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ECCENTRICITY SPECTRUM OF JOIN OF CENTRAL GRAPHS AND ECCENTRICITY WIENER INDEX OF GRAPHS

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Abstract. The eccentricity matrix of a simple connected graph is obtained from its distance matrix by preserving the largest non-zero distance in each row and column, while setting all other entries to zero. This article examines the ϵ-spectrum and ϵ-spectral radius of central graphs of triangle-free regular graphs. The study further explores the irreducibility of the eccentricity matrix of central graphs. Moreover, we investigate the ϵ-spectrum and the irreducibility of various central graph operations, such as the central vertex join, central edge join, and central vertex-edge join. Also, we compute the ϵ-energy of specific graphs. These findings allow us to construct new families of ϵ-cospectral graphs and non ϵ-cospectral ϵ-equienergetic graphs. Additionally, we estimate certain upper and lower bounds for the eccentricity Wiener index of graphs and an upper bound for the eccentricity energy of self-centered graphs.

Keywords: eccentricity matrix, ϵ-spectrum, ϵ- energy, central graph, ϵ-Wiener index.

AMS Subject Classification: 05C50, 05C76.

Introduction

All graphs considered in this paper are undirected, finite, and simple. Let $G = (V(G), E(G))$ be a (p, q) graph with vertex set $V(G) = \{v_1, \ldots, v_p\}$ and edge set $E(G) = \{e_1, \ldots, e_q\}$. The *adjacency matrix* $A(G) = (a_{ij})$ of G is a $p \times p$ matrix whose $(ij)^{th}$ entry a_{ij} is 1 if the i^{th} vertex is adjacent to the j^{th} vertex ($i.e, v_i \sim v_j$) of G and 0 otherwise. The eigenvalues of $A(G)$ are called the A-eigenvalues of G. The collection of all A-eigenvalues of G together with their multiplicities is called the A-spectrum. The sum of absolute values of A-eigenvalues of G is the adjacency energy of G, denoted as $E_A(G)$. The *incidence matrix* $R(G) = (r_{ij})$ of G is a $p \times q$ matrix whose $(ij)^{th}$ entry r_{ij} is 1 if the i^{th} vertex of G is incident with the j^{th} edge of G and 0 otherwise. The *distance matrix* $D(G) = (d_{ij})$ of a graph G is a $p \times p$ matrix whose $(ij)^{th}$ entry d_{ij} is the distance between the vertices v_i and v_j. The distance between the vertices v_i and v_j is denoted by $d(v_i, v_j)$. *The degree of a vertex* v_i, denoted by $deg(v_i)$, is the number of edges incident with vertex v_i. The eccentricity of a vertex v_i is defined as $e(v_i) = \max\{d(v_i, v_j) : v_j \in V(G)\}$. The vertex v_k

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is an eccentric vertex of v_i if $d(v_i, v_k) = e(v_i)$. The minimum and maximum eccentricities of all vertices of G are called *radius* $(r(G))$ and *diameter* $(diam(G))$ of G, respectively. A graph G is *self-centered* if $r(G)$ and $diam(G)$ are the same. The *total eccentricity* of a graph G is defined as $\varepsilon^*(G) = \sum_{i=1}^{p} e(v_i)$, and the *eccentric connectivity index* of G is, $\zeta(G) = \sum_{i=1}^{p} deg(v_i)e(v_i)$.
The *eccentricity matrix* $\epsilon(G)$ of a graph G is a $p \times p$ matrix whose entries are defined as

$$\epsilon(G) = (\epsilon_{ij}) = \begin{cases} d(v_i, v_j) & \text{if } d(v_i, v_j) = \min\{e(v_i), e(v_j)\}, \\ 0 & \text{otherwise.} \end{cases}$$

A graph G is *ϵ-regular* if $\epsilon(i) = k$ (say), $i = 1, 2, \dots p$ where $\epsilon(i) = \sum_{j=1}^{p} \epsilon_{ij}$ [20]. Note that $\epsilon(G)$ is a real symmetric matrix. The eigenvalues of $\epsilon(G)$ are called the ϵ-eigenvalues of G. Let $\epsilon_1, \epsilon_2, \dots, \epsilon_k$ be the distinct eigenvalues of $\epsilon(G)$, arranged in non-increasing order, that is, $\epsilon_1 > \epsilon_2 > \dots > \epsilon_k$. Then, the ϵ-spectrum of $\epsilon(G)$ is defined as

$$spec_\epsilon(G) = \begin{pmatrix} \epsilon_1 & \epsilon_2 & \cdots & \epsilon_k \\ m_1 & m_2 & \cdots & m_k \end{pmatrix},$$

where m_i denotes the algebraic multiplicity of the eigenvalue ϵ_i, for each $i = 1, 2, \dots k$. Let J and I represent the all-one matrix and the identity matrix of appropriate order, respectively, and Q represent the equitable quotient matrix [1]. A square matrix M is *reducible*, if there is a permutation matrix P such that

$$M = P^T \begin{pmatrix} N_{11} & N_{12} \\ 0 & N_{22} \end{pmatrix} P,$$

where N_{11} and N_{22} are square block matrices. Otherwise, M is *irreducible*. Unlike adjacency and distance matrices, the eccentricity matrix is not always irreducible. Certain classes of graphs are discussed with respect to whether their eccentricity matrices are irreducible or reducible in [9, 13, 14, 21, 23].

The concept of eccentricity matrix was introduced and studied in [21], which is also known as the *D_{MAX} matrix* [16]. The *ϵ-spectral radius*, $\rho_\epsilon(G)$ of G is the largest absolute value among the ϵ-eigenvalues, while the *ϵ-energy*, $E_\epsilon(G)$ of G is the sum of the absolute values of the ϵ-eigenvalues.

Two nonisomorphic graphs with the same order and the same ϵ-spectrum are called *the ϵ-cospectral* graphs. Otherwise, they are *non ϵ-cospectral*. Two graphs are said to be *ϵ-equienergetic* if they have the same ϵ-energy. Clearly, all ϵ-cospectral graphs are ϵ-equienergetic, but the converse need not be true. Construction of non ϵ-cospectral ϵ-equienergetic graphs is an interesting problem in spectral graph theory. In this article, we explore several families of non ϵ-cospectral ϵ-equienergetic graphs.

The *Wiener index* of a graph is an important and well studied topological index in mathematical chemistry. It is defined as $W(G) = \frac{1}{2}\sum_{u,v \in V(G)} d(u, v)$. In a similar way, in [11], the eccentricity Wiener index (ϵ-Wiener index) of G is defined. The *eccentricity Wiener index* of a graph G is defined as $W_\epsilon(G) = \frac{1}{2}\sum_{ij} \epsilon_{ij}$. The inertia of the eccentricity matrices for a few graphs has been discussed in the literature, see [4, 9, 10, 12, 14]. In this work, we study the eccentricity inertia of various graphs.

In graph theory, operations on graphs play a significant role in constructing new graphs from existing ones. The studies on the eccentricity spectra of some graph operations can be found in [13, 14].

Jahfar and Chithra [8] defined the central vertex (edge) join operation on two graphs and examined the adjacency spectra of the resulting graphs when they are regular. In

[7], Haritha and Chithra computed the distance spectra of the central vertex join and the central edge join of two graphs, where the first graph is a triangle-free regular graph and the second is a regular graph. A new graph operation, the central vertex-edge join, was introduced in [5], and its distance spectrum was estimated. Motivated by these studies, we investigate the ϵ-spectrum of central graphs, central vertex joins, central edge joins, and central vertex-edge joins of graphs.

Darabi et al. [3] established the relationship between the Wiener index and the total eccentricity of a graph. In[11, 17], the authors have found lower bounds for the eccentricity spectral radius and the eccentricity spectral spread of a graph G using the ϵ-Wiener index of G. The ϵ-Wiener index of a graph, derived from the eccentricity matrix, is a relatively less studied topological index. In this paper, we focus on estimating some bounds for the ϵ-Wiener index of a graph in terms of its total eccentricity, and its eccentric connectivity index.

This article is organized as follows: Section 2 covers preliminary results. In Section 3, we explore the ϵ-spectrum, irreducibility, and ϵ-spectral radius of the central graph of a graph. Also, we estimate the ϵ-spectrum and ϵ-energy of the complement of the central graph. Furthermore, we determine the ϵ-spectrum, irreducibility and ϵ-Wiener index of the central vertex (edge) join of graphs, where the first graph is a triangle free regular graph and the second is a regular graph. Additionally, we compute the ϵ-spectrum of the central vertex-edge join operation of a triangle-free regular graph with regular graphs. Using these graph operations, we construct new families of ϵ-cospectral graphs and non ϵ-cospectral ϵ-equienergetic graphs. Section 4 presents lower and upper bounds for the ϵ-Wiener index of G, as well as a Nordhus-Guddum type upper bound for the ϵ-Wiener index. Moreover, we present an upper bound for the ϵ-Wiener index for trees and the ϵ-energy of a self-centered graph.

1. Preliminaries

This section contains some definitions and results that will be used in the next sections.

Definition 1.1. [2] *The line graph $L(G)$ of a graph G is a graph whose vertex set is same as the edge set of G, and two vertices in $L(G)$ are adjacent if the corresponding edges in G share a common endpoint.*

The line graph of $L(G)$ is denoted as $L^2(G)$.The adjacency matrix of $L(G)$ is represented as $B(G)$.

Definition 1.2. [2] *Let G be a graph on p vertices and q edges. The central graph of G is a graph $C[G]$, is obtained by subdividing each edge of G exactly once (named as subdivision vertex, denoted by $I(G)$) and joining all the non-adjacent vertices in G.*

Lemma 1.1. [2] *Let G be an r-regular, (p, q) graph with the adjacency matrix $A(G)$ and the incidence matrix $R(G)$. Let $L(G)$ be the line graph of G. Then $R(G)R(G)^T = A(G) + rI$, $R(G)^T R(G) = B(G) + 2I$, where $B(G)$ is the adjacency matrix of $L(G)$. Also, if J is an all-one matrix of appropriate order, then $JR(G) = 2J = R(G)^T J$ and $JR(G)^T = rJ = R(G)J$.*

Lemma 1.2. [2] *Let G be an r-regular, (p, q) graph with $spec(G) = \{r, \lambda_2, \dots, \lambda_p\}$. Then*

$$spec(L(G)) = \begin{pmatrix} 2r-2 & \lambda_2 + r - 2 & \cdots & \lambda_p + r - 2 & -2 \\ 1 & 1 & \cdots & 1 & q-p \end{pmatrix}.$$

Also, Z is an eigenvector corresponding to the eigenvalue -2 if and only if $RZ = 0$.

Definition 1.3. [8] *Let G_1 and G_2 be any two graphs on p_1, p_2 vertices and q_1, q_2 edges respectively. The central vertex join of G_1 and G_2 is the graph $C[G_1]\dot{\vee}G_2$, is obtained from $C[G_1]$ and G_2 by joining each vertex of G_1 with every vertex of G_2.*

Definition 1.4. [8] *Let G_1 and G_2 be any two graphs on p_1, p_2 vertices and q_1, q_2 edges respectively. The central edge join of G_1 and G_2 is the graph $C[G_1] \veebar G_2$ is obtained from $C[G_1]$ and G_2 by joining each vertex corresponding to the edges of G_1 ($I(G_1)$) with every vertex of G_2.*

Lemma 1.3. [8] *Let G be an r-regular graph on p vertices and $r = \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_p$ be the adjacency eigenvalues of G. Then,*

$$spec(C[G]) = \begin{pmatrix} 0 & \frac{(p-1-r)\pm\sqrt{(p-1-r)^2+8r}}{2} & \frac{-1-\lambda_i\pm\sqrt{(1+\lambda_i)^2+4(\lambda_i+r)}}{2} \\ \frac{p(r-2)}{2} & 1 & 1 \end{pmatrix},$$

$i = 2, \ldots, p$.

Definition 1.5. [5] *Let G_1, G_2, G_3 be any three graphs of orders p_1, p_2, p_3 and sizes q_1, q_2, q_3, respectively. The central vertex-edge join of G_1 with G_2 and G_3 is the graph $C[G_1] \vee (G_2^V \cup G_3^E)$, which is obtained from $C[G_1]$, G_2 and G_3 by joining each vertex of G_1 with every vertex of G_2 and each vertex corresponding to the edges of G_1 with every vertex of G_3.*

Lemma 1.4. [15] *Let G be a connected graph on $p \geq 3$ vertices. Then $diam(L(G)) = 1$ if and only if $G = K_3$ or $G = K_{1,p-1}$.*

Lemma 1.5. [15] *Let G be a connected graph, none of the three graphs F_1, F_2 and F_3 (see FIGURE 1) are an induced subgraph of G if and only if $diam(L(G)) \leq 2$.*

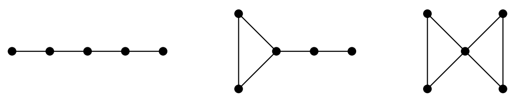

FIGURE 1. F_1, F_2 and F_3

The *complement* $\overline{G}$ of a graph G has $V(G)$ as its vertex set and two vertices are adjacent in $\overline{G}$ if and only if they are nonadjacent in G.

Lemma 1.6. [18] *If for any two adjacent vertices u and v of a graph G, there is a third vertex w that is not adjacent to either u or v (Property(†)), then,*

(1) *$\overline{G}$ is connected,*
(2) *$diam(\overline{G}) = 2$.*

For any connected graph G, the *eccentric graph*, G^e, is defined as the graph with $V(G^e) = V(G)$ and any two vertices $u, v \in V(G^e)$ are adjacent if and only if the distance detween them is $\min\{e(u), e(v)\}$ in G.

Theorem 1.1. [19] *Let G be a (p, q) graph. Then the matrix $\epsilon(G)$ is irreducible if and only if G^e is a connected graph.*

Lemma 1.7. [22] *Let G be a connected graph on p vertices. If $diam(G) = 2$ and the maximum degree of G, denoted as $\Delta(G)$, satisfies $\Delta(G) < p - 1$, then $A(\overline{G}) = \frac{1}{2}\epsilon(G)$.*

Lemma 1.8. [22] *Let G be an r-regular (p, q) graph with $diam(G) = 2$ and $e(v_i) > 1$, for every vertex $v_i \in V(G)$, then*

$$spec_\epsilon(G) = \begin{pmatrix} 2(p-r-1) & -2(1+\lambda_2) & \cdots & -2(1+\lambda_p) \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

where $r = \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_p$ are the A-eigenvalues of G.

Lemma 1.9. [11] *Let G be a connected (p, q) graph. Then $\rho_\epsilon(\epsilon(G)) \geq 2\frac{W_\epsilon}{p}$. Moreover, the equality holds if and only if G is ϵ-regular.*

Lemma 1.10. [24] *Let N be the equitable quotient matrix of M. If N is irreducible and nonnegative matrix, then M and N have the same spectral radius.*

2. Eccentricity spectrum of central graph operations of graphs

This section investigates the ϵ-spectral properties of the central graph and its complement. In addition, we discuss the ϵ-spectral properties of certain central graph operations such as the central vertex join, the central edge join, and the central vertex-edge join. Additionally, we apply these concepts to construct new families of ϵ-cospectral and non ϵ-cospectral ϵ-equienergetic graphs.

Theorem 2.1. *Let G be a triangle-free, r-regular$(r \geq 2)$, (p, q) graph. Then the ϵ-spectrum of $\epsilon(C[G])$ consists of*

(1) 3 *with multiplicity* $q - p$,
(2) $\frac{(3-(3r+\lambda_i)\pm\sqrt{(5\lambda_i+3(r-1))^2+16(\lambda_i+r)}}{2}$, $i = 2, \ldots, p$,
(3) $\frac{(3q-4r+3)\pm\sqrt{(3q-8r+3)^2+16(p-2)(q-r)}}{2}$,

where $r = \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_p$ are the eigenvalues of $A(G)$.

Proof. Let $V(C[G]) = V(G) \cup I(G)$. Then we have,

$$\begin{aligned} e(u_i) &= 2, \text{ for every } u_i \in V(G), \\ \text{and } e(u'_j) &= 3, \text{ for every } u'_j \in I(G). \end{aligned}$$

Now, by suitable labeling of vertices, the eccentricity matrix of $C[G]$ is of the form,

$$\epsilon(C[G]) = \begin{pmatrix} 2A(G) & 2(J - R(G)) \\ 2(J - R(G))^T & 3(J - I - B(G)) \end{pmatrix}.$$

Let $U_i(G)$, $i = 2, \ldots, p$ and $W_j(G)$, $j = 1, \ldots, q-p$ be the eigenvectors of $A(G)$ and $B(G)$ corresponding to the eigenvalues λ_i and -2, respectively. Using Lemma 1.1, $R(G)^T U_i(G)$ is an eigenvector of $B(G)$ corresponding to the eigenvalue $\lambda_i + r - 2$.

Let ζ be an eigenvalue of $\epsilon(C[G])$. Then, find a real number s such that $\epsilon(C[G])\chi_i(G) = \zeta\chi_i(G)$, where $\chi_i(G) = \begin{pmatrix} s\,U_i(G) \\ R(G)^T U_i(G) \end{pmatrix}$. From the equation $\epsilon(C[G])\chi_i(G) = \zeta\chi_i(G)$, we get

$$\begin{aligned} 2s\lambda_i - 2(\lambda_i + r) &= \zeta s, \\ -2s - 3(\lambda_i + r - 1) &= \zeta, \\ 2s^2 + s(\lambda_i + 3(\lambda_i + r - 1)) - 2(\lambda_i + r) &= 0. \end{aligned} \tag{1}$$

Therefore, $s = \frac{-(5\lambda_i+3(r-1))\pm\sqrt{(5\lambda_i+3(r-1))^2+16(\lambda_i+r)}}{4}$.
Hence from (1),

$$\zeta = \frac{(3-(3r+\lambda_i))\pm\sqrt{(5\lambda_i+3(r-1))^2+16(\lambda_i+r)}}{2}, i = 2,\ldots,p.$$

Thus, we get $2(p-1)$ number of eigenvalues of $\epsilon(C[G])$.

Since $\epsilon(C[G])\begin{pmatrix}0\\W_j(G)\end{pmatrix} = 3\begin{pmatrix}0\\W_j(G)\end{pmatrix}$, we get 3 is an eigenvalue of $\epsilon(C[G])$ with multiplicity $q-p$. The remaining two eigenvalues of $\epsilon(C[G])$ is obtained from the equitable quotient matrix $Q = \begin{pmatrix}2r & 2(q-r)\\2(p-2) & 3(q-2r+1)\end{pmatrix}$ of $\epsilon(C[G])$, which are

$$\frac{(3q-4r+3)\pm\sqrt{(3q-8r+3)^2+16(p-2)(q-r)}}{2}.$$

□

The following corollary is an immediate consequence of Theorem 2.1 and Lemma 1.10.

Corollary 2.1. *Let G be a triangle-free, r-regular ($r \geq 2$), (p,q) graph. Then the ϵ-spectral radius of $C[G]$ is*

$$\rho_\epsilon(C[G]) = \frac{(3q-4r+3)+\sqrt{(3q-8r+3)^2+16(p-2)(q-r)}}{2}.$$

The subsequent Lemma indicates that the graph $C[G]$ holds the property(†). To determine the eccentricity spectrum of $\overline{C[G]}$, we use the following Lemma.

Lemma 2.1. *Let G be a connected graph on p ($p \geq 3$) vertices. If u and v are two adjacent vertices in $C[G]$, then there will be a third vertex w in $C[G]$ that is not adjacent to both u and v.*

Proof. Let $V(C[G]) = V(G) \cup I(G)$, u and v be two adjacent vertices in $C[G]$.

Case 1: Let $u, v \in V(G)$. Since u and v are adjacent in $C[G]$, then they are not adjacent in G. Let P be a $u-v$ path in G. If P is of length 2, then the internal vertex of P is the required vertex. If the length of P greater than 2, then a subdivision vertex in between any two internal vertices of P is the required vertex.

Case 2: Let $u \in V(G)$ and $v \in I(G)$. If u is a pendant vertex in G, consider the vertex u_1 adjacent to u in G, and then choose an adjacent vertex u_2 of u_1 in G. Then the subdivision vertex corresponding to the edge u_1u_2 is the required vertex. If u is not a pendant vertex in G, then there exists at least two vertices u_3 and u_4 in $V(G)$ such that they are adjacent to u in G. Without loss of generality, if we consider v to be the subdivision vertex of the edge uu_3 then u_4 is the required vertex. □

From Lemmas 1.7, 1.6, 1.3 and 2.1 we obtain the following Corollaries.

Corollary 2.2. *Let G be an r-regular graph on p vertices and $r = \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_p$ be the adjacency eigenvalues of G. Then, $spec_\epsilon(\overline{C[G]})$ is*

$$\begin{pmatrix}0 & (p-1-r)\pm\sqrt{(p-1-r)^2+8r} & (-1-\lambda_i)\pm\sqrt{(1+\lambda_i)^2+4(\lambda_i+r)}\\ \frac{p(r-2)}{2} & 1 & 1\end{pmatrix},$$

$i = 2,\ldots,p$.

Let M be a symmetric matrix. The *inertia* of $M, In(M) = (n_+(M), n_-(M), n_0(M))$, describes the number of positive, negative, and zero eigenvalues of M, respectively.

Corollary 2.3. *Let G be an r-regular graph on p vertices. Then,*

(1) $In(\overline{C[G]}) = (p, p, \frac{p(r-2)}{2})$,
(2) $E_\epsilon(\overline{C[G]}) = 2E_A(C[G])$.

Theorem 2.2. *Let G be a (p, q) graph and $V(Tr')$ be the collection of all vertices in G that are not part of any triangle in G. Then,*

(1) *$\epsilon(C[G])$ is irreducible if G is triangle-free,*
(2) *$\epsilon(C[G])$ is irreducible if G is not triangle-free and $V(Tr') \neq \emptyset$.*

Proof. Case 1: Let G be a triangle-free graph. If $G \cong K_1$, K_2, then clearly $(C[G])^e$ is connected. If $G \ncong K_1$, K_2, consider $C[G]$, with $V(C[G]) = V(G) \cup I(G)$. Let u_i and u'_j denote the vertices in $V(G)$ and $I(G)$, respectively.
In $C[G]$,

$$e(u_i) = 2, \text{ for every } u_i \in V(G),$$
$$2 \leq e(u'_j) \leq 3 \text{ for every } u'_j \in I(G).$$

If $u_i \sim u_k$ in G, then $d(u_i, u_k) = 2 = \min\{e(u_i), e(u_k)\}$ in $C[G]$. Therefore, u_i is adjacent to u_k in $(C[G])^e$. Since G is a connected graph, any two vertices which are adjacent in G is also adjacent in $(C[G])^e$.
Also, since G is a triangle free graph, without loss of generality, there exists a vertex u_l such that $u_i \sim u_l$ and $u_k \nsim u_l$. Then, $u_l \sim u_k$ in $C[G]$. If u'_j is the subdivision vertex of the edge (u_i, u_k), then $d(u_l, u'_j) = 2 = \min\{e(u_l), e(u'_j)\}$in $C[G]$.This implies that every vertex u'_j in $I(G)$ is adjacent to u_l in $V(G)$. Hence $(C[G])^e$ is connected. Therefore, $\epsilon(C[G])$ is irreducible.

Case 2: Let $V(Tr)$ be the collection of vertices of K_3's in G, $V(Tr') = V(G) \setminus V(Tr)$, $V(I(Tr))$ be the collection of all newly added vertices of K_3's in $C[G]$, and $V(I(Tr')) = V(I(G)) \setminus V(I(Tr))$. If $V(Tr') \neq \emptyset$, then

$$e(v_j) = 2 \text{ for every } v_j \in V(Tr'),$$
$$e(u_i) = 3 \text{ for every } u_i \in V(Tr),$$
$$3 \leq e(u'_l) \leq 4 \text{ for every } u'_l \in V(I(Tr)),$$
$$e(v'_k) = 3 \text{ for every } v'_k \in V(I(Tr')).$$

In order to prove that, $\epsilon(C[G])$ is irreducible, it is enough to show that $(C[G])^e$ is connected. Let v_j be an arbitrary vertex in $V(Tr')$. Since every vertex $u'_l \in V(I(Tr))$ is an eccentric vertex of v_j, we have $e(v_j) = d(v_j, u'_l) = 2$. Moreover, since $\min\{e(v_j), e(u'_l)\} = 2$, it follows that v_j and u'_l are adjacent in $(C[G])^e$. Hence, every vertex in $V(Tr')$ is adjacent to every vertex in $V(I(Tr))$. Next, we prove that every vertex in $V(Tr)$ is adjacent to atleast one vertex in $V(I(Tr))$. For any vertex $u_i \in V(Tr)$, the vertex $u'_l \in V(I(Tr))$ which lies on edge of K_3 not incident to vertex u_i, implies that $d(u_i, u'_l) = 3$. Clearly, for each u_i in $V(Tr)$, there exists at least one vertex u'_l in $V(I(Tr))$ such that $d(u_i, u'_l) = \min\{e(u_i), e(u'_l)\}$. Therefore, $u_i \sim u'_l$ in $(C[G])^e$. Using a similar argument, for any vertex in $V(I(Tr'))$, there exists a vertex in $V(I(Tr))$ to which it is adjacent. Thus, $(C[G])^e$ is connected.

□

Theorem 2.3. *For $i = 1, 2$, let G_i be an r_i-regular, (p_i, q_i) graph, where G_1 is triangle-free. If $\{r_1, \lambda_2, \ldots, \lambda_{p_1}\}$ and $\{r_2, \beta_2, \ldots, \beta_{p_2}\}$ are the A-eigenvalues of G_1 and G_2 respectively. Then the ϵ-spectrum of $C[G_1]\dot{\vee}G_2$ consists of*

(1) 3 *with multiplicity* $q_1 - p_1$,
(2) $-2(1 + \beta_j)$, $j = 2, \ldots, p_2$,
(3) $-(2t_1 + 3(\lambda_i + r_1 - 1))$, $t_1 = \frac{-(3(r_1-1)+5\lambda_i)+\sqrt{(3(r_1-1)+5\lambda_i)^2+16(\lambda_i+r_1)}}{4}$, $i = 2, \ldots, p_1$,
(4) $-(2t_2 + 3(\lambda_i + r_1 - 1))$, $t_2 = \frac{-(3(r_1-1)+5\lambda_i)-\sqrt{(3(r_1-1)+5\lambda_i)^2+16(\lambda_i+r_1)}}{4}$, $i = 2, \ldots, p_1$,
(5) *the eigenvalues of the equitable quotient matrix of* $C[G_1]\dot{\vee}G_2$,

$$Q = \begin{pmatrix} 2r_1 & 2(q_1 - r_1) & 0 \\ 2(p_1 - 2) & 3(q_1 - 2r_1 + 1) & 2p_2 \\ 0 & 2q_1 & 2(p_2 - 1 - r_2) \end{pmatrix}.$$

Proof. By a proper labeling of vertices of $C[G_1]\dot{\vee}G_2$,

$$\epsilon(C[G_1]\dot{\vee}G_2) = \begin{pmatrix} 2A(G_1) & 2(J - R(G_1)) & 0 \\ 2(J - R(G_1)^T) & 3(J - I - B(G_1)) & 2J \\ 0 & 2J & 2(J - I - A(G_2)) \end{pmatrix}.$$

Let $W_j(G_1)$, $j = 1, 2, \ldots, q_1 - p_1$ be an eigenvector of $B(G_1)$ corresponding to the eigenvalue -2 with multiplicity $q_1 - p_1$. Then,

$$\epsilon(C[G_1]\dot{\vee}G_2) \begin{pmatrix} 0 \\ W_j(G_1) \\ 0 \end{pmatrix} = 3 \begin{pmatrix} 0 \\ W_j(G_1) \\ 0 \end{pmatrix}.$$

Thus, 3 is an eigenvalue of $\epsilon(C[G_1]\dot{\vee}G_2)$ with multiplicity $q_1 - p_1$.

Let $X_k(G_2)$ be an eigenvector of $A(G_2)$ corresponding to the eigenvalue β_k, for $k = 2, \ldots, p_2$. Then, $\epsilon(C[G_1]\dot{\vee}G_2) \begin{pmatrix} 0 \\ 0 \\ X_k(G_2) \end{pmatrix} = -2(1 + \beta_k) \begin{pmatrix} 0 \\ 0 \\ X_k(G_2) \end{pmatrix}$. Thus, $-2(1 + \beta_k)$, for $k = 2, \ldots, p_2$ is an eigenvalue of $\epsilon(C[G_1]\dot{\vee}G_2)$.

Let $U_l(G_1)$ be an eigenvector of $A(G_1)$ corresponding to the eigenvalue λ_l, for $l = 2, \ldots, p_1$. Using Lemma 1.1, $R(G_1)^T U_l(G_1)$ is an eigenvector of $B(G_1)$ corresponding to the eigenvalue $\lambda_l + r_1 - 2$. Thus, the vectors $U_l(G_1)$ and $R(G_1)^T U_l(G_1)$ are orthogonal to the vector J.

Now, consider the vector $\phi = \begin{pmatrix} t\,U_l(G_1) \\ R(G_1)^T U_l(G_1) \\ 0 \end{pmatrix}$. Next, determine the conditions under which ϕ is an eigenvector of $\epsilon(C[G_1]\dot{\vee}G_2)$. If μ is an eigenvalue of $\epsilon(C[G_1]\dot{\vee}G_2)$ corresponding to the eigenvector ϕ, then

$$\begin{aligned} 2t\lambda_i - 2(\lambda_i + r_1) &= \mu t, \\ -(2t + 3(\lambda_i + r_1 - 1)) &= \mu. \end{aligned}$$

From this we get,

$$2t^2 + t(3(\lambda_i + r_1 - 1) + 2\lambda_i) - 2(\lambda_i + r_1) = 0.$$

Therefore, $\mu = -(2t + 3(\lambda_i + r_1 - 1))$, where $t = \frac{-(3(r_1-1)+5\lambda_i)\pm\sqrt{(3(r_1-1)+5\lambda_i)^2+16(\lambda_i+r_1)}}{4}$. The remaining three eigenvalues are given by the equitable quotient matrix of $\epsilon(C[G_1]\dot{\vee}G_2)$,

$$Q = \begin{pmatrix} 2r_1 & 2(q_1 - r_1) & 0 \\ 2(p_1 - 2) & 3(q_1 - 2r_1 + 1) & 2p_2 \\ 0 & 2q_1 & 2(p_2 - 1 - r_2) \end{pmatrix}.$$

□

Corollary 2.4. *Let G_i be an r_i-regular, (p_i, q_i) graph, $i = 1, 2$. Then the ϵ-Wiener index of $\epsilon(C[G_1]\dot{\vee}G_2)$ is given by*

$$W_\epsilon(C[G_1]\dot{\vee}G_2) = 2q_1(p_1 - 1) + \frac{3}{2}q_1(q_1 - 2r_1 + 1) + p_2(2q_1 + p_2 - r_2 - 1).$$

Using Lemma 1.9 and Corollary 2.4, we get the following result.

Corollary 2.5. *Let G_i be an r_i-regular, (p_i, q_i) graph, $i = 1, 2$. Then,*

$$\rho_\epsilon(C[G_1]\dot{\vee}G_2) > \frac{4q_1(p_1 - 1) + 3q_1(q_1 - 2r_1 + 1) + 2p_2(2q_1 + p_2 - r_2 - 1)}{p_1 + p_2 + q_1}.$$

Proposition 2.1. *Let G_1 and G_2 be any two graphs, then $\epsilon(C[G_1]\dot{\vee}G_2)$ is always irreducible.*

Proof. Let $G = C[G_1]\dot{\vee}G_2$ and $V(G) = V(G_1) \cup I(G_1) \cup V(G_2)$. Let u_i, v_j and w_k denote the vertices in $V(G_1)$, $I(G_1)$ and $V(G_2)$ respectively. Then,

$$\begin{aligned} 2 \le e(u_i) \le 3, &\text{ for every } u_i \in V(G_1),\\ 3 \le e(v_j) \le 4, &\text{ for every } v_j \in I(G_1),\\ e(w_k) = 2, &\text{ for every } w_k \in V(G_2). \end{aligned}$$

In G^e, every vertex in $V(G_2)$ is adjacent to every vertex in $I(G_1)$, since $d(w_k, v_j) = 2 = \min\{e(w_k), e(v_j)\}$. Also, for each vertex u_i there is a vertex v_j such that $d(u_i, v_j) = e(u_i) = \min\{e(u_i), e(v_j)\}$. Therefore, every vertex in $V(G_1)$ will be adjacent to at least one vertex of $I(G_1)$ in G^e. Hence, G^e is connected. Thus, by Theorem 1.1, $\epsilon(C[G_1]\dot{\vee}G_2)$ is irreducible. □

Theorem 2.4. *Let G_i be an r_i-regular, $i = 1, 2$, (p_i, q_i) graph where G_1 is triangle-free. If $\{r_1, \lambda_2, \ldots, \lambda_{p_1}\}$ and $\{r_2, \beta_2, \ldots, \beta_{p_2}\}$ are the A-eigenvalues of G_1 and G_2 respectively, then the ϵ-spectrum of $C[G_1] \veebar G_2$ consists of*

1. *-2 with multiplicity $q_1 - p_1$,*
2. *$-2(1 + \beta_j)$, $j = 2, \ldots, p_2$,*
3. *$(\lambda_i - 1) \pm \sqrt{(\lambda_i + 1)^2 + 4(\lambda_i + r_1)}$, $i = 2, \ldots, p_1$,*
4. *the eigenvalues of the equitable quotient matrix of $C[G_1] \veebar G_2$,*

$$Q = \begin{pmatrix} 2r_1 & 2(q_1 - r_1) & 2p_2 \\ 2(p_1 - 2) & 2(q_1 - 1) & 0 \\ 2p_1 & 0 & 2(p_2 - 1 - r_2) \end{pmatrix}.$$

Proof. By a proper labeling of vertices of $C[G_1] \veebar G_2$ we get

$$\epsilon(C[G_1] \veebar G_2) = \begin{pmatrix} 2A(G_1) & 2(J - R(G_1)) & 2J \\ 2(J - R(G_1))^T & 2(J - I) & 0 \\ 2J & 0 & 2(J - I - A(G_2)) \end{pmatrix}.$$

Let $Z_l(G_1)$, $l = 1, 2 \ldots, q_1 - p_1$ be an eigenvector of $B(G_1)$ corresponding to the eigenvalue -2 with multiplicity $q_1 - p_1$. Then,

$$\epsilon(C[G_1] \veebar G_2) \begin{pmatrix} 0 \\ Z_l(G_1) \\ 0 \end{pmatrix} = -2 \begin{pmatrix} 0 \\ Z_l(G_1) \\ 0 \end{pmatrix}.$$

Thus, -2 is an eigenvalue of $\epsilon(C[G_1] \veebar G_2)$with multiplicity $q_1 - p_1$.

Let $X_j(G_2)$ be an eigenvector of $A(G_2)$ corresponding to the eigenvalue β_j $(j = 2, \ldots, p_2)$. Then, $\epsilon(C[G_1] \veebar G_2) \begin{pmatrix} 0 \\ 0 \\ X_j(G_2) \end{pmatrix} = -2(1+\beta_j) \begin{pmatrix} 0 \\ 0 \\ X_j(G_2) \end{pmatrix}$. Thus, $-2(1+\beta_j)$ is an eigenvalue of $\epsilon(C[G_1] \veebar G_2)$.

Let $U_k(G_1)$ be an eigenvector of $A(G_1)$ corresponding to the eigenvalue λ_k, $k = 2, \ldots, p_1$. Then, $R(G_1)^T U_k(G_1)$ is an eigenvector of $B(G_1)$ corresponding to the eigenvalue $\lambda_k + r_1 - 2$. Consider the vector $\phi = \begin{pmatrix} t\, U_k(G_1) \\ R(G_1)^T U_k(G_1) \\ 0 \end{pmatrix}$. Next, examine the conditions under which ϕ is an eigenvector of $\epsilon(C[G_1] \veebar G_2)$. Let μ be an eigenvalue of $\epsilon(C[G_1] \veebar G_2)$ such that $\epsilon(C[G_1] \veebar G_2)\phi = \mu\phi$. Solving this equation yields two possible values for t,

$$t = \frac{-(1+\lambda_i) \pm \sqrt{(1+\lambda_i)^2 + 4(r_1+\lambda_i)}}{2}.$$

Therefore,

$$\mu = (\lambda_i - 1) \pm \sqrt{(\lambda_i+1)^2 + 4(\lambda_i + r_1)}.$$

The remaining three eigenvalues are obtained from the equitable quotient matrix of $\epsilon(C[G_1] \veebar G_2)$,

$$Q = \begin{pmatrix} 2r_1 & 2(q_1 - r_1) & 2p_2 \\ 2(p_1 - 2) & 2(q_1 - 1) & 0 \\ 2p_1 & 0 & 2(p_2 - 1 - r_2) \end{pmatrix}.$$

□

Corollary 2.6. *Let G_i be an r_i-regular, (p_i, q_i) graph for $i = 1, 2$, where G_1 is triangle-free. Then, the ϵ-Wiener index of $\epsilon(C[G_1] \veebar G_2)$ is,*

$$W_\epsilon(C[G_1] \veebar G_2) = q_1(2p_1 + q_1 - 3) + p_2(2p_1 + p_2 - 1 - r_2).$$

Corollary 2.7. *Let G_i be an r_i-regular, (p_i, q_i) graph for $i = 1, 2$, where G_1 is triangle-free. Then,*

$$\rho_\epsilon(C[G_1] \veebar G_2) > \frac{2q_1(2p_1 + q_1 - 3) + 2p_2(2p_1 + p_2 - 1 - r_2)}{p_1 + p_2 + q_1}.$$

Proposition 2.2. *Let G_1 and G_2 be two graphs, then $\epsilon(C[G_1] \veebar G_2)$ is always irreducible.*

The following Corollaries are consequences of Theorems 2.3 and 2.4 which describe the construction of ϵ-cospectral graphs and non ϵ- cospectral ϵ-equienergetic graphs.

Corollary 2.8. *Let G be a regular triangle-free graph and S_1, S_2 be two cospectral graphs. Then,*

1. *$C[G]\dot{\vee}S_1$ and $C[G]\dot{\vee}S_2$ are ϵ-cospectral,*
2. *$C[G] \veebar S_1$ and $C[G] \veebar S_2$ are ϵ-cospectral.*

Corollary 2.9. *Let $G_1 = L^2(H_1)$ and $G_2 = L^2(H_2)$, where H_1 and H_2 are two non-cospectral, 3-regular, $(2t, 3t)$ graphs such that $t \geq 3$. Let G be a regular triangle-free graph. Then,*

1. *$C[G]\dot{\vee}G_1$ and $C[G]\dot{\vee}G_2$ are non ϵ-cospectral ϵ-equienergetic,*
2. *$C[G] \veebar G_1$ and $C[G] \veebar G_2$ are non ϵ-cospectral ϵ-equienergetic.*

Theorem 2.5. *For $i = 1, 2, 3$, let $r_i = \lambda_{i1} \geq \lambda_{i2} \geq \ldots \geq \lambda_{ip_i}$ be the A-eigenvalues of the r_i-regular graphs G_i, on p_i vertices and q_i edges, where G_1 is a triangle-free graph. Then, the ϵ-spectrum of $C[G_1] \vee (G_2^V \cup G_3^E)$ consists of,*

(1) $(-1+\lambda_{1j}) \pm \sqrt{(1+\lambda_{1j})^2+4(\lambda_{1j}+r_1)}, j=2,3,\dots,p_1,$
(2) -2 *with multiplicity* q_1-p_1,
(3) 0 *with multiplicity* p_2-p_3-2,
(4) *the eigenvalues of the equitable quotient matrix of* $C[G_1]\vee(G_2^V\cup G_3^E)$,

$$Q=\begin{pmatrix} 2r_1 & 2(q_1-r_1) & 0 & 2p_3\\ 2(p_1-2) & 2(q_1-1) & 2p_2 & 0\\ 0 & 2q_1 & 0 & 3p_3\\ 2p_1 & 0 & 3p_2 & 0\end{pmatrix}.$$

Proof. By a proper labeling of vertices,

$$\epsilon(C[G_1]\vee(G_2^V\cup G_3^E))=\begin{pmatrix} 2A(G_1) & 2(J-R(G_1)) & 0 & 2J\\ 2(J-(R(G_1))^T) & 2(J-I) & 2J & 0\\ 0 & 2J & 0 & J\\ 2J & 0 & 3J & 0\end{pmatrix}.$$

The rest of the proof follows a similar approach to that of Theorem 2.3.

□

Corollary 2.10. *Let* G *be a triangle-free, regular graph and* $G_i, i=2,3,4,5$ *be an* r*-regular graph with* p *vertices. Then, the graphs* $C[G_1]\vee(G_2^V\cup G_3^E)$ *and* $C[G_1]\vee(G_4^V\cup G_5^E)$ *are* ϵ*-cospectral.*

Proposition 2.3. *Let* G_i, $i=1,2,3$ *be any three graphs, where* G_1 *is triangle-free. Then* $\epsilon(C[G_1]\vee(G_2^V\cup G_3^E))$ *is irreducible.*

Proposition 2.4. *Let* G *be an* r*-regular graph on* p $(p\geq 4)$ *vertices such that none of the three graphs* F_1, F_2 *and* F_3 *are induced subgraph of* G*. If the smallest* A*-eigenvalue of* G *is greater than or equal to* $1-r$*, then*

$$E_\epsilon(L(G))=4p(r-1)+4(1-2r).$$

Proof. Using Lemmas 1.4 and 1.5, $diam(L(G))=2$. Hence by Lemma 1.8

$$spec_\epsilon(L(G))=\begin{pmatrix} pr-4r+2 & -2(\lambda_i+r-1) & 2\\ 1 & 1 & \frac{p(r-2)}{2}\end{pmatrix},$$

where $r\geq\lambda_2\geq\dots\geq\lambda_p$ are the adjacency eigenvalues of G. □

Example 2.1. *Let* K_p *be the complete graph on* p *vertices,* $K_{p,p}$ *be the complete bipartite graph on* $2p$ *vertices, and* $CP(k)$ *be the cocktail party graph (with regularity* $r=2k-2$ *and number of vertices* $p=2k$*). The smallest* A*-eigenvalue for* K_p *and* $CP(k)$ $(k\geq 3)$ *is at least* $1-r$*. In addition, none of the three graphs* F_1*,* F_2*, and* F_3 *in Figure 1 is an induced subgraph of these graphs. From Proposition 2.4,*

(1) $E_\epsilon(L(K_p))=4(p-1)(p-3),\ p\geq 3$
(2) $E_\epsilon(L(CP(k)))=6k(2k-3),\ k\geq 3$
(3) $E_\epsilon(L(K_{p,p}))=8(p-1)^2$.

Corollary 2.11. *Let* G *and* H $(G_1,G_2\neq K_3)$*be two* r*-regular graphs on* p *vertices such that none of the three graphs* F_1*,* F_2 *and* F_3 *is an induced subgraph of* G *and* H*. Then,* G *and* H *are cospectral if and only if* $L(G)$ *and* $L(H)$ *are* ϵ*-cospectral.*

Corollary 2.12. *Let* G *and* H *be* r*-regula, non-complete, non-cospectral graphs on* p *vertices such that none of the three graphs* F_1 F_2 *and* F_3 *is an induced subgraph of* G *and* H*. If the smallest* A*-eigenvalue of* G_i*,* $i=1,2$ *greater than or equal to* $1-r$*, then* $L(G)$ *and* $L(H)$ *are non* ϵ*-cospectral* ϵ*-equienergetic.*

The following Theorem is a consequence of Lemmas 1.2, 1.7 and 1.6.

Theorem 2.6. *Let G be an r-regular graph on p vertices and $L(G)$ be the line graph of G. If the smallest A-eigenvalue of G is greater than or equal to $2-r$ and $L(G)$ hold property (†), then*

$$E_\epsilon(\overline{L(G)}) = 4p(r-2).$$

Example 2.2. *The graphs $CP(3)$ and $CP(4)$ have the smallest A-eigenvalue greater than or equal to $2-r$, and their corresponding line graphs satisfy property (†). Thus, $E_\epsilon(\overline{L(CP(3))}) = 96$ and $E_\epsilon(\overline{L(CP(4))}) = 128$.*

3. Some bounds for eccentricity Wiener index of graphs

This section is devoted to determining the lower and upper bounds of the ϵ- Wiener index of graphs. Let G be a (p,q) graph with *diameter* 2. If G is self-centered, then the number of entries equal to 2 in $\epsilon(G)$ is p^2-p-2q. When G is not self-centered, $\epsilon(G)$ contains $l(p-1)+(p-l)l$ entries equal to 1 and p^2-p-2q entries equal to 2, where l denotes the number of vertices v_i satisfying $e(v_i)=1$. From the definition of ϵ-Wiener index we have the following,

$$W_\epsilon(G) = \begin{cases} p(p-1) - 2q + \frac{l}{2}(2p-l-1), & \text{if } G \text{ is not self centered,} \\ p(p-1) - 2q, & \text{if } G \text{ is self centered.} \end{cases}$$

Now, applying Lemma 1.9, we obtain the following proposition.

Proposition 3.1. *Let G be a (p,q) graph of diameter 2. Then,*

$$\rho_\epsilon(G) \geq \begin{cases} \dfrac{2(p(p-1) - 2q + \frac{l}{2}(2p-l-1))}{p}, & \text{if } G \text{ is not self centered,} \\ \dfrac{2(p(p-1) - 2q)}{p}, & \text{if } G \text{ is self centered,} \end{cases}$$

where l is the number of vertices v_i in G such that $e(v_i)=1$.

The following Proposition is obtained from Lemma 1.6.

Proposition 3.2. *Let G be a (p,q) graph having property (†). Then, $W_\epsilon(\overline{G}) = 2q$.*

The following Corollary is a consequence of Proposition 3.2 and Lemma 1.9.

Corollary 3.1. *Let G be a (p,q) graph with property (†) and girth $g(\geq 5)$. Then,*

$$\rho_\epsilon(\overline{G}) \geq \frac{4q}{p}.$$

Now, we consider the graphs with diameter greater than 2. Determining the bounds for $W_\epsilon(G)$ based on the number of vertices and edges of G is challenging. All upper bounds of the Wiener index of G will also be upper bounds for $W_\epsilon(G)$. Here, we obtain certain bounds for $W_\epsilon(G)$ that are tighter than the bounds of the Wiener index of G by using the total eccentricity of G ($\varepsilon^*(G)$) and the eccentricity connectivity index $\zeta(G)$ of G.

Proposition 3.3. *Let G be a graph, then $W_\epsilon(G) \geq \dfrac{\varepsilon^*(G)}{2}$. The equality holds if G is an even cycle.*

Proposition 3.4. *Let G be a (p,q) graph with $e(v_i) > 1$ for every $v_i \in V(G)$. Then,*

$$W_\epsilon(G) \leq \frac{(p-1)\varepsilon^*(G) - \zeta(G)}{2}. \tag{2}$$

The equality holds if and only if $diam(G) = 2$.

Proof. From the definition of the eccentricity Wiener index, we get

$$2W_\epsilon(G) \leq \sum_{v_i}(p-1-deg(v_i))e(v_i) = (p-1)\varepsilon^*(G) - \zeta(G).$$

□

Using Proposition 3.4 we provide a Nordhaus-Gaddum type upper bound for the ϵ-Wiener index of G.

Proposition 3.5. *Let G be a (p,q) graph with $e(v_i) > 1$, $v_i \in V(G)$. If $\overline{G}$ is also connected, then*

$$W_\epsilon(G) + W_\epsilon(\overline{G}) \leq \frac{1}{2}\Big((p-1)(\varepsilon^*(G) + \varepsilon^*(\overline{G})) - (\zeta(G) - \zeta(\overline{G}))\Big).$$

Proposition 3.6. *Let T be a tree on p vertices. Then,*

$$W_\epsilon(T) \leq \frac{k\varepsilon^*(T) + dk(p-k-1) - k(p-k)}{2},$$

where k and d represent the number of pendant vertices and the diameter of T, respectively. Moreover, equality holds if $T \cong K_{1,p-1}$.

Proof. Let $v_1, \ldots, v_k$ be the pendant vertices of T and $v_{k+1}, \ldots, v_p$ be the non pendant vertices of T. Then,

$$\epsilon(v_i) \leq e(v_i)(k-1) + (e(v_i)-1)(p-k), \text{ for } i = 1, 2, \ldots, k. \tag{3}$$

$$\epsilon(v_j) \leq e(v_j)k, \text{ for } j = k, k+1, \ldots, n. \tag{4}$$

Hence

$$2W_\epsilon(T) \leq k\varepsilon^*(T) + (p-k-1)(e(v_1) + e(v_2) + \ldots + e(v_k)) - (p-k)k \tag{5}$$

$$\leq k\varepsilon^*(T) + (p-k-1)kd - (p-k)k. \tag{6}$$

If $T \cong K_{1,p-1}$, then equality holds in equations (3), (4), (5) and (6).

□

Next, we provide an upper bound for the eccentricity energy of self-centered graphs in terms of its ϵ-Wiener index.

Theorem 3.1. *Let G be a self-centered graph on p vertices and diameter d. Then,*

$$E_\epsilon(G) \leq \frac{2W_\epsilon(G)}{p} + \sqrt{2(p-1)W_\epsilon(G)\left(d - \frac{2W_\epsilon(G)}{p^2}\right)}. \tag{7}$$

The equality holds in (7) if and only if G is a ϵ-regular graph with two distinct ϵ-eigenvalues $\frac{2W_\epsilon(G)}{p}$ and $-\sqrt{\frac{||\epsilon(G)||_2^2 - \frac{4W_\epsilon(G)^2}{p^2}}{p-1}}$, or three distinct ϵ-eigenvalues $\frac{2W_\epsilon(G)}{p}$, $-\sqrt{\frac{||\epsilon(G)||_2^2 - \frac{4W_\epsilon(G)^2}{p^2}}{p-1}}$ and $\sqrt{\frac{||\epsilon(G)||_2^2 - \frac{4W_\epsilon(G)^2}{p^2}}{p-1}}$.

Proof. Let $\epsilon_1 \geq \epsilon_2 \geq \ldots \geq \epsilon_p$ be the ϵ-eigenvalues of G. Then, by using the Cauchy-Schwarz inequality,

$$E_\epsilon(G) = \sum_{i=1}^{p}|\epsilon_i| \leq \epsilon_1 + \sqrt{(p-1)\sum_{i=2}^{p}|\epsilon_i|^2} \tag{8}$$

$$= \epsilon_1 + \sqrt{(p-1)(||\epsilon(G)||_2^2 - |\epsilon_1|^2)}.$$

Now, consider the function $h(x) = x + \sqrt{(p-1)(||\epsilon(G)||_2^2 - x^2)}$, is strictly decreasing in the interval $[\frac{||\epsilon(G)||_2}{\sqrt{p}}, ||\epsilon(G)||_2]$. The following inequality is obtained from Lemma 1.9 and G is self-centered,

$$\frac{||\epsilon(G)||_2}{\sqrt{p}} \leq \frac{2W_\epsilon(G)}{p} \leq \epsilon_1 < ||\epsilon(G)||_2.$$

Therefore,

$$\epsilon_1 + \sqrt{(p-1)(||\epsilon(G)||_2^2 - \epsilon_1^2)} \leq \frac{2W_\epsilon(G)}{p} + \sqrt{(p-1)\left(||\epsilon(G)||_2^2 - \frac{4W_\epsilon(G)^2}{p^2}\right)}. \qquad (9)$$

Hence,

$$\begin{aligned} E_\epsilon(G) &\leq \frac{2W_\epsilon(G)}{p} + \sqrt{(p-1)\left(||\epsilon(G)||_2^2 - \frac{4W_\epsilon(G)^2}{p^2}\right)} \\ &= \frac{2W_\epsilon(G)}{p} + \sqrt{(p-1)\left(2dW_\epsilon(G) - \frac{4W_\epsilon(G)^2}{p^2}\right)}. \end{aligned}$$

Now, we examine the equality in (7). If the equality in (8) holds, then $|\epsilon_2| = |\epsilon_3| = \ldots = |\epsilon_p| = \sqrt{\frac{||\epsilon(G)||_2^2 - \frac{4W_\epsilon(G)^2}{p^2}}{p-1}}$. The equality in (9) implies that $\epsilon_1 = \frac{2W_\epsilon(G)}{p}$. Then, either G is ϵ-regular with 2 distinct ϵ-eigenvalues or G is ϵ-regular with 3 distinct ϵ-eigenvalues. One can easily verify the converse. This completes the proof. □

4. Conclusion

Graph operations such as the central graph of a graph, the central vertex join, the central edge join, and the central vertex-edge join have been used to expand the collection of existing graphs for which the ϵ-spectrum, irreducibility, the ϵ-inertia, and ϵ-energy are known. In addition, the ϵ-energies of certain classes of graphs are estimated. These results allow us to construct new families of ϵ-cospectral graphs and non ϵ-cospectral ϵ-equienergetic graphs. Furthermore, the lower and upper bounds for the ϵ-Wiener index are determined using the eccentricity connectivity index and the total eccentricity of a graph. An upper bound for the ϵ-energy of self-centered graphs is also provided.

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On behalf of all authors, the corresponding author states that there is no conflict of interest.

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