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AN EFFICIENT NUMERICAL ALGORITHM FOR SOLVING THE LANE-EMDEN TYPE FUNCTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. In this article, we analyze collocation points and shifted Chebyshev polynomials based strategy to approximate the solution of the Lane-Emden type functional differential equations subjected to three-point boundary conditions. Shifted Chebyshev polynomials are used to reduce the problem into a matrix form, and then, collocation points are used to transform the matrix form into a system of nonlinear algebraic equations. The simplicity of the mathematical formulation and ease of code computation, demonstrate the accessibility and flexibility of the proposed numerical technique. The outcomes clearly show that the proposed approach achieves rapid convergence, exhibits a high level of computational efficiency, and delivers precise approximations. Finally, numerous examples are included to illustrate and confirm the applicability, validity and superiority of the proposed approach over the existing methods.

Keywords: Shifted Chebyshev Polynomials; Collocation Points; Lane-Emden Type Functional Differential Equations; Convergence Analysis.

AMS Subject Classification: 65T60, 34A12, 65L10.

1. Introduction

The present work is focused on a complex mathematical model involving a functional-type differential equation (DE) subject to three-point boundary conditions, which arises in the several phenomena of astrophysics, physiology, and other areas of science and technology. In this paper, we propose a novel numerical strategy to solve Lane-Emden functional-type differential equations (LEFDEs) subject to three-point boundary conditions (BCs), formulated as follows:

$$S''(\gamma_1\zeta+\gamma_2)+\frac{\alpha}{\zeta}S'(\beta_1\zeta+\beta_2)+p(\zeta)f(S(\alpha_1\zeta+\alpha_2))=g(\zeta), \quad 0<\zeta\leq 1, \tag{1}$$

subject to BCs

$$S'(0)=\mu, \quad S(1)=mS(\nu), \tag{2}$$

where $\nu\in(0,1)$, α_1, α_2, β_1, β_2, γ_1, γ_2, μ, and m are the real constants, and f, g are known continuous functions. Here, $\zeta=0$ is the singular point of LEFDE.

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The DE (1) can be reduced to the well-known Lane-Emden equation (see [21, 27] and the references therein) in the case $\alpha_1 = \beta_1 = \gamma_1 = 1$, $\alpha_2 = \beta_2 = \gamma_2 = 0$, $f(S) = S^n$, and $g(\zeta) = 0$. Thus, the DE (1) can be regarded as a Lane-Emden functional-type differential equation. Here, the term 'functional' refers to the dependence of the dependent variable not only on the independent variable but also through a functional relationship involving the function itself. In other words, the equation may include the terms, where the dependent variable or its derivatives appear with the dependency of another function, which makes it as a functional differential equation.

Unlike standard boundary conditions, the boundary value problem (BVP) (1)–(2) can be regarded as a non-local Lane-Emden type functional boundary value problem (LEF-BVP). The current article generalizes the studies presented in [20, 23, 25, 26]. The study of linear functional-type differential equations has been considered in [5]. Motivated by these papers, nonlinear functional-type differential equations are investigated throughout the article. Thus, our work also generalizes the study presented in [5].

In the recent past, some analytical works have been carried out on singular functional-type differential equations. A first-order functional-type differential equation $x'(t) = Ax(t) + f(t, x(t - \rho(x(t))))$, $t \in [0, b]$, subject to the condition $x(t) = \phi(t)$, $t \in [-r, 0]$, was studied in [1], where the existence of a solution in a Banach space has been shown. The existence of multiple solutions for the second-order singular delay differential equation $y'' + f(t, y(t - \tau)) = 0$, $t \in (0, 1) \setminus \{\tau\}$, subject to the boundary conditions $y(t) = \eta(t)$ for $t \in [-\tau, 0]$ and $y(1) = 0$, allowing singularities at $t = 0, 1$ and $y = 0$, has been carried out in [30]. Existence results for second-order singular functional differential equations in Hardy-Lebesgue space is discussed in [12].

In addition to these analytical works, a parallel numerical studies have also been carried out in the past few decades. A lot of researchers have expressed their interest in solving singular differential equations, specifically the Lane-Emden type DEs (see [10, 28] and reference therein) numerically. A variety of physical phenomena, including catalytic diffusion reactions, isotropic continuous media, morphogenesis, isothermal gas spheres, and stellar structure, have been extensively modeled using the Lane-Emden type DEs. Infinite Taylor series method [8], adomian decomposition method [18], hybrid function approximation method [6], homotopy perturbation method [3, 4], homotopy analysis method [15, 18, 19], Haar wavelet quasilinearization method [17], variational iteration method [24], and Bernstein collocation method [11] etc. are the few numerical techniques that have been developed in the literature to solve Lane-Emden type DEs.

Recently, a linear LEFDE with initial conditions has been solved using the Laguerre polynomial approach [5]. It has been pointed out that there is very limited literature for such a vast area of research on non-linear LEFBVPs. Mathematical models that presume a given behavior or event, depend on both the past and present states of a system, employ functional-type differential equations. To put it another way, the past directly affects the present and future. Thus, the functional-type differential equations (FDEs) are more useful than the ordinary differential equations (ODEs), where the future behavior depends only indirectly on the past. In this way, LEFBVPs become a matter of great interest for numerous scientific communities, working in the fields of astrophysics, physiology, and thermal conduction in the human head. So, an accurate and efficient numerical algorithm for the numerical solution of non-linear LEFBVPs is the most demanded algorithm among the researchers.

Upon reviewing the vast literature, it is found that the equation (1) has not yet been studied by any researcher subjected to the non-local three-point boundary condition (2). In the context of modeling of various physical processes, non-local boundary conditions are especially helpful when the value of the solution at the boundaries is unknown but the solution at the boundaries is directly linked with the solution inside the given domain. Materials that show non-local transport phenomena or have long-range interactions, like some fluids or plasmas, are the examples of these type of BCs. These type of boundary conditions are essential for increasing the precision and application of mathematical models that enable scientists to better predict the outcomes of their simulations and capture a wider variety of physical occurrences. These facts motivate us to create numerical algorithms for such type of non-local boundary conditions. Hence, the Chebyshev collocation-based methodology is used throughout this work to solve LEFDE associated with the non-local three-point boundary conditions (2).

The purpose of this paper is to present a numerical solution for a recently formulated LEFBVPs using the collocation method and shifted Chebyshev polynomials. The proposed method uses truncated shifted Chebyshev polynomials to transform the boundary value problems (BVPs) (1)-(2) into a matrix form. Moreover, the points of collocation have been used to the matrix form of BVPs and converted it into an algebraic system of non-linear equations. The Newton-Raphson iterative method is used to solve the system of non-linear algebraic equations. The approximate solution of the BVPs is given with the help of solution of the system of algebraic equations. This simplicity of the methodology provides reasonable fidelity to the suggested numerical technique. The convergence analysis of the suggested numerical scheme has been presented in terms of absolute error norms to ensure the reliability of the method. We have considered a number of test examples to simulate the proposed numerical schemes. The supremacy of the proposed schemes over existing method has also been demonstrated with the help of these test examples.

The remainder of the paper is structured as follows: Section 2, reviews the basics of the shifted Chebyshev polynomials and their estimates, which are useful for the numerical schemes. Section 3 presents the solution methodology for the BVPs (1)- (2). Section 4 provides the details of convergence analysis of the numerical scheme. To examine the multiple instances of functional differential equations of the Lane-Emden type, the suggested approach's performance is compared to the existing numerical techniques in section 5. Finally, section 6 concludes the article with valid justification, limitation of the methodology and importance of the paper.

Notation and Symbols. To ensure clarity and uniformity of notations throughout the manuscript, the main symbols and terms used in the paper are defined as follows:

- $S(\zeta)$: Dependent variable, representing the unknown function to be solved.
- $\zeta \in [0,1]$: Independent variable (radial coordinate in Lane–Emden type problems).
- $S'(\zeta), S''(\zeta)$: First and second order derivatives of $S(\zeta)$ with respect to ζ.
- $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \mu, m, \nu$: The real constants/parameters appearing in the boundary value problem.
- $f(\cdot), g(\zeta)$: The continuous functions defined in the given functional differential equation.
- $P_n(\zeta)$: Chebyshev polynomial of the first kind with degree n.
- $P_n^*(\zeta)$: Shifted Chebyshev polynomial, which has been mapped to the interval $[0,1]$.
- c_n: Coefficients in the series expansion of $S(\zeta)$ in terms of $P_n^*(\zeta)$.
- $S_N(\zeta)$: Truncated approximation of $S(\zeta)$ using N terms in the shifted Chebyshev polynomial expansion.
- $S_N^{(r)}(\zeta)$: r^{th} order derivative of the approximate solution $S_N(\zeta)$.
- $X(\zeta)$: Vector of monomials $[1, \zeta, \zeta^2, \ldots, \zeta^N]$ used in matrix representations.

- K: Coefficient vector $[c_0, c_1, \dots, c_N]^T$.
- M: Transformation matrix which provides the relation between $X(\zeta)$ and $P_n^*(\zeta)$.
- V^T: Matrix used to compute derivatives in matrix form; $(V^T)^r$ denotes its r-th power for r-th derivatives.
- $A(\alpha_i, \alpha_j)$: Transformation matrix for argument shifts in functional terms.
- N: Truncation degree or in other word number of collocation points used in the numerical scheme.
- ζ_i: Chebyshev-Gauss collocation points corresponding to the roots of $P_N^*(\zeta)$.

These symbols are used consistently in equations, tables, and figures, and ensures the clarity for the reader and addresses uniformity of notations throughout the paper.

2. Fundamental Concept of Chebyshev Polynomials

The first-kind Chebyshev polynomial $P_n(\zeta)$ with degree n, can be expressed as

$$P_n(\zeta) = \cos(n\theta), \text{ where } \zeta = \cos\theta. \tag{3}$$

To confine the independent variable into the semi-interval [0, 1] instead of [-1, 1], we utilize shifted Chebyshev polynomials, which have been defined as follows:

$$P_n^*(\zeta) = P_n(2\zeta - 1), \ \zeta \in [0,1], \quad n = 0, 1, 2, \dots \tag{4}$$

So, We have first few $P_n^*(\zeta)$, as follows:

$$P_0^*(\zeta) = 1, \quad P_1^*(\zeta) = 2\zeta - 1, \quad P_2^*(\zeta) = 8\zeta^2 - 8\zeta + 1. \tag{5}$$

$P_n^*(\zeta)$ possess numerous properties (see, [2, 7] and references therein), the most significant properties, which have been used to establish the numerical scheme, are as follows:

(1) Every roots $\zeta_i \in [0,1]$ of $P_n^*(\zeta)$ are real. Its value is given by

$$\zeta_i = \frac{1}{2}\left(1 + \cos\left(\frac{\left(n - i + \frac{3}{2}\right)\pi}{n+1}\right)\right), \ i = 1, 2, 3, ..., n, \tag{6}$$

Here, we utilize these points as collocation points to establish the numerical scheme for the solution of given functional type boundary value problems (BVPs) (1)-(2) numerically. These specific points are referred to as Chebyshev-Gauss points.

(2) The nth power of ζ in terms of $P_n^*(\zeta)$ which has been examine by [9] , is given by

$$\zeta^n = 2^{-2n+1} {\sum_{k=0}^{n}}' \begin{pmatrix} 2n \\ n-k \end{pmatrix} P_k^*(\zeta), \ 0 \le \zeta \le 1, \tag{7}$$

To represent the first term is halved, symbol $\sum'$ has been used.

An approximation of any function $S(\zeta) \in L^2[0,1]$ in terms of shifted Chebyshev polynomials, given as follows:

$$S(\zeta) = {\sum_{n=0}^{\infty}}' c_n P_n^*(\zeta). \tag{8}$$

Here

$$c_n = \langle S(\zeta), P_n^*(\zeta) \rangle = \frac{2}{\pi} \int_0^1 \frac{1}{\sqrt{\zeta(1-\zeta)}} S(\zeta) P_n^*(\zeta) d\zeta, \quad n = 0, 1, \dots . \tag{9}$$

Now, we investigate the approximate solution of the boundary value problems (BVPs) (1)-(2), which can be expressed by a linear combination of truncated shifted Chebyshev polynomials. Mathematically, it can be expressed as follows:

$$S_N(\zeta) = {\sum_{n=0}^{N}}' c_n P_n^*(\zeta), n \in \mathbb{N}, \tag{10}$$

where $\mathbb{N}$ be set of natural numbers. The matrix form of the summation (10) and its higher-order derivatives are given as follows:

$$S_N(\zeta) = P^*(\zeta)K, \quad S_N^{(r)}(\zeta) = P^{*(r)}(\zeta)K, \quad r = 1, 2, \dots, N, \tag{11}$$

where

$$P^*(\zeta) = [P_0^*(\zeta), P_1^*(\zeta), P_2^*(\zeta), ..., P_N^*(\zeta)], \; K = \left[\frac{1}{2}c_0, c_1, ..., c_N\right]^T \tag{12}$$

and $P^{*(r)}(\zeta)$ represents the r^{th} order derivatives of $P^*(\zeta)$. Let $X(\zeta) = [1, \; \zeta, \; \zeta^2, \; \dots, \zeta^N]$, Utilizing the formula (7), we have

$$(X(\zeta))^T = M(P^*(\zeta))^T \text{ or } X(\zeta) = (P^*(\zeta))M^T, \tag{13}$$

where,

$$M = \begin{bmatrix} 2^0\binom{0}{0} & 0 & 0 & \dots & 0 \\ 2^{-2}\binom{2}{1} & 2^{-1}\binom{2}{0} & 0 & \dots & 0 \\ 2^{-4}\binom{4}{2} & 2^{-3}\binom{4}{1} & 2^{-3}\binom{4}{0} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2^{-2N}\binom{2N}{N} & 2^{-2N+1}\binom{2N}{N-1} & 2^{-2N+1}\binom{2N}{N-2} & \dots & 2^{-2N+1}\binom{2N}{0} \end{bmatrix}.$$

Now, Multiplying by $(M^{-1})^T$ in equation (13), we get

$$P^*(\zeta) = X(\zeta)(M^{-1})^T \tag{14}$$

and r^{th} order derivative of $P^*(\zeta)$ can be expressed as follows:

$$P^{*(r)}(\zeta) = X^{(r)}(\zeta)(M^{-1})^T. \tag{15}$$

Furthermore, the relationship between $X(\zeta)$ and its r^{th} order derivatives, denoted by $X^r(\zeta)$, can be expressed as follows:

$$X^r(\zeta) = X(\zeta)\left(V^T\right)^r, \tag{16}$$

where

$$V = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & N & 0 \end{bmatrix}. \tag{17}$$

Here $(V^T)^r$ is the r^{th} exponent of V^T. Using equations (14)-(16) in (11), we get

$$S_N(\zeta) = X(\zeta)(M^{-1})^T K, \tag{18}$$

and

$$S_N^{(r)}(\zeta) = X(\zeta)(V^T)^r(M^{-1})^T K, \quad r = 1, 2, \ldots, N. \tag{19}$$

Let α_1 and α_2 are real constant, then $S(\alpha_1\zeta + \alpha_2)$ can be written as

$$S_N(\alpha_1\zeta + \alpha_2) = X(\alpha_1\zeta + \alpha_2)(M^{-1})^T K. \tag{20}$$

where

$$X(\alpha_1\zeta + \alpha_2) = X(\zeta)A_{(\alpha_1,\alpha_2)} \tag{21}$$

Here

$$A_{(\alpha_1,\alpha_2)} = \begin{bmatrix} 1 & \alpha_2 & \binom{2}{2}(\alpha_2)^2 & \binom{3}{3}(\alpha_2)^3 & \cdots & \binom{N}{N}(\alpha_2)^N \\ 0 & \alpha_1 & \binom{2}{1}(\alpha_1\alpha_2) & \binom{3}{2}(\alpha_1\alpha_2^2) & \cdots & \binom{N}{N-1}(\alpha_1\alpha_2^N) \\ 0 & 0 & \binom{2}{0}(\alpha_1^2) & \binom{3}{1}(\alpha_1^2\alpha_2) & \cdots & \binom{N}{N-2}(\alpha_1^2\alpha_2^{N-2}) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \binom{N}{0}\alpha_1^N \end{bmatrix}. \tag{22}$$

Hence

$$S_N(\alpha_1\zeta + \alpha_2) = X(\zeta)A_{(\alpha_1,\alpha_2)}(M^{-1})^T K \tag{23}$$

$$S'_N(\beta_1\zeta + \beta_2) = X(\zeta)A_{(\beta_1,\beta_2)}(V^T)(M^{-1})^T K \tag{24}$$

$$S''_N(\gamma_1\zeta + \gamma_2) = X(\zeta)A_{(\gamma_1,\gamma_2)}(V^T)^2(M^{-1})^T K \tag{25}$$

3. Iterative Scheme for their approximate solution of Functional type BVPs:

In this section, we build the fundamental matrix corresponding to functional type BVPs (1)-(2). To achieve this, we substitute the matrix representation provided in (23)-(25) into the differential equation (1).

$$X(\zeta)A_{(\gamma_1,\gamma_2)}\left(V^T\right)^2\left(M^T\right)^{-1}K + \frac{\alpha}{\zeta}X(\zeta)A_{(\beta_1,\beta_2)}V^T\left(M^T\right)^{-1}K +$$

$$p(\zeta)f(X(\zeta)A_{(\alpha_1,\alpha_2)}\left(M^T\right)^{-1}K) = g(\zeta). \tag{26}$$

Using collocation points ζ_i, the system of matrix equations is given by

$$X(\zeta_i)A_{(\gamma_1,\gamma_2)}\left(V^T\right)^2\left(M^T\right)^{-1}K + \frac{\alpha}{\zeta_i}X(\zeta_i)A_{(\beta_1,\beta_2)}V^T\left(M^T\right)^{-1}K +$$

$$p(\zeta_i)f(X(\zeta_i)A_{(\alpha_1,\alpha_2)}\left(M^T\right)^{-1}K) = g(\zeta_i). \tag{27}$$

Hence, the fundamental matrix representation of differential equation (1) can also be written as

$$X(\zeta)A_{(\gamma_1,\gamma_2)}\left(V^T\right)^2(M^T)^{-1}K + RX(\zeta)A_{(\beta_1,\beta_2)}V^T(M^T)^{-1}K + \tag{28}$$

$$Pf(X(\zeta)A_{(\alpha_1},\alpha_2)(M)^T)^{-1}K) = G \tag{29}$$

$$P = \begin{bmatrix} p(\zeta_0) & 0 & 0 & \cdots & 0 \\ 0 & p(\zeta_1) & 0 & \cdots & 0 \\ 0 & 0 & p(\zeta_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p(\zeta_N) \end{bmatrix}, \quad R = \begin{bmatrix} \frac{\alpha}{\zeta_0} & 0 & 0 & \cdots & 0 \\ 0 & \frac{\alpha}{\zeta_1} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\alpha}{\zeta_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\alpha}{\zeta_N} \end{bmatrix},$$

$$X=\begin{bmatrix} 1 & \zeta_0 & \zeta_0^2 & \cdots & \zeta_0^N \\ 1 & \zeta_1 & \zeta_1^2 & \cdots & \zeta_1^N \\ 1 & \zeta_2 & \zeta_2^2 & \cdots & \zeta_2^N \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \zeta_N & \zeta_N^2 & \cdots & \zeta_N^N \end{bmatrix}, \; G=\begin{bmatrix} g(\zeta_0) \\ g(\zeta_1) \\ g(\zeta_2) \\ \vdots \\ g(\zeta_N) \end{bmatrix}$$

Moreover, using matrix equations (18) and (19), the matrix form of the boundary condition (2) is expressed as

$$X(0)V^T\left(M^{-1}\right)^T K=\mu, \quad X(1)\left(M^{-1}\right)^T K-mX(\nu)\left(M^{-1}\right)^T K=0. \tag{30}$$

The above procedure provides a system of N+1 non-linear algebraic equations, which represents a discretization of the given functional type DE (1). The matrix form of these systems of equations is given in (29). Additionally, we obtain two sets of algebraic equations, as presented in (30) as per the BC (2). To obtain the singular functional type BVP's solution , replace equations (30) for any two of the equations provided in (29) based on the appropriate boundary conditions. Newton's iterative method is used to solve the system of $N+1$ non-linear algebraic equations for getting the values of c_r, $r=0,1,2,...N$. Now, we have substituted the values of the Chebyshev coefficient (c_r) into the series equation(10) for the approximate solution of BVPs (1)-(2). It can be observed from the last section of research article extremely non-linear terms are used in most of the test examples. So, it becomes essential to demonstrate the estimation of the positive integer power of S using the collocation and reduced shifted Chebyshev polynomials. The estimation of S^n has been discussed in Section 2.

3.1. Numerical Algorithm of the methodology and its Flowchart.

The steps of the proposed numerical algorithm are as follows:

1. **Input:** Degree N, constants $\alpha_1, \alpha_2, \ldots$, real constants μ, m, ν of boundary conditions, functions $f(\cdot), g(\zeta)$.
2. **Construction of shifted Chebyshev polynomials:** $P_n^*(\zeta)$, $n=0,1,\ldots,N$.
3. **Computation of collocation points:** Chebyshev-Gauss points ζ_i, $i=1,\ldots,N$.
4. **Approximation of the solution:** Approximate solution $S_N(\zeta)=\sum_{n=0}^{N} c_n P_n^*(\zeta)$.
5. **Computation of the derivatives:** Computation of $S_N'(\zeta), S_N''(\zeta)$ using matrix relations.
6. **Application of boundary conditions:** $S'(0)=\mu$, $S(1)=mS(\nu)$.
7. **Formulation of nonlinear algebraic system of equations:** Solve system of equations for coefficients c_n.
8. **Output:** Approximate solution $S_N(\zeta)$ and optionally its derivatives.

The methodology's flowchart is illustrated in Figure 1.

4. CONVERGENCE ANALYSIS

A convergence analysis is performed to verify the reliability and legality of the proposed methodology and its implementation to construct its numerical scheme for the solution. The existence and uniqueness of the solution for the LEFBVPs are also addressed. We discuss these via some theorems and their constructive proof.

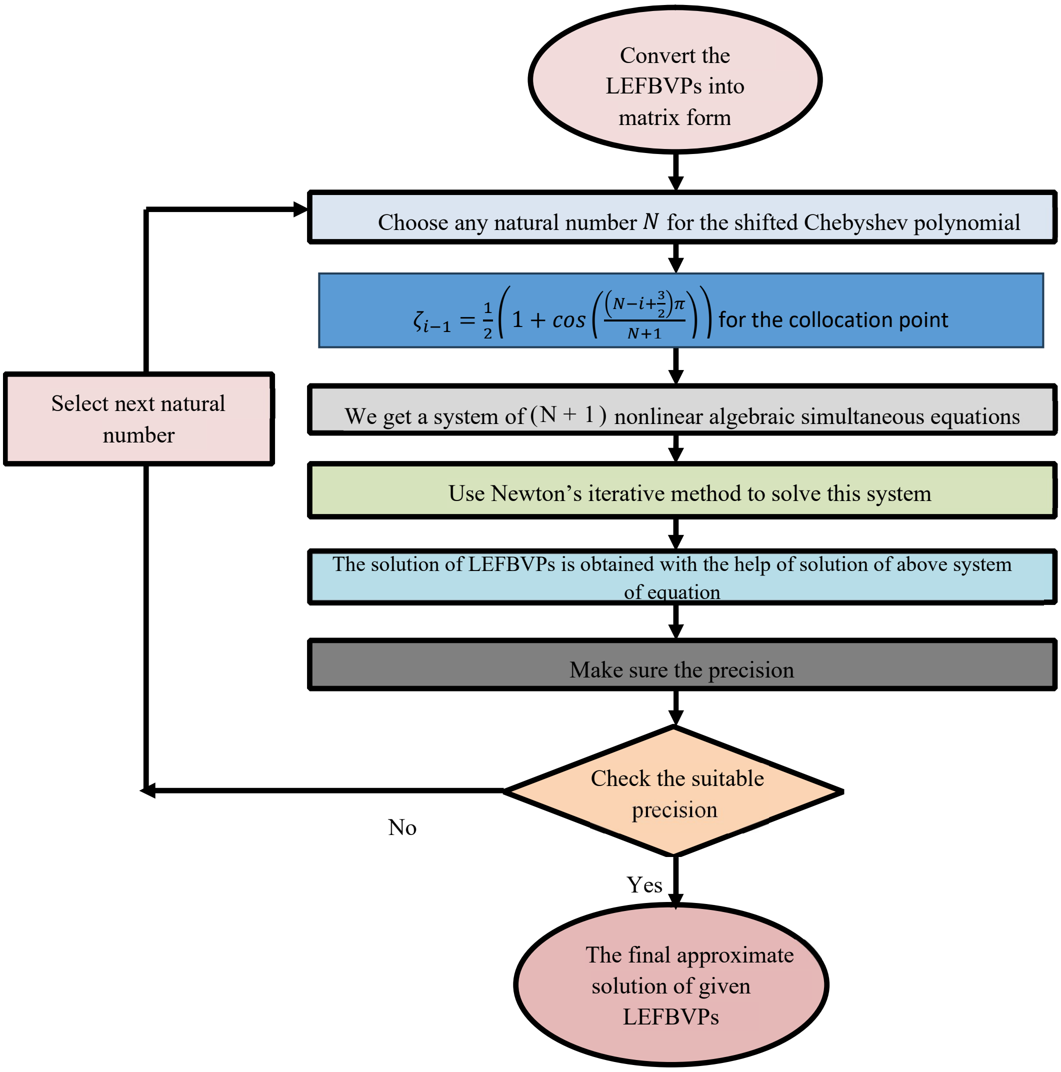


FIGURE 1. Diagrammatic illustration of the newly constructed methodology for the solution of LEFBVPs

Theorem 4.1. *In case* $\gamma_i = \beta_i = \alpha_i = \delta_i$, $i = 1, 2$ *and* $f(\zeta, S) = F(\zeta)$, *the LEFDE* (1) *with BCs* (2) *is equivalent to integral equation of the form*

$$S(\zeta) = \frac{m}{m-1}\int_{\nu}^{1}\left[\left(\frac{\delta_1}{t-\delta_2}\right)^{\rho}\int_0^{\frac{t-\delta_2}{\delta_1}} x^{\rho}F(x)dx\right]dt - \int_{\zeta}^{1}\left[\left(\frac{\delta_1}{t-\delta_2}\right)^{\rho}\int_0^{\frac{t-\delta_2}{\delta_1}} x^{\rho}F(x)dx\right]dt. \tag{31}$$

Proof. Integrate the problem (1) from 0 to ζ, we get

$$\zeta^{\rho}S'(\delta_1\zeta + \delta_2) + c = \int_0^{\zeta} x^{\rho}F\left(x\right)) \, dx. \tag{32}$$

Using BC $S'(0) = 0$, we get $c = 0$. Hence, equation (32) can be written as

$$S'(\delta_1\zeta + \delta_2) = \frac{1}{\zeta^{\rho}}\int_0^{\zeta} x^{\rho}F(x)dx. \tag{33}$$

Now, replace ζ to $\frac{\zeta-\delta_2}{\delta_1}$ in above equation, we get

$$S'(\zeta) = \frac{1}{\left(\frac{\zeta-\delta_2}{\delta_1}\right)^\rho} \int_0^{\frac{\zeta-\delta_2}{\delta_1}} x^\rho F(x)dx. \tag{34}$$

Again integrating above from ζ to 1, we get

$$S(\zeta) = c' - \int_\zeta^1 \left(\frac{\delta_1}{t-\delta_2}\right)^\rho \int_0^{\frac{t-\delta_2}{\delta_1}} x^\rho F(x)dxdt. \tag{35}$$

Use the boundary condition S(1) $= m$ S(ν), we get

$$c' = \frac{m}{m-1} \int_\nu^1 \left(\frac{\delta_1}{t-\delta_2}\right)^\rho \int_0^{\frac{t-\delta_2}{\delta_1}} x^\rho F(x)dxdt. \tag{36}$$

Put the value of c' in (35), and we get

$$S(\zeta) = \frac{m}{m-1} \int_\nu^1 \left(\frac{\delta_1}{t-\delta_2}\right)^\rho \int_0^{\frac{t-\delta_2}{\delta_1}} x^\rho F(x)dxdt - \int_\zeta^1 \left(\frac{\delta_1}{t-\delta_2}\right)^\rho \int_0^{\frac{t-\delta_2}{\delta_1}} x^\rho F(x)dxdt. \tag{37}$$

This is the equivalent integral form of the given linear form of LEFBVPs (1)-(2).

□

4.1. **Existence and Uniqueness Result for the solution of LEFBVPs.** In this section, we discuss the existence of a unique solution of LEFBVPs (1)-(2), with the help of integral equation (37). Firstly, we define an operator equation, $T : E \to E$ as follows

$$T(S) = \frac{m}{m-1} \int_\nu^1 \left(\frac{\delta_1}{t-\delta_2}\right)^\rho \int_0^{\frac{t-\delta_2}{\delta_1}} x^\rho f(x, S(\delta_1 x + \delta_2))dxdt -$$
$$\int_\zeta^1 \left(\frac{\delta_1}{t-\delta_2}\right)^\rho \int_0^{\frac{t-\delta_2}{\delta_1}} x^\rho f(x, S(\delta_1 x + \delta_2))dxdt. \tag{38}$$

Here $E = C[I, \mathbb{R}]$ be a Banach space with $I = [0, 1]$ and under the norms defined as $\|S\| = \max_{\zeta \in I} S(\zeta)$, $S \in E$. The following theorem deals the existence of a unique solution concerned with the BVPs (1)-(2).

Theorem 4.2. *Assume that f(ζ, S) ba a Lipschitz continuous function in variable S with Lipschitz constant L, then LEFBVPs (1)-*(2) *has a unique solution in E, whenever* $L(M_1 + M_2) < 1$, *where*

$$M_1 = \max_{\zeta\in I} \left| \frac{m(1-\nu)(1+\nu+2\delta_2)}{2\delta_1(m-1)(\rho+1)} \right|.$$

$$M_2 = \max_{\zeta\in I} \left| \frac{(1-\zeta)(1+\zeta-2\delta_2)}{2\delta_1(\rho+1)} \right|.$$

Proof. Suppose that S_1 and $S_2 \in C[I, \mathbb{R}]$. Then, we have

$$\|T(S_1) - T(S_2)\| =$$

$$\max_{\zeta\in I} \left| \frac{m}{m-1} \int_\nu^1 \left(\frac{\delta_1}{t-\delta_2}\right)^\rho \int_0^{\frac{t-\delta_2}{\delta_1}} x^\rho \left[f(x, S_1(\delta_1 x + \delta_2)) - f(x, S_2(\delta_1 x + \delta_2))\right] dxdt \right.$$
$$\left. - \int_\zeta^1 \left(\frac{\delta_1}{t-\delta_2}\right)^\rho \int_0^{\frac{t-\delta_2}{\delta_1}} x^\rho \left[f(x, S_1(\delta_1 x + \delta_2)) - f(x, S_2(\delta_1 x + \delta_2))\right] dx\, dt \right|. \tag{39}$$

Since $f(\zeta, S)$ is Lipschitz continuous, with Lipschitz constant L, So

$$\|f(\zeta, S_1) - f(\zeta, S_2)\| \leq L\|S_1 - S_2\|.$$

Hence, the equation (39) can be written as

$$\|T(S_1) - T(S_2)\| \leq L(M_1 + M_2)\|S_1 - S_2\|.$$

Being $L(M_1 + M_2) < 1$, T is a contraction map, and hence, from the Banach contraction principle, it has a unique non-negative fixed point in E , which is also the unique solution of the LEFBVPs (1)-(2).

□

4.2. **Error Analysis:** This section is devoted for the error bound of the solution of the LEFBVPs (1)-(2) using proposed methodology given in section (3). With the help of the following theorem, we establish the error estimation of Chebyshev polynomial approximation.

Theorem 4.3. *Let us consider $\psi(\zeta) \in C^{n+1}(I)$. If $\psi_n(\zeta)$ denote the best square approximation to $\psi(\zeta)$, then an upper bound of the error is obtained as*

$$\|\psi(\zeta) - \psi_n(\zeta)\|_{w(\zeta)} \leq \frac{M_\infty}{(n+1)!\sqrt{2n+3}},$$

where $w(\zeta) = e^{-\zeta}$ is a weight function and $M_\infty = \max_{\zeta \in I} \psi^{n+1}(\zeta)$.

Proof. Using the Taylor's expansion about $\zeta = 0$, the expression for $\psi(\zeta)$ is given by

$$\psi(\zeta) = \psi(0) + \zeta\psi'(0) + \frac{\zeta^2}{2!}\psi''(0) + \cdots + \frac{\zeta^n}{n!}\psi^n(0) + \frac{\zeta^{n+1}}{(n+1)!}\psi^{n+1}(\epsilon), \ \epsilon \in I$$

Suppose $\psi_n^*(\zeta) = \sum_{i=0}^{n} \frac{\zeta^i}{i!}\psi^i(0)$. So, we have

$$|\psi(\zeta) - \psi_n^*(\zeta)| = \frac{1}{(n+1)!}\left|\psi^{n+1}(\epsilon)\zeta^{n+1}\right|$$

Let W be the space spanned by $[P_0^*(\zeta), P_1^*(\zeta), P_2^*(\zeta), \ldots, P_n^*(\zeta)]$. Since $\psi_n(\zeta)$ is the best approximation in W, we have

$$\|\psi(\zeta) - \psi_n(\zeta)\|_{w(\zeta)} \leq \|\psi(\zeta) - S(\zeta)\|_{w(\zeta)}, \forall S \in W$$

Particularly, for $S = \psi_n^*$, we have

$$\begin{aligned} \|\psi - \psi_n\|^2_{w(\zeta)} &\leq \|\psi(\zeta) - \psi_n^*(\zeta)\|^2_{w(\zeta)} \\ &= \int_0^1 |\psi(\zeta) - \psi_n^*(\zeta|^2 w(\zeta) d\zeta \\ &= \int_0^1 \left|\frac{\zeta^{n+1}}{(n+1)!}\psi^{n+1}(\epsilon)\right|^2 e^{-\zeta} d\zeta \\ &= \left(\frac{M_\infty}{(n+1)!}\right)^2 \int_0^1 \zeta^{2n+2} e^{-\zeta} d\zeta. \end{aligned}$$

Thus we have,

$$\begin{aligned} \|\psi - \psi_n\|^2_{w(\zeta)} &\leq \left(\frac{M_\infty}{(n+1)!}\right)^2 \int_0^1 \zeta^{2n+2} d\zeta \\ &= \left(\frac{M_\infty}{(n+1)!}\right)^2 \frac{1}{(2n+3)}. \end{aligned}$$

Finally, taking the square root, the proof is complete. □

Theorem 4.4. *Assume that* $S(\zeta)$ *is the exact solution and* $S_n(\zeta)$ *is the approximate solution of LEFBVPs* (1)-(2) *using the proposed method. Let all the assumptions given in theorem (4.2) hold then the error bound is estimated as*

$$\|S - S_n\| \leq \frac{LM_\infty}{(n+1)!}\sqrt{\frac{M_1^2 + M_2^2}{2n+3}},$$

where,

$$M_1 = \max_{\zeta \in I}\left|\frac{m(1-\nu)(1+\nu+2\delta_2)}{2\delta_1(m-1)(\rho+1)}\right|,$$

and

$$M_2 = \max_{\zeta \in I}\left|\frac{(1-\zeta)(1+\zeta-2\delta_2)}{2\delta_1(\rho+1)}\right|.$$

Proof. Since $S(\zeta)$ is the exact solution and $S_n(\zeta)$ is the best approximate solution of given LEFBVPs (1)-(2), we have

$$\begin{aligned}\|S - S_n\|^2 = \int_0^1 \Bigg| \frac{m}{m-1}\int_\nu^1 \left(\frac{\delta_1}{t-\delta_2}\right)^\rho \int_0^{\frac{t-\delta_2}{\delta_1}} x^\rho \left[f(x,S) - f(x,S_n)\right] dx\, dt \\ - \int_\zeta^1 \left(\frac{\delta_1}{t-\delta_2}\right)^\rho \int_0^{\frac{t-\delta_2}{\delta_1}} x^\rho \left[f(x,S) - f(x,S_n)\right] dx\, dt\Bigg|^2 dy. \qquad (40)\end{aligned}$$

Applying the Lipschitz continuity of f, the above expression is reduced to

$$\|S - S_n\|_2^2 \leq L^2(M_1^2 + M_2^2)\int_0^1 |S(y) - S_n(y)|^2 dy$$

Using theorem 4.3, we have

$$\|S - S_n\|_2 \leq \frac{LM_\infty}{(n+1)!}\sqrt{\frac{M_1^2 + M_2^2}{2n+3}}$$

Hence the approximate solution $S_n(\zeta)$ of the given problem converges uniformly to the exact solution $S(\zeta)$ as n tends to ∞. □

5. Illustrative example

In this section, we demonstrate the accuracy and efficacy of the shifted Chebyshev collocation method ($Cheb(N)$) with a set of various numerical test examples. Maple 18 is used to perform the all the calculations on a computer. The absolute errors (e) are determined by the error equation $e = |\text{Exact solution} - \text{Approximate solution}|$. Residual error is also used in some test examples to justify the numerical estimation through proposed methodology when exact solution is not known. The internal configuration of the PC used for computational work on MAPLE 18 is Intel(R) Core(TM) i3-10110U CPU @ 2.10GHz(2.59GHz) and RAM 8.00 GB.

Example 5.1. *Assume the LEFDE*

$$-S''(2\zeta-1) - \frac{2}{\zeta}S'(3\zeta) + cosS(\zeta-1) = cos((\zeta-1)^2 - (\zeta-1)^3) + 66\zeta - 20,\ 0 < \zeta \leq 1, \quad (41)$$

subject to BCs

$$S'(0) = 0,\ S\left(\frac{1}{2}\right) = \frac{27}{16}S\left(\frac{1}{3}\right) \qquad (42)$$

The exact solution of this problem is $S(\zeta) = \zeta^2 - \zeta^3$. Using the proposed numerical scheme, discussed in section 3, the matrix form of (41) is given by

$$-X(\zeta)A_{(2,-1)}\left(V^T\right)^2\left(M^T\right)^{-1}K - \left(\frac{2}{\zeta}\right)X(\zeta)A_{(3,0)}V^T\left(M^T\right)^{-1}K + \frac{1}{2}(P^*K)X(\zeta)A_{(1,-1)}$$

$$\left(M^T\right)^{-1}K + \frac{1}{24}(P^*K)^3X(\zeta)A_{(1,-1)}\left(M^T\right)^{-1}K = cos((\zeta-1)^2-(\zeta-1)^3)+66\zeta-21. \quad (43)$$

Also, the matrix form of the BCs (42) is given by

$$X(0)V^T(M^T)^{-1}K = 0. \quad (44)$$

$$X(1)\left(M^T\right)^{-1}K - \frac{27}{16}X\left(\frac{1}{3}\right)\left(M^T\right)^{-1}K = 0 \quad (45)$$

Utilize the collocation points (6) in (43), we have

$$[-XA_{(2,-1)}\left(V^T\right)^2 - RXA_{(3,0)}V^T + \frac{1}{2}(P^*K)XA_{(1,-1)} + \frac{1}{24}(P^*K)^3XA_{(1,-1)}$$

$$]\left(M^T\right)^{-1}K - G = 0. \quad (46)$$

Here, V^T, $\left(M^T\right)^{-1}$ and K are described in the section 2 and X is described in the section 3. The matrices R and G can be addressed as

$$R = \begin{bmatrix} \frac{2}{\zeta_0} & 0 & 0 & \cdots & 0 \\ 0 & \frac{2}{\zeta_1} & 0 & \cdots & 0 \\ 0 & 0 & \frac{2}{\zeta_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & \cdots & \frac{2}{\zeta_N} \end{bmatrix}, G = \begin{bmatrix} cos((\zeta_0-1)^2-(\zeta_0-1)^3)+66\zeta_0-21 \\ cos((\zeta_1-1)^2-(\zeta_1-1)^3)+66\zeta_1-21 \\ cos((\zeta_2-1)^2-(\zeta_2-1)^3)+66\zeta_2-21 \\ \vdots \\ cos((\zeta_N-1)^2-(\zeta_N-1)^3)+66\zeta_N-21 \end{bmatrix}.$$

Now, we solve system of any $N-1$ non linear equations obtained from equation (46), and two equations (44) and (45) for the values of Chebyshev coefficient c_r, $r = 0, 1, 2, .., N$. Then, we substitute these values of the Chebyshev coefficient in series (10) for getting the numerical estimation of the solution using Chebyshey sifted polynomial $(Cheb(N))$ and compare the result with the Bernoulli collocation method(BCM) [22]. The numerical values of the solution are tabulated and presented, along with a comparison to BCM in Figure (2) and Table (1) for $N = 3$. To verify the correctness of the method, absolute errors are given in Table (1). The values of c_r are provided in Table 2 for $N = 3$. According to Tables 1, 2 and Figure 2, It is easy to conclude that the suggested method has a lot of potential for solving LEFBVPs and higher precision than the existing method (BCM).

Example 5.2. *Consider the LEFDE*

$$S''(2\zeta-1) + \frac{2}{\zeta}S'(3\zeta) + S(\zeta-1) + S^3(\zeta-1) + S^5(\zeta-1) = (\zeta^2-2\zeta+16) + (\zeta^2-2\zeta+2)^3 +$$

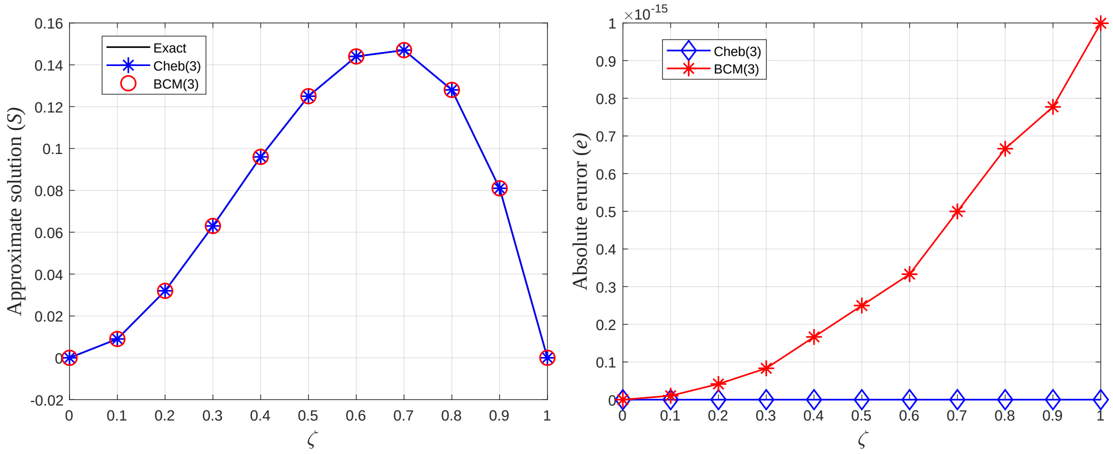
$$(\zeta^2-2\zeta+2)^5, \ 0 < \zeta \leq 1, \quad (47)$$

with BCs

$$S'(0) = 0, \ S(1) = \frac{8}{5}S\left(\frac{1}{2}\right) \quad (48)$$

Table 1. Quantitative estimation of numerical results using Cheb(N) and BCM at $N = 3$ for Example 1 with CPU time = 0.06 seconds and 1.04 seconds, respectively.

ζ	Exact	Cheb(3)	e	BCM(3)	e
0.0	0	-1.00×10^{-25}	1.0×10^{-25}	0	0
0.1	9.00×10^{-3}	9.00×10^{-3}	0	9.00×10^{-3}	1.1×10^{-17}
0.2	3.20×10^{-3}	3.20×10^{-3}	0	3.20×10^{-3}	4.2×10^{-17}
0.3	6.30×10^{-3}	6.30×10^{-3}	0	6.30×10^{-3}	8.3×10^{-17}
0.4	9.60×10^{-3}	9.60×10^{-3}	0	9.60×10^{-3}	1.7×10^{-16}
0.5	1.25×10^{-3}	1.25×10^{-3}	0	1.25×10^{-3}	2.5×10^{-16}
0.6	1.44×10^{-3}	1.44×10^{-3}	0	1.44×10^{-3}	3.3×10^{-16}
0.7	1.47×10^{-3}	1.47×10^{-3}	0	1.47×10^{-3}	5.0×10^{-16}
0.8	1.28×10^{-3}	1.28×10^{-3}	0	1.28×10^{-3}	6.7×10^{-16}
0.9	8.10×10^{-3}	8.10×10^{-3}	0	8.10×10^{-3}	7.8×10^{-16}
1.0	0	0	0	-1.0×10^{-15}	9.9×10^{-16}



(a) Approximate solution (b) Absolute error

Figure 2. Graphical representation of approximate solution and respective error estimation for Example 1.

Table 2. Values of Chebyshev coefficient (c_r) at $N = 3$ for Example 1.

	c_0	c_1	c_2	c_3
$N = 3$	1.00000000	-0.75000000	0.50000000	-0.25000000

The exact solution of this problem is $S(\zeta) = \zeta^2 + 1$.

Using the proposed numerical scheme discussed in Section 3, the matrix form of (47) is given by

$$X(\zeta)A_{(2,-1)}\left(V^T\right)^2\left(M^T\right)^{-1}K + \left(\frac{2}{\zeta}\right)X(\zeta)A_{(3,0)}V^T\left(M^T\right)^{-1}K$$

$$X(\zeta)A_{(1,-1)}\left(M^T\right)^{-1}K + (P^*K)^2X(\zeta)A_{(1,-1)}\left(M^T\right)^{-1}K$$

$$+(P^*K)^4X(\zeta)A_{(1,-1)}\left(M^T\right)^{-1}K = (\zeta^2-2\zeta+16)+(\zeta^2-2\zeta+16)^3+(\zeta^2-2\zeta+16)^5. \quad (49)$$

Also, the matrix form of the BCs is given by (48)

$$X(0)V^T(M^T)^{-1}K = 0. \quad (50)$$

$$X(1)\left(M^T\right)^{-1}K-\frac{8}{5}X\left(\frac{1}{2}\right)\left(M^T\right)^{-1}K=0. \tag{51}$$

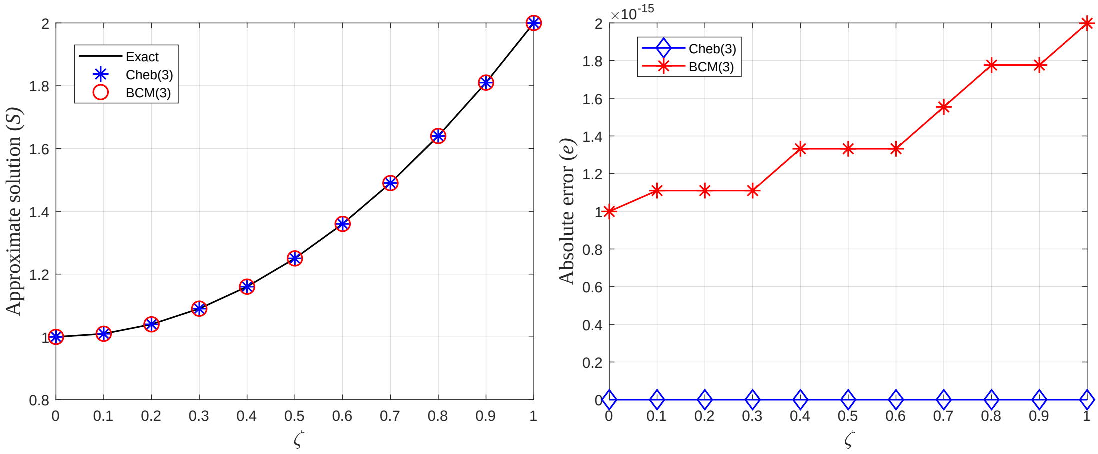
Now, using the collocation points (6), in (49) we have

$$[XA_{(2,-1)}\left(V^T\right)^2-RXA_{(3,0)}V^T+XA_{(1,-1)}+(P^*C)^2XA_{(1,-1)}+(P^*C)^4XA_{(1,-1)}$$
$$]\left(M^T\right)^{-1}K-G=0. \tag{52}$$

Here, V^T, $\left(M^T\right)^{-1}$ and K are described in the section 2 and X is described in the section 3. The matrices R and G can be addressed as

$$R=\begin{bmatrix} \frac{2}{\zeta_0} & 0 & 0 & \cdots & 0 \\ 0 & \frac{2}{\zeta_1} & 0 & \cdots & 0 \\ 0 & 0 & \frac{2}{\zeta_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & \cdots & \frac{2}{\zeta_N} \end{bmatrix},$$

$$G=\begin{bmatrix} (\zeta_0^2-2\zeta_0+16)+(\zeta_0^2-2\zeta_0+2)^3+(\zeta_0^2-2\zeta_0+2)^5 \\ (\zeta_1^2-2\zeta_1+16)+(\zeta_1^2-2\zeta_1+2)^3+(\zeta_1^2-2\zeta_1+2)^5 \\ (\zeta_2^2-2\zeta_2+16)+(\zeta_2^2-2\zeta_2+2)^3+(\zeta_2^2-2\zeta_2+2)^5 \\ \vdots \\ (\zeta_N^2-2\zeta_N+16)+(\zeta_N^2-2\zeta_N+2)^3+(\zeta_N^2-2\zeta_N+2)^5 \end{bmatrix}.$$



(a) Numerical vs BCM (b) Absolute error (e)

FIGURE 3. Graphical representation of approximate solution and respective error estimation for Example 5.2.

Now, we solve system of any $N-1$ non linear equations obtained from equation (52),and two equations (50) and (51) for the value of Chebyshev coefficient c_r, $r=0,1,2,..,N$. Then we substitute these values of the Chebyshev coefficient in series (10) for getting the numerical estimation of the solution and compare the result with the Bernoulli collocation method (BCM) [22]. The numerical values of the solution are tabulated and presented, along with a comparison to BCM in Figure (3) and Table (3) for $N=3$. To verify the correctness of the method, absolute errors are given in Table (3). The values of c_r are provided in Table 4 for $N = 3$. According to Tables (3), (4) and Figure 3, It is easy to conclude that the suggested method has a lot of potential for solving LEFBVPs and high accurate solution are obtained over the existing method (BCM).

Table 3. Quantitative estimation of numerical results using Cheb(N) and BCM at $N = 3$ for Example 5.2 with CPU time = 0.07 seconds and 0.09 seconds, respectively.

ζ	Exact	Cheb(3)	e	BCM	e
0.0	1.00	1.00	0.00	1.00	9.9×10^{-16}
0.1	1.01	1.01	0.00	1.01	1.1×10^{-15}
0.2	1.04	1.04	0.00	1.04	1.1×10^{-15}
0.3	1.09	1.09	0.00	1.09	1.1×10^{-15}
0.4	1.16	1.16	0.00	1.16	1.3×10^{-15}
0.5	1.25	1.25	0.00	1.25	1.3×10^{-15}
0.6	1.36	1.36	0.00	1.36	1.3×10^{-15}
0.7	1.49	1.49	0.00	1.49	1.5×10^{-15}
0.8	1.64	1.64	0.00	1.64	1.7×10^{-15}
0.9	1.81	1.81	0.00	1.81	1.7×10^{-15}
1.0	2.00	2.00	0.00	2.00	1.9×10^{-15}

Table 4. The values of c_r at $N = 3$ for Example 5.2.

	c_0	c_1	c_2	c_3
N = 3	3.00000000	0.00000000	0.50000000	0.00000000

Example 5.3. *Consider the LEFDE*

$$S^{''}(2\zeta - 1) + \frac{2}{\zeta}S^{'}(3\zeta) + \zeta S(\zeta - 1) = \zeta^4 - 5\zeta^3 + 7\zeta^2 + 63\zeta - 34,\ 0 < \zeta \leq 1 \tag{53}$$

subject to BCs

$$S'(0) = 0,\ S(1) = \frac{8}{3}S(\frac{1}{2}). \tag{54}$$

The exact solution of this BVPs (53)-(54) is $S = \zeta^3 - 2\zeta^2$.

Example 5.4. *Let us assume that the BVP*

$$S^{''}\left(\frac{\zeta}{2}\right) + \frac{6}{\zeta}S^{'}\left(\frac{\zeta}{2}\right) + 2e^{S} = 2e^{1+\zeta^3} + \frac{15}{2}\zeta,\ 0 < \zeta \leq 1, \tag{55}$$

subject to BCs

$$S^{'}(0) = 0,\ S(1) = \frac{27}{14}S\left(\frac{1}{3}\right) \tag{56}$$

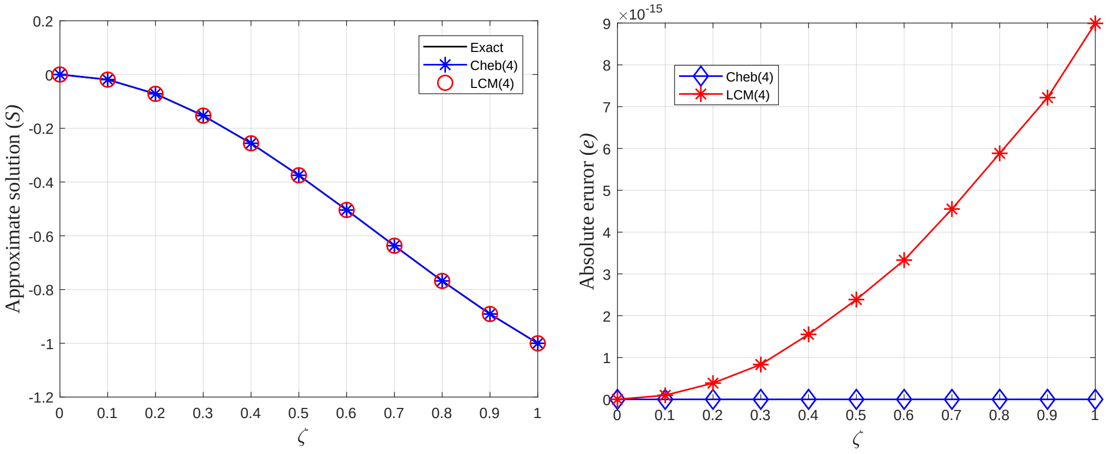
The exact solution of this BVPs (55)-(56) is $S = \zeta^3 + 1$. In Figure 5(a) and Table 7 we matched approximate solution with an exact solution. The absolute error is also calculated in the same Table 7 to verify the method's accuracy. It can be seen from Figure 5(a) the approximate and exact solutions are in good agreement. The errors drastically decreased as N changed from 3 to 5, as shown in Table 7 and Figure 5(b). The values of c_r are provided in Table 8 for this above mentioned value N. It is evident that the suggested approach computes extremely accurate results and matches the precise results well.

TABLE 5. Quantitative estimation of numerical results using Cheb(N) and LCM at $N = 4$ for Example 5.3.

ζ	Exact	Cheb(4)	e	LCM(4)	e
0.0	0.000000	9.6×10^{-25}	9.6×10^{-25}	-4.0×10^{-24}	4.0×10^{-24}
0.1	-0.019000	-0.019000	0	-0.019000	9.71×10^{-17}
0.2	-0.072000	-0.072000	0	-0.072000	3.89×10^{-16}
0.3	-0.153000	-0.153000	0	-0.153000	8.33×10^{-16}
0.4	-0.256000	-0.256000	0	-0.256000	1.55×10^{-15}
0.5	-0.375000	-0.375000	0	-0.375000	2.39×10^{-15}
0.6	-0.504000	-0.504000	0	-0.504000	3.33×10^{-15}
0.7	-0.637000	-0.637000	0	-0.637000	4.55×10^{-15}
0.8	-0.768000	-0.768000	0	-0.768000	5.88×10^{-15}
0.9	-0.891000	-0.891000	0	-0.891000	7.22×10^{-15}
1.0	-1.000000	-1.000000	0	-1.000000	8.99×10^{-15}

TABLE 6. Chebyshev coefficients at $N = 4$ for Example 5.3.

	c_0	c_1	c_2	c_3	c_4
$N = 4$	-2.00000000	0.75000000	-1.00000000	0.25000000	0.00000000



(a) Approximate vs Exact

(b) Absolute error (e)

FIGURE 4. Graphical representation of approximate solution and respective error estimation for Example 5.3.

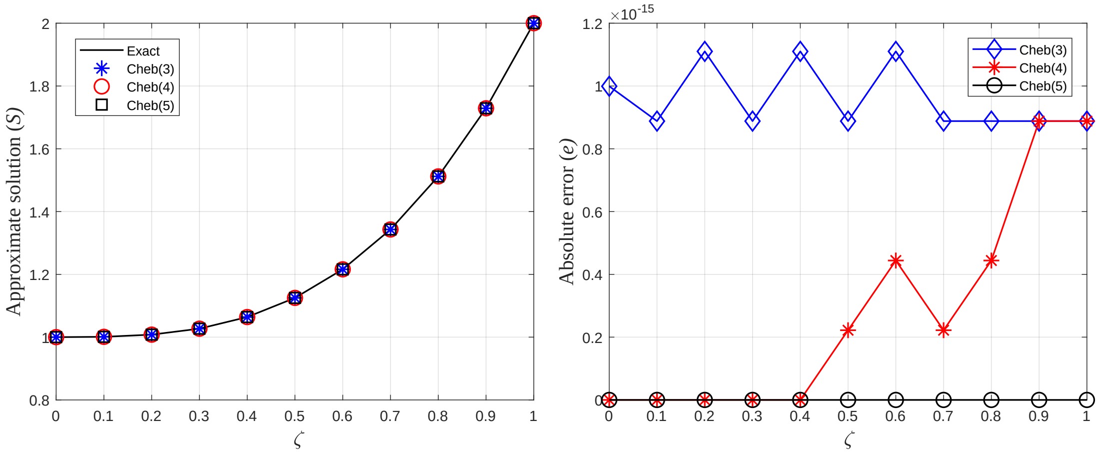
Example 5.5. *Consider the LEFDE*

$$S^{''}(3\zeta+7)+\frac{2}{\zeta}S^{'}(5\zeta-2)+sinS(\zeta+1)=sin((\zeta+1)^2)-\frac{8}{\zeta}+22,\ \ 0<\zeta\leq 1, \tag{57}$$

subject to BCs

$$S'(0)=0,\ \ S(1)=4S\left(\frac{1}{2}\right) \tag{58}$$

The exact solution of LEFBVP (57)-(58) is $S = \zeta^2$. The exact and approximate solution using existing method BCM are compared with the approximate solution obtained from proposed method (Cheb(3)) graphically in Figure (6) in addition to the quantitatively in



(a) Numerical vs Exact (b) Absolute error (e)

FIGURE 5. Graphical representation of approximate solution and respective error estimation for Example 5.4.

TABLE 7. Quantitative estimation of numerical results using Cheb(N) at $N = 3, 4, 5$ for Example 5.4.

ζ	Exact	Cheb(3)	e	Cheb(4)	e	Cheb(5)	e
0.0	1.000	1.000	9.9×10^{-16}	1.000	0	1.000	0
0.1	1.001	1.001	8.8×10^{-16}	1.001	0	1.001	0
0.2	1.008	1.008	1.1×10^{-15}	1.008	0	1.008	0
0.3	1.027	1.027	8.8×10^{-16}	1.027	0	1.027	0
0.4	1.064	1.064	1.1×10^{-15}	1.064	0	1.064	0
0.5	1.125	1.125	8.8×10^{-16}	1.125	2.20×10^{-16}	1.125	0
0.6	1.216	1.216	1.1×10^{-15}	1.216	4.44×10^{-16}	1.216	0
0.7	1.343	1.343	8.8×10^{-16}	1.343	2.20×10^{-16}	1.343	0
0.8	1.512	1.512	8.8×10^{-16}	1.512	4.40×10^{-16}	1.512	0
0.9	1.729	1.729	8.8×10^{-16}	1.729	8.80×10^{-16}	1.729	0
1.0	2.000	2.000	8.8×10^{-16}	2.000	8.80×10^{-16}	2.000	0

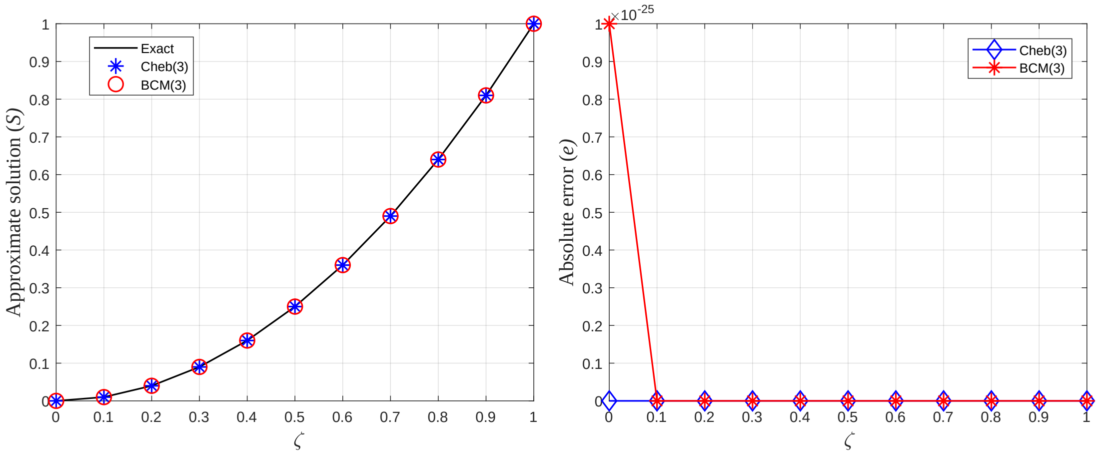
TABLE 8. The values of c_r for $N = 3, 4, 5$ for Example 5.4.

	c_0	c_1	c_2	c_3	c_4	c_5
$N = 3$	2.00000000	0.74999999	0.00000000	0.24999999		
$N = 4$	2.00000000	0.74999999	0.00000000	0.24999999	0.00000000	
$N = 5$	1.99999999	0.75000000	0.00000000	0.25000000	0.00000000	0.00000000

Table (9) at different values of ζ. Table (9) and Figure (6) also provide absolute errors (e). The values of c_r are given in Table (10) at $N = 3$. The numerical solutions with BCM at $N = 3$ are given in Figure 6 and Table 9. These plots demonstrate accuracy of the suggested method coincides with the precise solutions.

Example 5.6. *Consider the LEFBVP*

$$S''(2\zeta - 1) + \frac{2}{\zeta}S'(3\zeta) + S^2(\zeta - 1) = \zeta^6 - 6\zeta^5 + 15\zeta^4 - 18\zeta^3 + 9\zeta^2 + 66\zeta - 6, \tag{59}$$



(a) Approximate vs Exact (b) Absolute error (e)

Figure 6. Graphical representation of approximate solution and respective error estimation for Example 5.5.

Table 9. Quantitative estimation of numerical results using Cheb(N) and BCM at $N = 3$ for Example 5.5.

ζ	Exact	Cheb(3)	e	BCM(3)	e
0.0	0.000	0.000	0	1×10^{-25}	1×10^{-25}
0.1	0.010	0.010	0	0.010	0
0.2	0.040	0.040	0	0.040	0
0.3	0.090	0.090	0	0.090	0
0.4	0.160	0.160	0	0.160	0
0.5	0.250	0.250	0	0.250	0
0.6	0.360	0.360	0	0.360	0
0.7	0.490	0.490	0	0.490	0
0.8	0.640	0.640	0	0.640	0
0.9	0.810	0.810	0	0.810	0
1.0	1.000	1.000	0	1.000	0

Table 10. The values of c_r at $N = 3$ for Example 5.5.

	c_0	c_1	c_2	c_3
$N=3$	1.000000	0.000000	0.500000	0.000000

subject to BCs

$$S'(0) = 0,\ \ S(1) = \frac{27}{14} S\left(\frac{1}{3}\right). \tag{60}$$

The exact solution of LEFBVP (59)-(60) is $S(\zeta) = 1+\zeta^3$. In Table 11 and Figure 7(a), we matched approximate solution with exact solution. The absolute error is also calculated in the same Table 11 to verify the accuracy of method. It can be seen from Figure 7(a), the approximate and exact solution are in good agreement. The errors drastically decreased as N change from 6 to 12, as shown in table 11 and figure 7(b). The values of c_r are provided in Table 12 for this above mentioned value N.

Example 5.7. *Finally, we consider the following LEFBVP*

$$-S''\left(\frac{\zeta}{2}\right) - \frac{2}{\zeta} S'\left(\frac{\zeta}{2}\right) = 1 - 2S^3\left(\frac{\zeta}{2}\right) \tag{61}$$

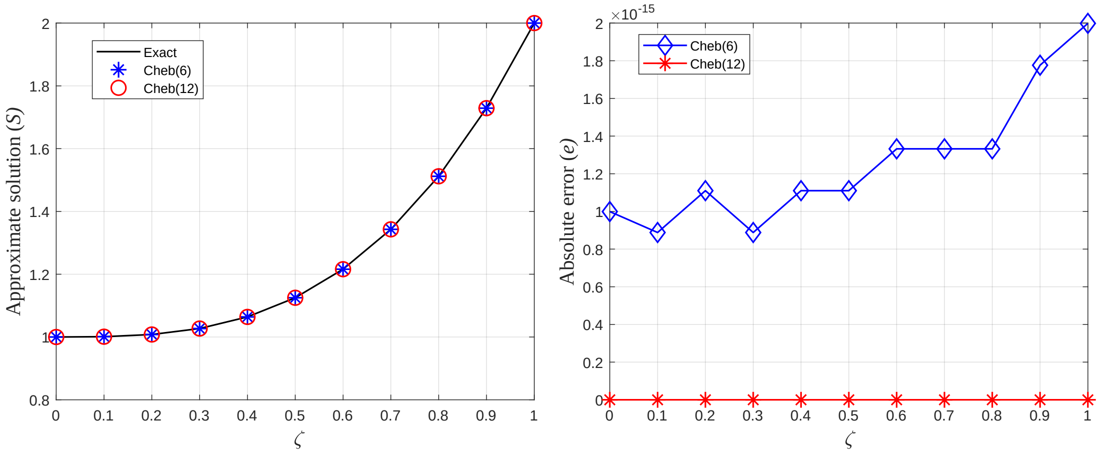
TABLE 11. Comparison of Cheb(N) with exact solution for Example 5.6.

ζ	Exact	Cheb(6)	e	Cheb(12)	e
0.0	1.000	1.000	9.9×10^{-16}	1.000	0
0.1	1.001	1.001	8.8×10^{-16}	1.001	0
0.2	1.008	1.008	1.1×10^{-15}	1.008	0
0.3	1.027	1.027	8.8×10^{-16}	1.027	0
0.4	1.064	1.064	1.1×10^{-15}	1.064	0
0.5	1.125	1.125	8.8×10^{-16}	1.125	0
0.6	1.216	1.216	1.3×10^{-15}	1.216	0
0.7	1.343	1.343	1.3×10^{-15}	1.343	0
0.8	1.512	1.512	1.3×10^{-15}	1.512	0
0.9	1.729	1.729	1.7×10^{-15}	1.729	0
1.0	2.000	2.000	1.9×10^{-15}	2.000	0

TABLE 12. Chebyshev coefficients at $N = 6, 12$ for Example 5.6.

	c_0	c_1	c_2	c_3	c_4	c_5	c_6
N = 6	2.00000000	0.74999999	0.00000000	0.24999999	0.00000000	0.00000000	0.00000000
N =12	2.00000000	0.75000000	0.00000000	0.25000000	0.00000000	0.000000000	0.00000000

	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}
	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.000000000



(a) Numerical vs Exact

(b) Absolute error (e)

FIGURE 7. Graphical representation of approximate solution and respective error estimation for Example 5.6.

subject to BCs

$$S'(0) = 0, \;\; S(1) = \frac{1}{3} S\left(\frac{1}{4}\right). \tag{62}$$

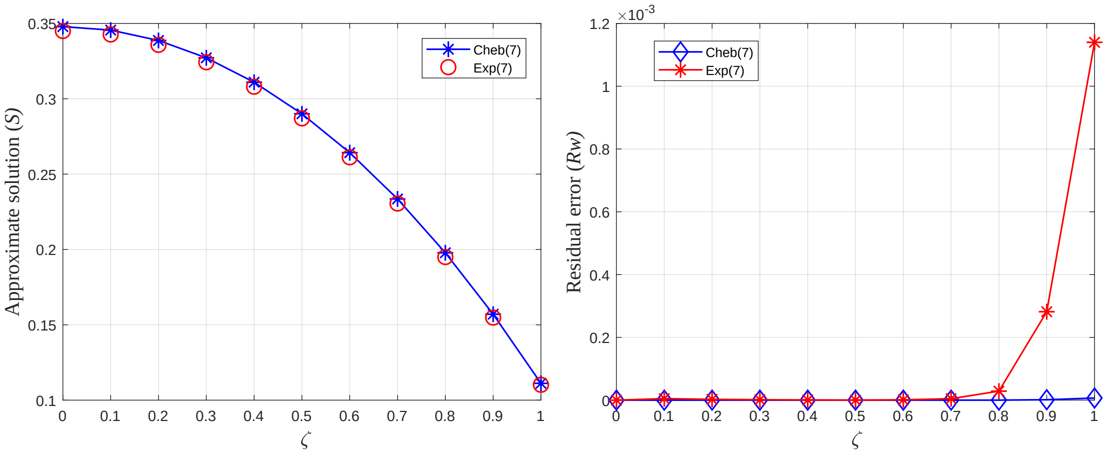
To the best of our knowledge, exact solution of LEFBVP (59)-(60) is not available in literature. The approximate solution (Cheb(N)) has been presented in comparison with exponential collocation method (EXP(N)) [29] for $N = 7$, both in terms of quantity and quality in Table 13 and Figure 8, respectively. To test the accuracy of method, the residual error $Rw = \left| -S''\left(\frac{\zeta}{2}\right) - \frac{2}{\zeta} S'\left(\frac{\zeta}{2}\right) - 1 + 2S^3\left(\frac{\zeta}{2}\right) \right|$ has calculated in Table 13 and Figure 8. The values of the Chebyshev coefficient for $N = 7$ provides in Table 14. Figure 8 and Table 13 show better accuracy of the proposed methodology over exponential collocation method [29].

TABLE 13. Quantitative estimation of numerical results using Cheb(N) and BCM at $N = 7$ for Example 5.7.

ζ	Cheb(7)	Rw	EXP(7)	Rw
0.0	3.47874×10^{-1}	2.0×10^{-28}	3.44974×10^{-1}	1.0×10^{-12}
0.1	3.45583×10^{-1}	4.0×10^{-9}	3.42678×10^{-1}	4.8×10^{-6}
0.2	3.38699×10^{-1}	2.6×10^{-9}	3.35779×10^{-1}	2.5×10^{-6}
0.3	3.27186×10^{-1}	1.7×10^{-9}	3.24241×10^{-1}	1.3×10^{-6}
0.4	3.10987×10^{-1}	9.9×10^{-10}	3.08008×10^{-1}	6.1×10^{-7}
0.5	2.90030×10^{-1}	1.1×10^{-20}	2.87011×10^{-1}	1.7×10^{-11}
0.6	2.64233×10^{-1}	2.6×10^{-9}	2.61181×10^{-1}	1.0×10^{-6}
0.7	2.33514×10^{-1}	1.4×10^{-8}	2.30486×10^{-1}	4.6×10^{-6}
0.8	1.97798×10^{-1}	1.1×10^{-7}	1.94969×10^{-1}	2.8×10^{-5}
0.9	1.57027×10^{-1}	1.3×10^{-6}	1.54779×10^{-1}	2.8×10^{-4}
1.0	1.11175×10^{-1}	6.7×10^{-6}	1.10198×10^{-1}	1.1×10^{-3}

TABLE 14. Chebyshev coefficients at $N = 3$ for Example 5.7.

	c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7
$N = 7$	0.113480	0.000005	-0.028534	0.000003	0.000007	0.000001	-0.000002	0.000000



(a) Approximate vs Exact (b) Residual error (Rw)

FIGURE 8. Graphical representation of approximate solution and respective error estimation for Example 5.7.

6. CONCLUSION AND REMARKS

In the presence of shifted Chebyshev polynomials and collocation points, this article offers a reliable and improved methodology for estimating the numerical solution of LEF-BVPs. The numerical scheme of the proposed methodology is relatively more straightforward due to its simplicity and less complexity as it uses shifted Chebyshev polynomial of lower order (Specifically $N = 3, 4, 5, 6, 7$ and 12 for our cases). If the variable coefficients of derivatives in LEFDE (1) and, the terms $p(\zeta)$ and $f(\zeta)$, are expandable in terms of shifted Chebyshev series in the interval $[0, 1]$, then there exists a solution $S(\zeta)$; otherwise the solution will not be exist. This is a necessary condition for the existence of solution using proposed methodology. A sufficiently large truncation limit of N is required to obtain the approximate solution of the proposed BVPs. Since the precision of the solution $S(\zeta)$ depends on the choice of N, the value of N may be large and hence computational effort may increase. However, it is worth noting that the computational cost required by present methodology is lower.

Using the Banach contraction principle on equivalent integral operator, sufficient conditions for the unique solution are also presented. The given comparisons of residual and absolute error of existing techniques ([29, 22] and reference therein) and superior outcomes demonstrate the effectiveness and accuracy of the proposed collocation method over the exponential collocation method, Bernoulli collocation method and Laguerre polynomial approach. Convergence analysis of the proposed numerical scheme is also discussed to examine the legality and applicability of the proposed method.

Although the proposed methodology has been used with excellent accuracy and efficiency for the considered class of LEFBVPs, some limitations still remain. The present work is restricted to one dimensional formulations with non-local three point BCs. It can be extended to multi-dimensional models with nonlinear boundary conditions. It can also be extended to coupled systems of nonlinear functional differential equations which provides a good frame work for analyzing more complex phenomena in engineering and applied science.

7. Conflict of Interest

The authors certify that there are no conflicts of interest to this work.

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