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GIBBS MEASURES FOR THE ASYMMETRIC CLOCK MODEL ON A CAYLEY TREE

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ABSTRACT. We consider the p-state clock model on the Cayley tree and derive a system of functional equations governing this model. Each positive solution of the system corresponds to a splitting Gibbs measure. Utilizing this framework, we investigate the translation-invariant splitting Gibbs measures (TISGMs) of the four-state clock model on the Cayley tree of order two.

Keywords: Cayley tree, asymmetric clock model, Gibbs measures, phase transitions.

AMS Subject Classification: 20B07, 20E06.

1. INTRODUCTION

One of the fundamental problems in the theory of Gibbs measures is the characterization of infinite-volume (or limiting) Gibbs measures associated with a given Hamiltonian. The existence of such measures for a broad class of Hamiltonians was established in the seminal work of Dobrushin (see, e.g., [26]). However, determining the full set of limiting Gibbs measures for a specific Hamiltonian remains a challenging problem [16, 7, 27]. The occurrence of multiple Gibbs measures for a given model signifies the presence of a phase transition [1, 7, 17, 13, 14, 15], a phenomenon that is central to statistical mechanics [7]. In recent years, numerous finite-state models of statistical mechanics on Cayley trees have been shown to exhibit phase transitions (see, e.g., [20, 21, 22, 25]).

The asymmetric p-state clock model, also known as the chiral model, was introduced by Ostlund and Huse in 1981 [4, 12, 3]. These works, however, are primarily intended for a physics audience and lack rigorous mathematical treatment. In [4], the asymmetric clock model on a Cayley tree was investigated, revealing the existence of both commensurate (periodic with the lattice) and incommensurate (modulated) phases. The transition lines

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were derived from stability conditions, and characteristic points in the phase diagram were analyzed through numerical iteration. Notably, a critical endpoint was identified for the three-state case, while a Lifshitz point was observed for the four-state case.

The XY chiral model generalizes the p-state clock model (see, e.g., [11]). In [11], forward quantum Markov chains (QMC) on a Cayley tree were studied, demonstrating the existence of a phase transition in the XY-model on a Cayley tree of order three within the QMC framework. The λ-model is another generalization of the p-state clock model [13, 14], although results for the λ-model have primarily been obtained for up to three spin values.

The p-state asymmetric clock model has also been extensively studied on the lattice $\mathbb{Z}^d$ (see, e.g., [2, 10, 5, 9]), revealing a range of intriguing properties. Notably, the standard nearest-neighbor large-p clock model in two dimensions exhibits a Kosterlitz-Thouless phase, characterized by slow decay and enhanced continuous symmetry in an intermediate temperature regime [5, 6]. Using the tensor renormalization group method based on higher-order singular value decomposition, the phase transitions of the five-state clock model on the square lattice were investigated in [2]. The temperature dependence of specific heat indicates the presence of two phase transitions, as confirmed through correlation function analysis. Moreover, in [9], it was proven that the dilute ferromagnetic nearest-neighbor p-state clock model on $\mathbb{Z}^d$ undergoes a phase transition for all $p \geq 2$ and $d \geq 2$.

In this study, we analyzed the four-state asymmetric clock model on a Cayley tree of order two, focusing on the existence and behavior of translation-invariant splitting Gibbs measures (TISGMs). By deriving and solving the corresponding functional equations, we identify the conditions under which multiple Gibbs measures emerge, thereby establishing the critical temperatures associated with phase transitions.

Our results demonstrate that the number of TISGMs varies significantly with temperature, reaching up to fifteen distinct solutions. This variation highlights the rich phase structure of the model and provides a rigorous mathematical foundation for understanding phase transitions in discrete spin systems on tree-like structures.

These findings contribute to the broader study of statistical mechanics on non-Euclidean lattices, offering potential applications in quantum information theory and complex network modeling. Future research may extend this approach to higher-order Cayley trees or incorporate external fields to explore additional critical phenomena.

The paper is organized as follows. Section 2 provides the necessary preliminaries, including key definitions and relevant theoretical background. The main results are presented in Section 3. In Section 4, we carry out a complete analysis of the solutions in the case $k = 2$, $p = 4$, and $\Delta = 0$. Finally, Section 5 summarizes the key findings and outlines potential directions for future research.

2. Preliminaries

The *Cayley tree* Γ^k of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, in which each vertex has exactly $k+1$ adjacent edges. Let $\Gamma^k = (V, L)$, where V denotes the set of vertices and L represents the set of edges. Two vertices x and y are said to be *nearest neighbors* if they are connected by an edge $l \in L$, which we denote as $l = \langle x, y\rangle$. A sequence of nearest-neighbor pairs $\langle x, x_1\rangle, \langle x_1, x_2\rangle, \dots, \langle x_{d-1}, y\rangle$ forms a *path* from x to y (see, e.g., [17, 19]).

The natural metric on this tree, denoted by $d(x, y)$, is defined as the number of nearest-neighbor pairs in the shortest path connecting vertices x and y.

For a fixed root vertex $x^0 \in V$, define

$$W_n = \{x \in V : d(x, x^0) = n\}, \quad V_n = \{x \in V : d(x, x^0) \leq n\}$$

to be the sphere and the ball of radius n centered at x^0, respectively. Furthermore, for any $x \in W_n$, define

$$S(x) = \{y \in W_{n+1} : d(y, x) = 1\}$$

as the set of direct successors of x.

2.1. **The Model.** We assume that the spin variables take values in the set $\Phi = \{1, 2, ..., p\}$. A configuration σ on a subset $A \subseteq V$ is defined as a function $x \in A \mapsto \sigma_A(x) \in \Phi$. The set of all configurations on A is denoted by $\Omega_A = \Phi^A$, while the set of all configurations on V is given by $\Omega = \Omega_V$ with $\sigma = \sigma_V$.

The p-state asymmetric clock model is defined by the Hamiltonian

$$H(\sigma) = -J \sum_{\langle x,y \rangle \in L} \cos\Big(\frac{2\pi}{p}\big(\sigma(x) - \sigma(y) - \Delta\big)\Big), \tag{1}$$

where $J \in \mathbb{R}$ is the coupling constant, and $\Delta \in \mathbb{R}$ is an additional parameter of the model.

2.2. **Functional Equations and Splitting Gibbs Measures.** We now derive a system of functional equations whose solutions correspond to Gibbs measures for the p-state clock model on the Cayley tree. For completeness, we recall the derivation of these equations based on the compatibility condition.

Let $h : x \mapsto h_x = (h_{1,x}, h_{2,x}, ..., h_{p,x}) \in \mathbb{R}^p$ be a vector function defined on $x \in V \setminus \{x^0\}$. Consider the probability distribution $\mu^{(n)}$ on Ω_{V_n}:

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp(-\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x),x}), \tag{2}$$

where $\sigma_n \in \Omega_{V_n}$, $\beta = 1/T$, with $T > 0$ denoting the temperature, and

$$Z_n = \sum_{\tilde{\sigma}_n \in \Omega_{V_n}} \exp(-\beta H(\tilde{\sigma}_n) + \sum_{x \in W_n} h_{\tilde{\sigma}(x),x}).$$

The probability distribution $\mu^{(n)}$ is said to be compatible if, for any $n \geq 1$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$, we have

$$\sum_{\omega_n \in \Omega_{W_n}} \mu^{(n)}(\sigma_{n-1} \vee \omega_n) = \mu^{(n-1)}(\sigma_{n-1}), \tag{3}$$

where $\sigma_{n-1} \vee \omega_n \in \Omega_{V_n}$ is the concatenation of the configurations.

In this case, by the Kolmogorov extension theorem (see [8]) there exists a unique measure μ on Ω such that

$$\mu(\{\sigma \in \Omega : \sigma |_{V_n} = \sigma_n\}) = \mu^{(n)}(\sigma_n), \quad n \in \mathbb{N}$$

for all n and $\sigma_n \in \Omega_{V_n}$. Such a measure is called a splitting Gibbs measure (SGM) associated with the Hamiltonian H and the function $x \mapsto h$ for $x \neq x^0$.

The following theorem provides a condition on h_x that ensures the compatibility of the probability distribution $\mu^{(n)}(\sigma_n)$.

Theorem 2.1. *The probability distributions $\mu^{(n)}(\sigma_n), n = 1, 2, \ldots,$ determined by the formula* (2)*, are compatible if and only if, for any $x \in V \setminus \{x^0\}$, the following equation holds:*

$$h_x^* = \sum_{y \in S(x)} F(h_y^*, p, \theta). \tag{4}$$

Here,

$$\theta = \exp(J\beta\cos(\frac{2\pi}{p}\Delta)), \tag{5}$$

where h_x^* *denotes the vector*

$$h_x^* = (h_{1,x} - h_{p,x}, h_{2,x} - h_{p,x}, \dots, h_{p-1,x} - h_{p,x})$$

and the vector function

$$F(\cdot, p, \theta) : \mathbb{R}^{p-1} \to \mathbb{R}^{p-1}$$

is given by

$$F(h, p, \theta) = (F_1(h, p, \theta), \dots, F_{p-1}(h, p, \theta)), \quad h = (h_1, h_2, \dots, h_{p-1}),$$

with components

$$F_i = \ln \frac{\sum\limits_{j=1}^{p-1} \exp(J\beta\cos(\frac{2\pi}{p}(i-j-\Delta)) + h_{j,y}) + \exp(J\beta\cos(\frac{2\pi}{p}(i-p-\Delta)))}{\sum\limits_{j=1}^{p-1} \exp(J\beta\cos(\frac{2\pi}{p}(p-j-\Delta)) + h_{j,y}) + \theta},$$

$i = 1, \dots, p-1$.

Proof. Necessity. Suppose that (3) holds; we aim to prove (4). Substituting (2) into (3), we obtain, for any configuration $\sigma_{n-1} : x \in V \mapsto \sigma_{n-1}(x) \in \Phi$:

$$\sum_{\omega_n \in \Omega_{W_n}} Z_n^{-1} \exp\Bigg(-\beta H_{n-1}(\sigma_{n-1}) + \sum_{x \in W_{n-1}} \sum_{y \in S(x)} \big(J\beta\cos(\frac{2\pi}{p}(\sigma_{n-1}(x) - \omega_n(y) - \Delta))$$

$$+ h_{\omega_n(y),y} \big) \Bigg) = Z_{n-1}^{-1} \exp\Bigg(-\beta H_{n-1}(\sigma_{n-1}) + \sum_{x \in W_{n-1}} h_{\sigma_{n-1}(x),x} \Bigg),$$

where $\omega_n : x \in W_n \mapsto \omega_n(x)$.

From this, we derive:

$$\frac{Z_{n-1}}{Z_n} \sum_{\omega_n \in \Omega_{W_n}} \prod_{x \in W_{n-1}} \prod_{y \in S(x)} \exp(J\beta\cos(\frac{2\pi}{p}(\sigma_{n-1}(x) - \omega_n(y) - \Delta)) + h_{\omega_n(y),y}) =$$

$$\prod_{x \in W_{n-1}} \exp(h_{\sigma_{n-1}(x),x}). \tag{6}$$

Consequently, for all $i \in \Phi$,

$$\exp(h_{i,x} - h_{p,x}) = \prod_{y \in S(x)} \frac{\sum\limits_{j \in \Phi} \exp(J\beta\cos(\frac{2\pi}{p}(i-j-\Delta)) + h_{j,y})}{\sum\limits_{j \in \Phi} \exp(J\beta\cos(\frac{2\pi}{p}(p-j-\Delta)) + h_{j,y})}. \tag{7}$$

Introducing $\theta = \exp(J\beta\cos(\frac{2\pi}{p}\Delta))$ as in (5) and denoting $h_{i,x}^* = h_{i,x} - h_{p,x}$, we obtain (4) from (7).

Sufficiency. From (4), we recover (7) and (6), leading to (2). This completes the proof. □

Remark 2.1. *There are several approaches to deriving equations that describe the limiting Gibbs measures for lattice models on the Cayley tree. In* [4]*, the system* (4) *is obtained via recurrent equations for partition functions. Our approach is based on the properties of Markov random fields on the Cayley tree. Notably, both approaches lead to the same equation (see* [17]*).*

From Theorem 2.1, it follows that for any $h = \{h_x, x \in V\}$ satisfying (4), there exists a unique SGM μ for the clock model.

3. Translation-Invariant SGMs

It is known (see, e.g., [17]) that there exists a one-to-one correspondence between the set V of vertices of the Cayley tree of order $k \geq 1$ and the group G_k of the free products of $k+1$ cyclic groups of the second order with generators $a_1,\ a_2, \dots,\ a_{k+1}$.

Definition 3.1. *Let K be a subgroup of G_k. We say that a collection of vectors $h = \{h_x : x \in G_k\}$ is K-periodic if $h_{yx} = h_x$ for all $x \in G_k$ and $y \in K$. A G_k-periodic function h is called translation-invariant.*

Definition 3.2. *A SGM is called K-periodic if it corresponds to a K-periodic collection h.*

A natural starting point is to consider translation-invariant (TI) solutions, where the vector $h_x^* = h \in \mathbb{R}^{p-1}$, $\forall x \in V$ is constant. In this case, Eq. (4) simplifies to

$$z_i = \left(\frac{\sum\limits_{j=1}^{p-1} \exp\Big(J\beta \cos\big(\frac{2\pi}{p}(i-j-\Delta)\big)\Big) z_j + \exp\Big(J\beta\cos\big(\frac{2\pi}{p}(i-p-\Delta)\big)\Big)}{\sum\limits_{j=1}^{p-1} \exp\Big(J\beta\cos\big(\frac{2\pi}{p}(p-j-\Delta)\big)\Big) z_j + \theta} \right)^k, \tag{8}$$

where $z_i = \exp(h_i)$, $i = 1, ..., p-1$. The vector $(z_1, ..., z_{p-1})$ is referred to as a translation-invariant law. More generally, even in the non-translation-invariant case, the quantities $l_x(i) = \exp(h_{i,x}^*)$ are commonly called boundary laws (see [7], p. 242).

From a physical perspective, the two-state clock model is equivalent to the Ising model when $\Delta = 0$ and $p = 2$. Notably, the following identity holds:

$$\cos\big(\pi(\sigma(x) - \sigma(y))\big) = (2\sigma(x) - 3)(2\sigma(y) - 3). \tag{9}$$

Similarly, the three-state clock model corresponds to the three-state Potts model when $\Delta = 0$ and $p = 3$. Specifically, the following relation holds:

$$\cos\frac{2\pi}{3}\Big(\sigma(x) - \sigma(y)\Big) = \frac{3}{2}\delta_{\sigma(x),\sigma(y)} - \frac{1}{2}, \tag{10}$$

where $\delta_{i,j}$ denotes the Kronecker symbol.

The Main Result We denote by μ_i, $i \in \{1, 2, ..., 15\}$ the TISGMs which correspond to the solutions of the system (8) for $k = 2$, $p = 4$ and $\Delta = 0$.

The following theorem is the main result of this paper

Theorem 3.1. *For the clock model with $p = 4$ and $\Delta = 0$ on the Cayley tree of order two, the following assertions hold. There exist critical values $\theta_{c,1} := 3 - 2\sqrt{2}$, $\theta_{c,2} := 3$, $\theta_{c,3} := 3 + 2\sqrt{2}$, and $\theta_{c,4} := 3 + 2\sqrt{3}$ such that:*

- *If $\theta < \theta_{c,1}$, there exist exactly three TISGMs, μ_i, where $i = 1, 2, 3$.*
- *If $\theta_{c,1} \leq \theta \leq \theta_{c,2}$, there exists a unique TISGM, μ_1.*
- *If $\theta_{c,2} < \theta \leq \theta_{c,3}$, there exist exactly nine TISGMs, μ_i, where $i = 1, 4, 5, 8, 9, \dots, 13$.*
- *If $\theta_{c,3} < \theta < \theta_{c,4}$, there exist exactly eleven such measures, μ_i, where $i = 1, 2, 3, 4, 5, 8, 9, \dots, 13$.*
- *If $\theta = \theta_{c,4}$, there exist exactly twelve such measures, μ_i, for $i \in \{1, 2, \dots, 15\} \setminus \{3, 7, 15\}$.*
- *If $\theta > \theta_{c,4}$, there exist exactly fifteen such measures, μ_i, where $i = 1, 2, \dots, 15$.*

Remark 3.1. *Analyzing solutions of Eq.* (8) *for* $p \geq 5$ *or* $\Delta \neq 0$ *seems to be challenging due to the appearance of polynomials of order greater than or equal to five, which cannot be solved by radicals. The same reasoning also applies to the model with higher order* $k \geq 3$. *To avoid such difficulties, in* [18] *the contour method is implemented to show that for the several* p*-component models on a Cayley tree of order two, at sufficiently low temperatures, there are at least* p *Gibbs measures.*

4. $k = 2$, $p = 4$ and $\Delta = 0$: Full analysis of solutions

From now on, we focus on the case $\Delta = 0$ and $p = 4$ on a Cayley tree of order two. Under these conditions, Eq. (8) reduces to

$$\begin{cases} z_1 = \left(\frac{\theta z_1+z_2+\frac{1}{\theta}z_3+1}{z_1+\frac{1}{\theta}z_2+z_3+\theta}\right)^2, \\ z_2 = \left(\frac{z_1+\theta z_2+z_3+\frac{1}{\theta}}{z_1+\frac{1}{\theta}z_2+z_3+\theta}\right)^2, \\ z_3 = \left(\frac{\frac{1}{\theta}z_1+z_2+\theta z_3+1}{z_1+\frac{1}{\theta}z_2+z_3+\theta}\right)^2. \end{cases} \tag{11}$$

Introducing the notations $\sqrt{z_1} = x$, $\sqrt{z_2} = y$, and $\sqrt{z_3} = z$, we rewrite the system as

$$\begin{cases} x = \frac{\theta x^2+y^2+\frac{1}{\theta}z^2+1}{x^2+\frac{1}{\theta}y^2+z^2+\theta}, \\ y = \frac{x^2+\theta y^2+z^2+\frac{1}{\theta}}{x^2+\frac{1}{\theta}y^2+z^2+\theta}, \\ z = \frac{\frac{1}{\theta}x^2+y^2+\theta z^2+1}{x^2+\frac{1}{\theta}y^2+z^2+\theta}. \end{cases} \tag{12}$$

Next, we define the operator $W : \mathbb{R}^3 \to \mathbb{R}^3$ as

$$\begin{cases} x' = \frac{\theta x^2+y^2+\frac{1}{\theta}z^2+1}{x^2+\frac{1}{\theta}y^2+z^2+\theta}, \\ y' = \frac{x^2+\theta y^2+z^2+\frac{1}{\theta}}{x^2+\frac{1}{\theta}y^2+z^2+\theta}, \\ z' = \frac{\frac{1}{\theta}x^2+y^2+\theta z^2+1}{x^2+\frac{1}{\theta}y^2+z^2+\theta}. \end{cases} \tag{13}$$

The following proposition establishes an invariant set for W:

Proposition 4.1. *The operator* W *has the invariant set:*

$$I_1 = \{(x, y, z) \in \mathbb{R}^3 : y = 1\}. \tag{14}$$

Proof. For any $v^* = (x^*, y^*, z^*)$ with $y^* = 1$, from (13) we obtain

$$\begin{cases} x' = \frac{\theta x^{*2}+y^{*2}+\frac{1}{\theta}z^{*2}+1}{x^{*2}+\frac{1}{\theta}y^{*2}+z^{*2}+\theta} = \frac{\theta x^{*2}+\frac{1}{\theta}z^{*2}+2}{x^{*2}+z^{*2}+\theta+\frac{1}{\theta}}, \\ y' = \frac{x^{*2}+\theta y^{*2}+z^2+\frac{1}{\theta}}{x^{*2}+\frac{1}{\theta}y^{*2}+z^{*2}+\theta} = \frac{x^{*2}+z^2+\theta+\frac{1}{\theta}}{x^{*2}+z^{*2}+\theta+\frac{1}{\theta}} = 1, \\ z' = \frac{\frac{1}{\theta}x^{*2}+y^{*2}+\theta z^{*2}+1}{x^{*2}+\frac{1}{\theta}y^{*2}+z^{*2}+\theta} = \frac{\frac{1}{\theta}x^{*2}+\theta z^{*2}+2}{x^{*2}+z^{*2}+\theta+\frac{1}{\theta}}. \end{cases}$$

i.e., $y' = 1$. Hence, $v' = W(v^*) \in I_1$. □

4.1. Case: I_1.

In this case, from the system of equations (12), we obtain

$$x = \frac{\theta x^2 + \frac{1}{\theta}z^2 + 2}{x^2 + z^2 + \frac{1}{\theta} + \theta}, \tag{15}$$

$$z = \frac{\frac{1}{\theta}x^2 + \theta z^2 + 2}{x^2 + z^2 + \frac{1}{\theta} + \theta}. \tag{16}$$

From Eq. (15), we derive

$$z^2 = \frac{-\theta x^3 + \theta^2 x^2 - (\theta^2+1)x + 2\theta}{\theta x - 1}. \tag{17}$$

Eq. (16) can be rewritten as

$$z^2 = \left(\frac{\frac{1}{\theta}x^2 + \theta z^2 + 2}{x^2 + z^2 + \frac{1}{\theta} + \theta}\right)^2. \tag{18}$$

From Eqs. (17) and (18), it follows that

$$\frac{-\theta x^3 + \theta^2 x^2 - (\theta^2+1)x + 2\theta}{\theta x - 1} = \left(\frac{-\theta x^3 + (\theta^2+1)x^2 - \theta^2 x + 2\theta}{\theta x^2 + 1}\right)^2, \tag{19}$$

which is equivalent to

$$(x-1)\left(2\theta x^2 - (\theta-1)^2 x + 2\theta\right)\left(x^2 - (\theta-1)x + 1\right)\left(\theta^2 x^2 - (\theta^2+\theta)x + 2\theta + 1\right) = 0. \tag{20}$$

The following lemma provides a complete analysis of Eq. (20):

Lemma 4.1. *Let $\theta_{c,1} = 3 - 2\sqrt{2}$, $\theta_{c,2} = 3$, $\theta_{c,3} = 3 + 2\sqrt{2}$, $\theta_{c,4} = 3 + 2\sqrt{3}$. Then, the following assertions hold*

- *If $\theta < \theta_{c,1}$, the equation (20) has three positive solutions $x_i,\ i = 1,2,3$.*
- *If $\theta_{c,1} \le \theta \le \theta_{c,2}$, the equation (20) has one positive solution x_1.*
- *If $\theta_{c,2} < \theta \le \theta_{c,3}$, the equation (20) has three positive solutions $x_i,\ i = 1,4,5$.*
- *If $\theta_{c,3} < \theta \le \theta_{c,4}$, the equation (20) has five positive solutions $x_i,\ i = 1,2,\dots,5$.*
- *If $\theta > \theta_{c,4}$, the equation (20) has seven positive solutions $x_i,\ i = 1,2,\dots,7$ (see Fig. 1).*

where

$$\begin{aligned}
x_1 &= 1,\\
x_2 &= \frac{(\theta-1)^2 + (\theta+1)\sqrt{\theta^2 - 6\theta + 1}}{4\theta},\\
x_3 &= \frac{(\theta-1)^2 - (\theta+1)\sqrt{\theta^2 - 6\theta + 1}}{4\theta},\\
x_4 &= \frac{\theta - 1 + \sqrt{(\theta+1)(\theta-3)}}{2},\\
x_5 &= \frac{\theta - 1 - \sqrt{(\theta+1)(\theta-3)}}{2},\\
x_6 &= \frac{\theta + 1 + \sqrt{\theta^2 - 6\theta - 3}}{2\theta},\\
x_7 &= \frac{\theta + 1 - \sqrt{\theta^2 - 6\theta - 3}}{2\theta}.
\end{aligned}$$

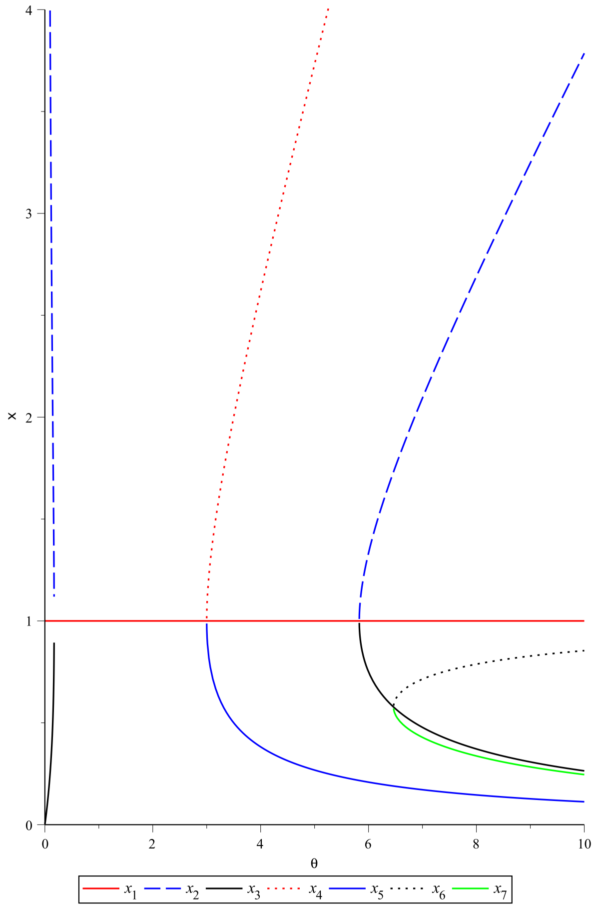

Figure 1. Plots of x_i, $i = 1, 2, ..., 7$.

Substituting x_i, $i = 1, 2, ..., 7$ values into Eq. (17), we obtain

$$z_1 = 1 = x_1,$$
$$z_2 = \frac{(\theta-1)^2 + (\theta+1)\sqrt{\theta^2 - 6\theta + 1}}{4\theta} = x_2,$$
$$z_3 = \frac{(\theta-1)^2 - (\theta+1)\sqrt{\theta^2 - 6\theta + 1}}{4\theta} = x_3,$$
$$z_4 = \frac{\theta - 1 - \sqrt{(\theta+1)(\theta-3)}}{2} = x_5,$$
$$z_5 = \frac{\theta - 1 + \sqrt{(\theta+1)(\theta-3)}}{2} = x_4,$$
$$z_6 = \frac{\theta + 1 - \sqrt{\theta^2 - 6\theta - 3}}{2\theta} = x_7,$$
$$z_7 = \frac{\theta + 1 + \sqrt{\theta^2 - 6\theta - 3}}{2\theta} = x_6.$$

4.2. **Case:** $\mathbb{R}^3 \setminus I_1$**.** In this case, from the system of equations (12), we obtain:

$$x = \frac{\theta x^2 + y^2 + \frac{1}{\theta}z^2 + 1}{x^2 + \frac{1}{\theta}y^2 + z^2 + \theta}, \tag{21}$$

$$y = \frac{x^2 + \theta y^2 + z^2 + \frac{1}{\theta}}{x^2 + \frac{1}{\theta} y^2 + z^2 + \theta}, \tag{22}$$

$$z = \frac{\frac{1}{\theta} x^2 + y^2 + \theta z^2 + 1}{x^2 + \frac{1}{\theta} y^2 + z^2 + \theta}. \tag{23}$$

From Eq. (22), we obtain either $y = 1$ or:

$$\theta x^2 = (\theta^2 - 1)y - y^2 - \theta z^2 - 1. \tag{24}$$

Eq. (23) can be rewritten as:

$$\theta(\theta - 1)(\theta + 1)(\theta yz - \theta z^2 + \theta z - y^2 - y - 1) = 0. \tag{25}$$

Eq. (25) implies:

$$\theta = \frac{y^2 + y + 1}{z(y - z + 1)}. \tag{26}$$

Substituting (24) and (26) into (21), we obtain:

$$(z^6(z-1)(y-z)(y-z^2)(y^2+y+1-z^2))(y+1-z)^6\cdot$$
$$(y^2 + yz - z^2 + y + z + 1)^2(y^2 - yz + z^2 + y - z + 1)^3 = 0. \tag{27}$$

1) Case $z = 1$: Eq. (26) simplifies to:

$$y^2 - (\theta - 1)y + 1 = 0. \tag{28}$$

The solutions of Eq. (28) are:

$$y_8 = \frac{\theta - 1 + \sqrt{(\theta+1)(\theta-3)}}{2}, \quad y_9 = \frac{\theta - 1 - \sqrt{(\theta+1)(\theta-3)}}{2}.$$

Substituting these into (24), we obtain:

$$x_8 = \frac{\theta - 1 + \sqrt{(\theta+1)(\theta-3)}}{2}, \quad x_9 = \frac{\theta - 1 - \sqrt{(\theta+1)(\theta-3)}}{2}.$$

2) Case $z = y$: From Eq. (26), we get:

$$y^2 - (\theta - 1)y + 1 = 0. \tag{29}$$

The solutions of Eq. (29) are:

$$y_{10} = z_{10} = \frac{\theta - 1 + \sqrt{(\theta+1)(\theta-3)}}{2}, \quad y_{11} = z_{11} = \frac{\theta - 1 - \sqrt{(\theta+1)(\theta-3)}}{2}.$$

The corresponding values of x are:

$$x_{10} = 1, \quad x_{11} = 1.$$

3) Case $y = z^2$: Substituting into Eq. (26), we obtain:

$$(z^2 - z + 1)(z^2 - (\theta - 1)z + 1) = 0.$$

Since $z^2 - z + 1 = 0$ has no positive solutions, we solve $z^2 - (\theta - 1)z + 1 = 0$, yielding:

$$z_{12} = \frac{\theta - 1 + \sqrt{(\theta+1)(\theta-3)}}{2}, \quad z_{13} = \frac{\theta - 1 - \sqrt{(\theta+1)(\theta-3)}}{2}.$$

The corresponding values of y and x are:

$$y_{12} = \frac{1}{4}\Big(\theta - 1 + \sqrt{(\theta+1)(\theta-3)}\Big)^2, \quad y_{13} = \frac{1}{4}\Big(\theta - 1 - \sqrt{(\theta+1)(\theta-3)}\Big)^2.$$

$$x_{12} = \frac{\theta - 1 + \sqrt{(\theta+1)(\theta-3)}}{2}, \quad x_{13} = \frac{\theta - 1 - \sqrt{(\theta+1)(\theta-3)}}{2}.$$

4) Case $y^2+y+1=z^2$: From Eq. (26), we obtain:

$$z=\frac{\theta(y+1)}{\theta+1}.$$

Substituting this into $y^2+y+1=\left(\frac{\theta(y+1)}{\theta+1}\right)^2$, we get:

$$(2\theta+1)y^2-(\theta^2-2\theta-1)y+2\theta+1=0. \tag{30}$$

The solutions of Eq. (30) are:

$$y_{14}=\frac{\theta^2-2\theta-1+(\theta+1)\sqrt{\theta^2-6\theta-3}}{4\theta+2}, \quad y_{15}=\frac{\theta^2-2\theta-1-(\theta+1)\sqrt{\theta^2-6\theta-3}}{4\theta+2}.$$

The corresponding values of z and x are:

$$z_{14}=\frac{2\theta}{\theta+1-\sqrt{\theta^2-6\theta-3}}, \quad z_{15}=\frac{2\theta}{\theta+1+\sqrt{\theta^2-6\theta-3}}.$$

$$x_{14}=\frac{2\theta}{\theta+1-\sqrt{\theta^2-6\theta-3}}, \quad x_{15}=\frac{2\theta}{\theta+1+\sqrt{\theta^2-6\theta-3}}.$$

Lemma 4.2. *There exist critical values $\theta_{c,2}=3$ and $\theta_{c,4}=3+2\sqrt{3}$ such that:*

- *If $\theta<\theta_{c,2}$, Eq. (27) does not have positive solutions.*
- *If $\theta=\theta_{c,2}$, Eq. (27) has one positive solution, $y_8=1$.*
- *If $\theta_{c,2}<\theta<\theta_{c,4}$, Eq. (27) has six positive solutions.*
- *If $\theta=\theta_{c,4}$, Eq. (27) has seven positive solutions.*
- *If $\theta>\theta_{c,4}$, Eq. (27) has eight positive solutions.*

Remark 4.1. *Note that equations $(y+1-z)^6=0$, $(y^2+yz-z^2+y+z+1)^2=0$ and $(y^2-yz+z^2+y-z+1)^3=0$ do not have positive solutions.*

Summarizing, we obtain a complete characterization of the solutions:

Proposition 4.2. *The set of solutions to the system of equations (12) varies with the parameter θ as follows. There exist critical values $\theta_{c,1}=3-2\sqrt{2}$, $\theta_{c,2}=3$, $\theta_{c,3}=3+2\sqrt{2}$, and $\theta_{c,4}=3+2\sqrt{3}$ such that*

- *If $\theta<\theta_{c,1}$, the system admits exactly three solutions, denoted as v_i, $i=1,2,3$.*
- *If $\theta_{c,1}\le\theta\le\theta_{c,2}$, the system has a unique solution v_1.*
- *If $\theta_{c,2}<\theta\le\theta_{c,3}$, the system has nine positive solutions, v_i, where $i=1,4,5,8,9,\ldots,13$.*
- *If $\theta_{c,3}<\theta<\theta_{c,4}$, the system has eleven positive solutions, v_i, where $i=1,2,3,4,5,8,9,\ldots,13$.*
- *If $\theta=\theta_{c,4}$, the system has twelve positive solutions, v_i, for $i\in\{1,2,\ldots,15\}\setminus\{3,7,15\}$.*
- *If $\theta>\theta_{c,4}$, the system has fifteen positive solutions, v_i, where $i=1,2,\ldots,15$ (see Fig. 2).*

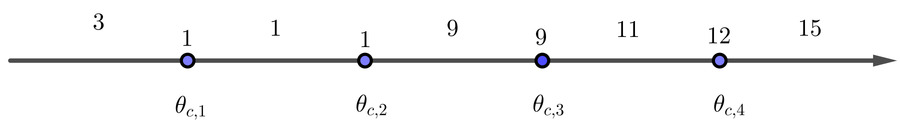

Figure 2. Distribution of the solutions of system of equations (12).

Remark 4.2. *The stability of the solutions is not addressed in the present paper. We intend to explore this issue in a subsequent work. For a related analysis of the stability of some solutions of the asymmetric clock model on a Cayley tree, we refer the reader to* [4].

5. Conclusion

In this paper, we investigated the four-state asymmetric clock model on a Cayley tree of order two, focusing on the structure and number of translation-invariant splitting Gibbs measures (TISGMs). By deriving a system of functional equations, we explored their solutions and identified critical temperature values marking phase transitions. Our results demonstrate that the number of TISGMs varies with temperature, reaching up to fifteen distinct solutions, highlighting the complex behavior of the model.

These findings provide a deeper mathematical understanding of phase transitions in discrete spin systems on tree-like structures. They also contribute to the broader study of Gibbs measures on non-Euclidean lattices, with potential applications in statistical mechanics, quantum information theory, and complex networks. Future research could extend these methods to higher-order Cayley trees or incorporate external fields to explore additional critical phenomena.

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