | $\frac{1}{2}$ | 25 | - - |

## 8. MAIN RESULTS

(i) In this paper, the moduli of continuity of a function $f \in H_h^s(\mathbb{R}),\ s > 0$

$$W\left(f - P_N f, \frac{1}{2^j}\right) = \begin{cases} O(1), & s \leq \frac{1}{2}, \\ O\left(\frac{1}{2^{N(s-\frac{1}{2})}}\right), & \frac{1}{2} < s < \frac{3}{2}, \\ O\left(\frac{1}{2^N}\right) & s \geq \frac{3}{2}, \end{cases}$$

has been estimated and it tends to 0 as $N \to \infty$.
(ii) From corollary (3.1), it is observed that $W\left(f - P_N f, \frac{1}{2^j}\right) \leq 2.E_{2^N-1}(f)$.
Therefore, moduli of continuity is more better and sharper than approximations.
(iii) Also, the Haar wavelet operational matrix of fractional integration has been obtained from Haar coefficient matrix for different values of fractional orders $\alpha$ and has been applied to solve fractional differential equation called Basset equation whose analytic solution is difficult to obtain.
(iii) The solutions obtained by Haar wavelet method have been compared with integer valued equations. The similar nature of the solutions validate the proposed method.
(iv) This method has also been extende to solve a system of fractional differential equations in which solutions $x(t)$ and $y(t)$ obtained are very close to the exact solution. This shows the effectiveness of the proposed method.

(v) The solutions obtained by Haar wavelet method have also been compared with other numerical methods as in tables 5 and 7. This shows that more accurate solutions are obtained using lesser number of terms in the series. Hence, this shows the computational efficiency of the proposed method.

**Acknowledgement.** Shyam Lal, one of the authors, likes to extend his gratitude to DST-CIMS for encouragement to this work. The authors thank the reviewer for his valuable and insightful comments which have significantly enhance the paper.

## References

[1] Arikoglu, A., Ozkol, I., (2007), Solution of fractional differential equations by using differential transform method, Chaos Soliton. Fract., 34, pp. 1473–1481.

[2] Angulo, P., (2022), J. Nonlinear dispersive equations: classical and new frameworks, São Paulo J. Math. Sci., 16, pp. 171–255.

[3] Atangana, A., (2018), Fractional Operators with Constant and Variable Order with Application to Geo-Hydrology, Academic Press.

[4] Basset, A. B., (1888), A Treatise on Hydrodynamics With Numerous Examples, Vol. 2, Cambridge [Eng.] Deighton, Bell and Co.

[5] 1. Agarwal, G., Yadav, L.K., Nisar, K.S, Alqarni, M.M., Mahmoud, E.E. , (2024), A Hybrid Method For The Analytical Solution Of Time Fractional Whitham-Broer-Kaup Equations, Applied And Computational Mathematics, 23(1), 2024, pp.3-17. .

[6] Bhrawy, A. H., Doha, E. H., Ezz-Eldien, S. S., Abdelkawy, M. A., (2015), A numerical technique based on the shifted Legendre polynomials for solving the time-fractional coupled KdV equation, Calcolo, 53, pp. 1-17.

[7] Chandra, P., (1982), On the generalised Fejér means in the metric of Hölder space, Mathematische Nachrichten, 109(1), pp. 39-45.

[8] Chen, C. F., Hsiao, C. H., (1997), Haar wavelet method for solving lumped and distributed parameter-systems, IEEE Proc-Control Theory Appication, 144(1), pp. 87–94.

[9] Chui, C. K., (1992), An Introduction to Wavelets (Wavelet Analysis and its Applications), Vol. I, Academic Press, Cambridge.

[10] 7. Eriqat, T., Oqielat, M.N., El-Ajou, A., Kumar, S., Momani, S., Arqub, O., (2025), Approximate solutions of time fractional Whitham-Broer-Kaup equations using Laplace-residual power series method, TWMS J. Pure Appl. Math., 16(2), pp. 176-199.

[11] Derakhshan, M. H., (2022), Existence, uniqueness, Ulam–Hyers stability and numerical simulation of solutions for variable order fractional differential equations in fluid mechanics, J. Appl. Math. Comput., 68, pp. 403–429.

[12] Diethelm, K., Ford, N. J., Freed, A. D., Luchko, Y., (2005), Algorithms for the fractional calculus: a selection of numerical methods, Computer methods in applied mechanics and engineering, 194(6-8), pp. 743-773.

[13] Ditzian, Z., (1981), Lipschitz classes and convolution approximation processes, Mathematical Proceedings of the Cambridge Philosophical Society, 90(1), pp. 51-61.

[14] El-Wakil, S. A., Elhanbaly, A., Abdou, M. A., (2006), Adomian decomposition method for solving fractional nonlinear differential equations, Appl. Math. Comput., 182, pp. 313–324.

[15] Ertürka, V. S., Momanib, S., (2008), Solving systems of fractional differential equations using differential transform method, Journal of Computational and Applied Mathematics, 215, pp. 142–151.

[16] Gorka, P., Kostrzewa, T., Reyes, E. G., (2014), Sobolev spaces on locally compact abelian groups:compact embeddings and local spaces, Journal of Function Spaces, Article ID 04738, 6 pages. http://dx.doi.org/10.1155/2014/404738

[17] Haar, A., (1910), Zur theorie der orthogonalen Funktionsysteme, Math. Annal, 69, pp. 331-371.

[18] Hariharan, G., Kannan, K., (2013), An Overview of Haar Wavelet Method for Solving Differential and Integral Equations, World Applied Sciences Journal, 23(12), pp. 1-14.

[19] Hemeda, A. A., (2013), Solution of Fractional Partial Differential Equations in Fluid Mechanics by Extension of Some Iterative Method, Abstract and Applied Analysis, Article ID 717540, 9 pages. https://doi.org/10.1155/2013/717540

[20] Izadi, M., Afshar, M., (2021), Solving the Basset equation via Chebyshev collocation and LDG methods, Journal of Mathematical Modeling, 9(1), pp. 61-79.

[21] Kilicman, A., Al Zhour, Z. A. A., (2007), Kronecker operational matrices for fractional calculus and some applications, Appl. Math. Comput., 187, pp. 250–265.

[22] Lazopoulos, K., Lazopoulos, A., (2017), Fractional vector calculus and fluid mechanics, Journal of the Mechanical Behavior of Materials, 26(1-2), pp. 43-54.

[23] Magin, R., Ortigueira, M. D., Podlubny,I., Trujillo,J., (2011), Review: On the fractional signals and systems, Signal Process, 91(3), pp. 350–371.

[24] Matlob, M. A., Jamali, Y., (2019), The Concepts and Applications of Fractional Order Differential Calculus in Modeling of Viscoelastic Systems: A Primer, Critical Reviews in Biomedical Engineering, 47(4), pp. 249-276.

[25] Mechee, M. S., Salman, Z. I., Alkufi, S. Y., (2023), Novel Definitions of $\alpha$-Fractional Integral and Derivative of the Functions, Iraqi Journal of Science, 64(11), pp. 5866-5877.

[26] Meerschaert, M., Tadjeran, C., (2006), Finite difference approximations for two-sided space-fractional partial differential equations, Appl. Numer. Math., 56, pp. 80-90.

[27] Miller, K. S., Ross, B., (1993), An Introduction to the Fractional Calculus and Fractional Differential Equations, New York, Wiley.

[28] Nouri, K., Elahi-Mehr, S., Torkzadeh, L., (2016), Investigation of the behavior of the fractional Bagley-Torvik AND Basset equations via numerical inverse Laplace Transform, Romanian Reports in Physics, 68(2), pp. 503–514.

[29] Oldham, K. B., (2010), Fractional differential equations in electrochemistry, Advances in Engineering Software, 41(1), pp. 9-12.

[30] Podlubny, I., (1999), Fractional Differential Equations, Academic Press, New York, USA.

[31] Rouzegar, J., Vazirzadeh, M., Heydari, M. H., (2020), A fractional viscoelastic model for vibrational analysis of thin plate excited by supports movement, Mechanics Research Communications, 110, Article ID 103618.

[32] Sohail, A., Arshad, S., Ehsan, Z., (2017), Numerical Analysis of Plasma KdV Equation: Time-Fractional Approach, Int. J. Appl. Comput. Math, 3(1), pp. 1325–1336.

[33] Talib, I., Özger, F., (2023), Orthogonal Polynomials Based Operational Matrices with Applications to Bagley-Torvik Fractional Derivative Differential Equations, IntechOpen. DOI: 10.5772/intechopen.1001144


**Prof. Shyam Lal** is retired Senior Professor of Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi, India. He received his D.Sc. and Ph.D. from the University of Allahabad, Allahabad, India and Banaras Hindu University, Varanasi, India respectively. His research interests include wavelet analysis, approximation theory, summability theory, Fourier Analysis and Fixed Point Theory.

**Abhilasha** is presently working as an Assistant Professor at IIHS, Kurukshetra University and currently pursuing a Ph.D. from the Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi, India. Her areas of research interest are approximation theory and numerical solution of physical problems using wavelet methods.