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# NONOSCILLATORY SOLUTIONS OF CERTAIN NONLINEAR HIGHER ORDER DIFFERENTIAL AND DELAY DIFFERENTIAL EQUATIONS WITH FORCING TERM

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Abstract. There are some nonlinear higher order differential and delay equations with forcing term that are taken into consideration. Using the Schauder fixed point theorem, the sufficient conditions for the existence of nonoscillatory solutions to these equations are derived. The application of theoretical findings is demonstrated by an example. Our results generalize some known results in the references.

Keywords: Nonoscillatory solution, Fixed point, Higher-order, Delay.

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## 1. Introduction

In this article, the authors aim to establish sufficient conditions for the existence of nonoscillatory solutions to three types of higher-order differential and delay differential equations with forcing term

$$\left[a(t)\left[y^{(n)}(t)\right]^{\gamma}\right]^{(m)}+f(t,y(t))=g(t), \tag{1}$$

$$\left[a(t)\left[y^{(n)}(t)\right]^{\gamma}\right]^{(m)}+f(t,y(t-\tau))=g(t), \tag{2}$$

and

$$\left[a(t)\left[y^{(n)}(t)\right]^{\gamma}\right]^{(m)}+\int_{a_1}^{b_1}f(t,y(t-\mu))d\mu=g(t), \tag{3}$$

where $\gamma$ is a ratio of odd positive integers, $n$ and $m$ are integers with $n\geq 2$ and $m\geq 1$, $a\in C([t_0,\infty),(0,\infty))$, $f\in C([t_0,\infty)\times\mathbb{R},\mathbb{R})$, $g\in C([t_0,\infty),\mathbb{R})$, $\tau>0$, $\mu\in[a_1,b_1]$ and $b_1>a_1\geq 0$.

Studying oscillatory and nonoscillatory solutions in differential and delay differential equations is important in both theory and practice. These solutions have the potential to

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bring stability, predictability, and practical benefits in various areas, including engineering, control systems, and physics. The nonoscillatory behavior of solutions to differential and delay differential equations has been the subject of study by several authors in the past. Specifically, Kusano and Naito [10, 11] investigated the existence of oscillatory and nonoscillatory solutions to fourth-order differential equations of the form:

$$[r(t)y''r(t)]'' + yF(y^2, t) = 0,$$

and their results based on whether the function $F(z,t)$ is nondecreasing or nonincreasing in the variable $z$. Li and Fei [14] focused on the existence of positive solutions to higher-order nonlinear delay differential equations:

$$(r(t)x^{(m-1)}(t))' + f(t, x(t-\tau)) = 0$$

and they introduced conditions related to the behavior of $f(t,x)/x$ is nonincreasing or nondecreading in $x$ for $x > 0$. Meanwhile, Candan [5] examined the existence of positive solutions to the specific case of equations (1)-(3) with $\gamma = 1$. He imposed conditions on the function $f(t,x)$, specifying whether it is a non-decreasing or non-increasing function in the variable $x$.

In this paper, we extend the results reported in [5] to a more general case. Unlike previous studies [5, 10, 11, 14], we impose mild conditions on the function $f$. Our approach involves converting the differential or delay differential equations to integral equations and applying the Schauder fixed point theorem.

Recently, there has been significant interest in the nonoscillatory behavior of solutions to nonlinear differential, delay differential, and delay dynamic equations [3, 4, 6, 8, 15, 17, 18]. For further developments in this area, we refer readers to the books [1, 2, 7, 9, 12].

As usual, by a solution of (1) we mean a function $y \in C([t_1,\infty),\mathbb{R})$, $t_1 \geq t_0$, which has the property $y \in C^n([t_1,\infty),\mathbb{R})$ and $a(t)\left[y^{(n)}(t)\right]^\gamma \in C^m([t_1,\infty),\mathbb{R})$ and such that (1) is satisfied for $t \geq t_1$. By a solution of (2) we mean a function $y \in C([t_1-\tau,\infty),\mathbb{R})$, $t_1 \geq t_0$, which has the property $y \in C^n([t_1,\infty),\mathbb{R})$ and $a(t)\left[y^{(n)}(t)\right]^\gamma \in C^m([t_1,\infty),\mathbb{R})$ and such that (2) is satisfied for $t \geq t_1$. By a solution of (3) we mean a function $y \in C([t_1-b_1,\infty),\mathbb{R})$, $t_1 \geq t_0$, which has the property $y \in C^n([t_1,\infty),\mathbb{R})$ and $a(t)\left[y^{(n)}(t)\right]^\gamma \in C^m([t_1,\infty),\mathbb{R})$ and such that (3) is satisfied for $t \geq t_1$.

The oscillatory character is considered in the usual sense; i.e., a solution of (1)-(3) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is considered nonoscillatory.

## 2. Main Results

**Theorem 2.1.** *Suppose that there exists an interval $[c_1, c_2] \subset \mathbb{R}^+$ such that*

$$\int_{t_0}^{\infty} s^{n-1}\left[\frac{1}{a(s)}\int_s^{\infty}\frac{r^{m-1}}{(m-1)!}\left[\sup_{c\in[c_1,c_2]}|f(r,c)| + |g(r)|\right]dr\right]^{1/\gamma} ds < \infty. \tag{4}$$

*Then, (1) has a bounded nonoscillatory solution.*

*Proof.* Due to (4), it can be chosen a $t_1 \geq t_0$ sufficiently large such that

$$\frac{1}{(n-1)!}\int_t^{\infty} s^{n-1}\left[\frac{1}{a(s)}\int_s^{\infty}\frac{r^{m-1}}{(m-1)!}\left[\sup_{c\in[c_1,c_2]}|f(r,c)| + |g(r)|\right]dr\right]^{1/\gamma} ds \tag{5}$$
$$\leq \frac{c_2 - c_1}{2}, \quad t \geq t_1.$$

Let $\Psi$ be the set of all continuous and bounded functions on $[t_1, \infty)$ with the supremum norm. Then $\Psi$ is a Banach space. We define a closed, bounded and convex subset $M$ of $\Psi$ as follows:

$$M = \{y \in \Psi : c_1 \leq y(t) \leq c_2, \quad t \geq t_1\}.$$

Consider the operator $F : M \longrightarrow \Psi$ defined by

$$(Fy)(t) = \frac{c_1 + c_2}{2}$$
$$- \frac{(-1)^{m+n}}{(n-1)!} \int_t^\infty (s-t)^{n-1} \left[ \frac{1}{a(s)} \int_s^\infty \frac{(r-s)^{m-1}}{(m-1)!} [f(r, y(r)) - g(r)]\, dr \right]^{1/\gamma} ds, \quad t \geq t_1.$$

The operator $F$ is constructed by integrating equation (1) $m$ times and then $n$ times from $t$ to $\infty$. A fixed point $y = Fy$ automatically satisfies the original equation when differentiated back. In order to use Schauder fixed point theorem, we need to show the following:

**i)** $F$ maps $M$ into $M$. For $t \geq t_1$ and $y \in M$, using (5) we have

$$\begin{aligned}
(Fy)(t) \leq{}& \frac{c_1 + c_2}{2} \\
&+ \frac{1}{(n-1)!} \int_t^\infty (s-t)^{n-1} \left[ \frac{1}{a(s)} \int_s^\infty \frac{(r-s)^{m-1}}{(m-1)!} \left[ \sup_{c \in [c_1, c_2]} |f(r, c)| + |g(r)| \right] dr \right]^{1/\gamma} ds \\
&\leq \frac{c_1 + c_2}{2} \\
&+ \frac{1}{(n-1)!} \int_t^\infty s^{n-1} \left[ \frac{1}{a(s)} \int_s^\infty \frac{r^{m-1}}{(m-1)!} \left[ \sup_{c \in [c_1, c_2]} |f(r, c)| + |g(r)| \right] dr \right]^{1/\gamma} ds \leq c_2,
\end{aligned}$$

and

$$\begin{aligned}
(Fy)(t) \geq{}& \frac{c_1 + c_2}{2} \\
&- \frac{1}{(n-1)!} \int_t^\infty (s-t)^{n-1} \left[ \frac{1}{a(s)} \int_s^\infty \frac{(r-s)^{m-1}}{(m-1)!} \left[ \sup_{c \in [c_1, c_2]} |f(r, c)| + |g(r)| \right] dr \right]^{1/\gamma} ds \\
&\geq \frac{c_1 + c_2}{2} \\
&- \frac{1}{(n-1)!} \int_t^\infty s^{n-1} \left[ \frac{1}{a(s)} \int_s^\infty \frac{r^{m-1}}{(m-1)!} \left[ \sup_{c \in [c_1, c_2]} |f(r, c)| + |g(r)| \right] dr \right]^{1/\gamma} ds \geq c_1.
\end{aligned}$$

Consequently, $F$ maps $M$ into $M$.

**ii)** $F$ is continuous. Let $\{y_k\} \subset M$ and $y_k(t) \to y(t)$ as $k \to \infty$. Because $M$ is closed,

$y \in M$. For $t \geq t_1$,

$$
\begin{aligned}
&|(Fy_k)(t) - (Fy)(t)| \\
&\qquad \leq \frac{1}{(n-1)!} \int_t^\infty \frac{(s-t)^{n-1}}{(a(s))^{1/\gamma}} \left| \left[ \int_s^\infty \frac{(r-s)^{m-1}}{(m-1)!} \left[f(r, y_k(r)) - g(r)\right] dr \right]^{1/\gamma} \right. \\
&\qquad \left. - \left[ \int_s^\infty \frac{(r-s)^{m-1}}{(m-1)!} \left[f(r, y(r)) - g(r)\right] dr \right]^{1/\gamma} \right| ds \\
&\qquad \leq \frac{1}{(n-1)!} \int_t^\infty \frac{s^{n-1}}{(a(s))^{1/\gamma}} G_k(s) ds,
\end{aligned}
$$

where

$$
\begin{aligned}
G_k(s) = &\left| \left[ \int_s^\infty \frac{(r-s)^{m-1}}{(m-1)!} \left[f(r, y_k(r)) - g(r)\right] dr \right]^{1/\gamma} \right. \\
&\left. - \left[ \int_s^\infty \frac{(r-s)^{m-1}}{(m-1)!} \left[f(r, y(r)) - g(r)\right] dr \right]^{1/\gamma} \right|.
\end{aligned}
$$

Hence, we have $\lim_{k\to\infty} G_k(s) = 0$ and then $\lim_{k\to\infty} \frac{1}{(n-1)!} \int_t^\infty \frac{s^{n-1}}{(a(s))^{1/\gamma}} G_k(s) ds = 0$ by the Lebesgue dominated convergence theorem. This implies that

$$
\lim_{k\to\infty} \|(Fy_k) - (Fy)\| = 0,
$$

and therefore $F$ is continuous.

**iii)** $FM$ is relatively compact. It is shown that the family of functions $\{Fy : y \in M\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$. For any given $\epsilon > 0$, and for all $y \in M$, in view of (4), we take $T \geq t_1$ sufficiently large such that

$$
\frac{1}{(n-1)!} \int_T^\infty s^{n-1} \left[ \frac{1}{a(s)} \int_s^\infty \frac{r^{m-1}}{(m-1)!} \left[ \sup_{c\in[c_1,c_2]} |f(r,c)| + |g(r)| \right] dr \right]^{1/\gamma} ds < \frac{\epsilon}{2}.
$$

For $y \in M$ and $T^* > T_* \geq T$, we obtain

$$
\begin{aligned}
&|(Fy)(T^*) - (Fy)(T_*)| \\
&\qquad \leq \frac{1}{(n-1)!} \int_{T_*}^\infty s^{n-1} \left[ \frac{1}{a(s)} \int_s^\infty \frac{r^{m-1}}{(m-1)!} \left[ \sup_{c\in[c_1,c_2]} |f(r,c)| + |g(r)| \right] dr \right]^{1/\gamma} ds \\
&\qquad + \frac{1}{(n-1)!} \int_{T^*}^\infty s^{n-1} \left[ \frac{1}{a(s)} \int_s^\infty \frac{r^{m-1}}{(m-1)!} \left[ \sup_{c\in[c_1,c_2]} |f(r,c)| + |g(r)| \right] dr \right]^{1/\gamma} ds \\
&\qquad < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}
$$

Observing that for $U > Z > 0$, we get

$$
U^n - Z^n = (U-Z)(U^{n-1} + U^{n-2}Z + \ldots + UZ^{n-2} + Z^{n-1}) \leq n(U-Z)U^{n-1}. \tag{6}
$$

For $y \in M$ and $t_1 \le T_* < T^* \le T$, because of (6) we have

$$
\begin{aligned}
&|(Fy)(T^*) - (Fy)(T_*)| \\
&\le \frac{1}{(n-1)!}\int_{T_*}^{T^*}(s-T_*)^{n-1}\left[\frac{1}{a(s)}\int_s^\infty \frac{(r-s)^{m-1}}{(m-1)!}\,|f(r,y(r))-g(r)|\,dr\right]^{1/\gamma} ds \\
&+ \frac{1}{(n-1)!}\int_{T^*}^{\infty}\left[(s-T_*)^{n-1}-(s-T^*)^{n-1}\right] \\
&\times \left[\frac{1}{a(s)}\int_s^\infty \frac{(r-s)^{m-1}}{(m-1)!}\,|f(r,y(r))-g(r)|\,dr\right]^{1/\gamma} ds \\
&\le \max_{T_*\le s\le T^*}\left\{\frac{s^{n-1}}{(n-1)!}\left[\frac{1}{a(s)}\int_s^\infty \frac{r^{m-1}}{(m-1)!}\left[\sup_{c\in[c_1,c_2]}|f(r,c)|+|g(r)|\right]dr\right]^{1/\gamma}\right\}(T^*-T_*) \\
&+ \frac{1}{(n-2)!}\int_{T^*}^{\infty}s^{n-2}\left[\frac{1}{a(s)}\int_s^\infty \frac{r^{m-1}}{(m-1)!}\left[\sup_{c\in[c_1,c_2]}|f(r,c)|+|g(r)|\right]dr\right]^{1/\gamma} ds\,(T^*-T_*).
\end{aligned}
$$

Therefore, there exists a $\delta > 0$ such that $|(Fy)(T^*) - (Fy)(T_*)| < \epsilon$ if $0 < T^* - T_* < \delta$. Thus, it can be concluded that $FM$ is relatively compact, and therefore there exists a $y \in M$ such that $Fy = y$ by the Schauder fixed point theorem. □

**Theorem 2.2.** *Suppose that (4) holds. Then, (2) has a bounded nonoscillatory solution.*

*Proof.* Due to (4), a $t_1 > t_0$ sufficiently large with $t_1 \ge t_0 + \tau$ can be chosen such that

$$
\begin{aligned}
&\frac{1}{(n-1)!}\int_t^\infty s^{n-1}\left[\frac{1}{a(s)}\int_s^\infty \frac{r^{m-1}}{(m-1)!}\left[\sup_{c\in[c_1,c_2]}|f(r,c)|+|g(r)|\right]dr\right]^{1/\gamma} ds \\
&\le \frac{c_2-c_1}{2}, \quad t \ge t_1.
\end{aligned}
\tag{7}
$$

Let $\Psi$ be the set of all continuous and bounded functions on $[t_0,\infty)$ with the supremum norm. Then $\Psi$ is a Banach space. We define a closed, bounded and convex subset $M$ of $\Psi$ as follows:

$$M = \{y \in \Psi : c_1 \le y(t) \le c_2, \quad t \ge t_0\}.$$

Consider the operator $F : M \longrightarrow \Psi$ defined by

$$
(Fy)(t) = \begin{cases}
\frac{c_1+c_2}{2} - \frac{(-1)^{m+n}}{(n-1)!}\int_t^\infty (s-t)^{n-1} & \\
\left[\frac{1}{a(s)}\int_s^\infty \frac{(r-s)^{m-1}}{(m-1)!}\left[f(r,y(r-\tau))-g(r)\right]dr\right]^{1/\gamma} ds, & t \ge t_1, \\
(Fy)(t_1), & t_0 \le t \le t_1.
\end{cases}
$$

We need to show the following in order to use the Schauder Fixed Point Theorem:
**i)** $F$ maps $M$ into $M$. Due to (7), it can be shown that

$$(Fy)(t) \le c_2, \quad (Fy)(t) \ge c_1,$$

where $t \ge t_1$ and $y \in M$. Hence, it can be concluded that $F$ maps $M$ into $M$.
**ii)** $F$ is continuous. Let $\{y_k\} \subset M$ and $y_k(t) \to y(t)$ as $k \to \infty$. Because $M$ is closed,

$y \in M$. For $t \geq t_1$,

$$
\begin{aligned}
&|(Fy_k)(t) - (Fy)(t)| \\
&\quad \leq \frac{1}{(n-1)!} \int_t^\infty \frac{(s-t)^{n-1}}{(a(s))^{1/\gamma}} \left| \left[ \int_s^\infty \frac{(r-s)^{m-1}}{(m-1)!} [f(r, y_k(r-\tau)) - g(r)]\, dr \right]^{1/\gamma} \right. \\
&\quad \left. - \left[ \int_s^\infty \frac{(r-s)^{m-1}}{(m-1)!} [f(r, y(r-\tau)) - g(r)]\, dr \right]^{1/\gamma} \right| ds \\
&\quad \leq \frac{1}{(n-1)!} \int_t^\infty \frac{s^{n-1}}{(a(s))^{1/\gamma}} G_k(s) ds,
\end{aligned}
$$

where

$$
\begin{aligned}
G_k(s) = &\left| \left[ \int_s^\infty \frac{(r-s)^{m-1}}{(m-1)!} [f(r, y_k(r-\tau)) - g(r)]\, dr \right]^{1/\gamma} \right. \\
&\left. - \left[ \int_s^\infty \frac{(r-s)^{m-1}}{(m-1)!} [f(r, y(r-\tau)) - g(r)]\, dr \right]^{1/\gamma} \right|.
\end{aligned}
$$

Due to the similarity to the proof of Theorem 2.1, the remaining part of *ii*) is omitted.
**iii)** $FM$ is relatively compact. It is shown that the family of functions $\{Fy : y \in M\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$. It is sufficient to show equicontinuity because it is clear that $\{Fy : y \in M\}$ is uniformly bounded. For $y \in M$ and any $\epsilon > 0$, in view of (4), we choose a $T \geq t_1$ sufficiently large such that

$$
\frac{1}{(n-1)!} \int_T^\infty s^{n-1} \left[ \frac{1}{a(s)} \int_s^\infty \frac{r^{m-1}}{(m-1)!} \left[ \sup_{c \in [c_1, c_2]} |f(r,c)| + |g(r)| \right] dr \right]^{1/\gamma} ds < \frac{\epsilon}{2}.
$$

We follow the same lines of the proof of Theorem 2.1 when $y \in M$ and $T^* > T_* \geq T$, $y \in M$ and $t_1 \leq T_* < T^* \leq T$. Finally, for $y \in M$ and $t_0 \leq T_* < T^* \leq t_1$ there exists a $\delta > 0$ such that $|(Fy)(T^*) - (Fy)(T_*)| < \epsilon$ if $0 < T^* - T_* < \delta$. Thus, it can be concluded that $FM$ is relatively compact and therefore there exists a $y \in M$ such that $Fy = y$ by the Schauder fixed point theorem. □

**Theorem 2.3.** *Suppose that there exists an interval $[c_1, c_2] \subset \mathbb{R}^+$ such that*

$$
\int_{t_0}^\infty s^{n-1} \left[ \frac{1}{a(s)} \int_s^\infty \frac{r^{m-1}}{(m-1)!} \left[ (b_1 - a_1) \sup_{c \in [c_1, c_2]} |f(r,c)|| + |g(r)| \right] dr \right]^{1/\gamma} ds < \infty. \tag{8}
$$

*Then, (3) has a bounded nonoscillatory solution.*

*Proof.* Suppose (8) holds. Hence, it can be chosen a $t_1 > t_0$ sufficiently large with $t_1 \geq t_0 + b_1$ such that

$$
\begin{aligned}
&\frac{1}{(n-1)!} \int_t^\infty s^{n-1} \left[ \frac{1}{a(s)} \int_s^\infty \frac{r^{m-1}}{(m-1)!} \left[ (b_1 - a_1) \sup_{c \in [c_1, c_2]} |f(r,c)| + |g(r)| \right] dr \right]^{1/\gamma} ds \\
&\leq \frac{c_2 - c_1}{2}, \quad t \geq t_1.
\end{aligned} \tag{9}
$$

Let $\Psi$ be the set as in the proof of Theorem 2.2. We define a closed, bounded and convex subset $M$ of $\Psi$ as follows:

$$
M = \{y \in \Psi : c_1 \leq y(t) \leq c_2, \quad t \geq t_0.\}
$$

Consider the operator $F : M \longrightarrow \Psi$ defined by

$$(Fy)(t) = \begin{cases} \frac{c_1+c_2}{2} - \frac{(-1)^{m+n}}{(n-1)!} \int_t^\infty (s-t)^{n-1} \\ \left[\frac{1}{a(s)} \int_s^\infty \frac{(r-s)^{m-1}}{(m-1)!} \left[\int_{a_1}^{b_1} f(r, y(r-\mu))d\mu - g(r)\right] dr\right]^{1/\gamma} ds, & t \geq t_1, \\ (Fy)(t_1), & t_0 \leq t \leq t_1. \end{cases}$$

We show that $F$ meets the requirements of Schauder fixed point theorem.

**i)** $F$ maps $M$ into $M$. For $t \geq t_1$ and $y \in M$, using (9) we obtain

$$\begin{aligned} (Fy)(t) &\leq \frac{c_1+c_2}{2} + \frac{1}{(n-1)!} \\ &\times \int_t^\infty s^{n-1} \left[\frac{1}{a(s)} \int_s^\infty \frac{r^{m-1}}{(m-1)!} \left[(b_1 - a_1) \sup_{c \in [c_1, c_2]} |f(r, c)| + |g(r)|\right] dr\right]^{1/\gamma} ds \\ &\leq c_2, \end{aligned}$$

and

$$\begin{aligned} (Fy)(t) &\geq \frac{c_1+c_2}{2} - \frac{1}{(n-1)!} \\ &\times \int_t^\infty s^{n-1} \left[\frac{1}{a(s)} \int_s^\infty \frac{r^{m-1}}{(m-1)!} \left[(b_1 - a_1) \sup_{c \in [c_1, c_2]} |f(r, c)| + |g(r)|\right] dr\right]^{1/\gamma} ds \\ &\geq c_1. \end{aligned}$$

Hence, $F$ maps $M$ into $M$.

**ii)** $F$ is continuous. Let $\{y_k\} \subset M$ and $y_k(t) \to y(t)$ as $k \to \infty$. Because $M$ is closed, $y \in M$. For $t \geq t_1$,

$$\begin{aligned} &|(Fy_k)(t) - (Fy)(t)| \\ &\leq \frac{1}{(n-1)!} \int_t^\infty \frac{(s-t)^{n-1}}{(a(s))^{1/\gamma}} \left| \left[\int_s^\infty \frac{(r-s)^{m-1}}{(m-1)!} \left[\int_{a_1}^{b_1} f(r, y_k(r-\mu))d\mu - g(r)\right] dr\right]^{1/\gamma} \right. \\ &\left. - \left[\int_s^\infty \frac{(r-s)^{m-1}}{(m-1)!} \left[\int_{a_1}^{b_1} f(r, y(r-\mu))d\mu - g(r)\right] dr\right]^{1/\gamma} \right| ds \\ &\leq \frac{1}{(n-1)!} \int_t^\infty \frac{s^{n-1}}{(a(s))^{1/\gamma}} G_k(s) ds, \end{aligned}$$

where

$$\begin{aligned} G_k(s) = &\left| \left[\int_s^\infty \frac{(r-s)^{m-1}}{(m-1)!} \left[\int_{a_1}^{b_1} f(r, y_k(r-\mu))d\mu - g(r)\right] dr\right]^{1/\gamma} \right. \\ &\left. - \left[\int_s^\infty \frac{(r-s)^{m-1}}{(m-1)!} \left[\int_{a_1}^{b_1} f(r, y(r-\mu))d\mu - g(r)\right] dr\right]^{1/\gamma} \right|. \end{aligned}$$

According to the Lebesgue dominated convergence theorem, we have $\lim_{k \to \infty} G_k(s) = 0$ and then

$\lim_{k\to\infty} \frac{1}{(n-1)!}\int_t^\infty \frac{s^{n-1}}{(a(s))^{1/\gamma}} G_k(s)ds = 0$. This implies that

$$\lim_{k\to\infty} \|(Fy_k) - (Fy)\| = 0,$$

and therefore $F$ is continuous.

**iii)** $FM$ is relatively compact. It is shown that the family of functions $\{Fy : y \in M\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$. It suffices to show equicontinuity because the uniformly boundedness of $\{Fy : y \in M\}$ is clear. For $y \in M$ and any $\epsilon > 0$, in view of (9), take $T \geq t_1$ sufficiently large such that

$$\frac{1}{(n-1)!}\int_T^\infty s^{n-1}\left[\frac{1}{a(s)}\int_s^\infty \frac{r^{m-1}}{(m-1)!}\left[(b_1-a_1)\sup_{c\in[c_1,c_2]}|f(r,c)| + |g(r)|\right]dr\right]^{1/\gamma} ds < \frac{\epsilon}{2}.$$

Due to the similarity to the proof of Theorem 2.2, the remaining part of the proof is omitted. □

**Example 2.1.** Consider the equation

$$\left[e^{2t}\left[y^{(4)}(t)\right]^3\right]^{(5)} + \int_1^2 e^{-t}y^3(t-\mu)d\mu$$
$$= \frac{3}{2}e^{-3t}[e^4-e^2] + 3e^{-2t}[e^2-e] + \frac{1}{3}e^{-4t}[e^6-e^3], \qquad (10)$$

and note that $n = 4$, $m = 5$, $a(t) = e^{2t}$, $\gamma = 3$, $a_1 = 1$, $b_1 = 2$, $f(t, y) = e^{-t}y^3$ and $g(t) = \frac{3}{2}e^{-3t}[e^4-e^2]+3e^{-2t}[e^2-e]+\frac{1}{3}e^{-4t}[e^6-e^3]$. Furthermore, we will now demonstrate the validity of (8) as follows:

$$\int_{t_0}^\infty s^{n-1}\left[\frac{1}{a(s)}\int_s^\infty \frac{r^{m-1}}{(m-1)!}\left[(b_1-a_1)\sup_{c\in[c_1,c_2]}|f(r,c)|| + |g(r)|\right]dr\right]^{1/\gamma} ds = \int_{t_0}^\infty s^3$$
$$\left[\frac{1}{e^{2s}}\int_s^\infty \frac{r^4}{4!}\left[\sup_{c\in[c_1,c_2]}|e^{-r}c^3| + \frac{3}{2}e^{-3r}[e^4-e^2] + 3e^{-2r}[e^2-e] + \frac{1}{3}e^{-4r}[e^6-e^3]\right]dr\right]^{1/3} ds$$
$$= \int_{t_0}^\infty s^3\left[\frac{1}{e^{2s}}\int_s^\infty \frac{r^4}{4!}\left[e^{-r}c_2^3 + \frac{3}{2}e^{-3r}[e^4-e^2] + 3e^{-2r}[e^2-e] + \frac{1}{3}e^{-4r}[e^6-e^3]\right]dr\right]^{1/3} ds$$
$$< \infty,$$

where $c_2 > 0$. The integral converges because, for large $s$, the exponential decay of the integrand dominates polynomial growth, guaranteeing convergence of the integral. This verifies that all conditions of Theorem 2.3 are satisfied. In fact, $y(t) = 1 + e^{-t}$ is a nonoscillatory solution of (10).

## 3. Conclusions

In the present paper, we studied the existence of nonoscillatory solutions to the nonlinear higher-order differential and delay differential equations with forcing term. We investigated sufficient conditions for the existence of nonoscillatory solutions to these equations. Our approach was based on converting the differential or delay differential equations to integral equations and applying the Schauder fixed point theorem. An example was provided to justify the main results.

When $\gamma = 1$, our results reproduce the conclusions of [5], but under significantly milder conditions on the function $f$. In the context of future research, this approach can be applied to explore the nonoscillatory behavior of neutral functional differential equations.

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