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PICTURE FUZZY CHARACTERISTIC AND PICTURE FUZZY DIVISOR OF ZERO IN A RING ALONG WITH THEIR APPLICATIONS IN REAL LIFE SITUATIONS

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Abstract. The paper provides the concept of a picture fuzzy ideal over a ring and illustrates it with example. Additionally, it introduces the notions of picture fuzzy characteristic and picture fuzzy divisor of zero in a ring with respect to some picture fuzzy subring, investigating their related properties. The conditions under which picture fuzzy characteristic and picture fuzzy divisor of zero coincide with ordinary characteristic and ordinary divisor of zero are explicitly stated. It is demonstrated that the unit element in a ring, under some certain conditions, is not a picture fuzzy divisor of zero. Moreover, the picture fuzzy characteristic of the Cartesian product of two rings over the Cartesian product of two picture fuzzy subrings is calculated. Lastly, applications of PFCh and PFD of zero in Customers' feedback and Customers' sentiment analysis are presented

Significant note of the work: Characteristic and divisor of zero are two foundational concepts in Abstract Algebra, playing a pivotal role in understanding the structure and behavior of rings. In this paper, these classical notions are extended and explored within the picture fuzzy environment, an advanced framework that builds upon fuzzy sets and intuitionistic fuzzy sets. By generalizing these concepts, the study bridges the gap between abstract algebraic theory and the flexible, nuanced reasoning enabled by picture fuzziness. Key properties of picture fuzzy characteristic and picture fuzzy divisor of zero are thoroughly investigated, revealing how these concepts retain their essence while adapting to the picture fuzzy context. Furthermore, the abstract notions of characteristic and divisor of zero are connected to real-world scenarios, demonstrating their relevance and applicability when interpreted through the lens of picture fuzziness. This work not only advances the theoretical understanding of algebraic structures but also opens new avenues for applying fuzzy algebra in solving practical problems, thereby inspiring innovative research in both mathematics and interdisciplinary fields.

Keyword: Fuzzy set, intuitionistic fuzzy set, picture fuzzy set, picture fuzzy subring, picture fuzzy ideal, picture fuzzy characteristic, picture fuzzy divisor of zero.

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1. Introduction

To address challenges involving imprecision and uncertainty, Zadeh [1] introduced the concept of fuzzy sets (FS), which revolutionized mathematical frameworks for handling vagueness. This foundational concept was extended to group theory by Rosenfeld [2], who investigated fuzzy subgroups. Subsequently, Ren [3] studied fuzzy ideals and quotient fuzzy subrings, further broadening the scope of fuzzy set applications. Atanassov [4] generalized the idea of fuzzy sets by introducing intuitionistic fuzzy sets, which incorporated an additional degree of uncertainty. This innovation inspired Biswas [5] to develop intuitionistic fuzzy sets based on subgroups. Hur et al. [6] extended these ideas by establishing intuitionistic fuzzy subrings, while Banerjee and Basnet [7] carried out additional research on intuitionistic fuzzy subrings and ideals. Paruchuri et al. [8] studied fuzzy ideals in matrix near-rings and examined their structural characteristics in fuzzy algebraic structures. Awad and Fakih [9] presented the notion of bipolar intuitionistic fuzzy matrices and analyzed their determinant to expand matrix theory in fuzzy environments.

Recognizing a novel type of uncertainty namely 'neutrality', Cuong [10] extended intuitionistic fuzzy sets into the framework of picture fuzzy sets, offering a more nuanced representation of real-world complexities. Dogra and Pal [11] further advanced this field by introducing picture fuzzy subrings over rings and exploring their core properties (based on intersection, union and Cartesian product of picture fuzzy sets). They showed that (θ, ϕ, ψ)-cut of a picture fuzzy subring form a ring. They also explored the conditions under which a picture fuzzy set over a ring qualifies as a picture fuzzy subring and analyzed the impact of ring homomorphisms on these structures, showing that both the inverse and bijective images of a picture fuzzy subring remain picture fuzzy subrings. Decision making problems under different types of uncertain environments were studied by several researchers [12, 13, 14, 16, 17, 18, 19, 20, 21, 22].

Despite these advancements, significant gaps remain in the study of algebraic structures under the picture fuzzy framework. While much research has focused on the fundamental properties of picture fuzzy subrings, such as their operations and homomorphic images, the extension of classical algebraic concepts like characteristic and divisor of zero to the picture fuzzy context remains unexplored. These concepts are critical in understanding the structure of rings and their generalizations, yet their behavior and implications in the picture fuzzy environment have not been adequately addressed. Specifically, the conditions under which picture fuzzy characteristic and picture fuzzy divisor of zero align with their classical counterparts, as well as their practical relevance, remain unclear.

In this paper, we aim to address these gaps by introducing the notions of picture fuzzy characteristic and picture fuzzy divisor of zero in a ring with respect to some picture fuzzy subring. We investigate key properties of these concepts, demonstrating how they generalize and extend their classical counterparts. Furthermore, we explore their real-life applications, illustrating the utility of picture fuzzy characteristic, and picture fuzzy divisors of zero in addressing practical problems. Through this study, we establish a bridge between abstract algebraic theory and real-world scenarios, advancing the scope of research in fuzzy algebra.

List of Abbreviations:

SU: Set of universe
SsU: Sets of universe
FS: Fuzzy set

FI: Fuzzy ideal
FSR: Fuzzy subring
FSRs: Fuzzy subrings
IFS: Intuitionistic fuzzy set
IFSR: Intuitionistic fuzzy subring
IFSRs: Intuitionistic fuzzy subrings
IFCh: Intuitionistic fuzzy characteristic
IFLD: Intuitionistic fuzzy left divisor
IFRD: Intuitionistic fuzzy right divisor
IFD: Intuitionistic fuzzy divisor
PFS: Picture fuzzy set
PFSs: Picture fuzzy sets
PFSR : Picture fuzzy subring
PFSRs : Picture fuzzy subrings
PFI : Picture fuzzy ideal
PFIs: Picture fuzzy ideals
MS: Membership
NMS: Non-membership
MMS: Measure of membership
MNMS: Measure of non-membership
MPMS: Measure of positive membership
MNeuMS: Measure of neutral membership
MNegMS: Measure of negative membership
CP: Cartesian product
PFCh: Picture fuzzy characteristic
PFLD: Picture fuzzy left divisor
PFRD: Picture fuzzy right divisor
PFD: Picture fuzzy divisor

2. Preliminaries

In this section, some basic concepts of fuzzy sets (FSs), intuitionistic fuzzy sets (IFSs), picture fuzzy sets (PFSs), operations on PFSs, intuitionistic fuzzy subring (IFSR), intuitionistic fuzzy characteristic (IFCh), intuitionistic fuzzy divisor (IFD) of zero, and picture fuzzy subring (PFSR) are briefly discussed which will be helpful for proposing the current study.

To address the challenges involving imprecision and uncertainty, Zadeh [1] introduced the concept of FS, which revolutionized the mathematical frameworks for handling vagueness.

Definition 2.1. [1] *A FS ω over a SU V is defined as: $\omega = \{(l, \omega_{MS}(l)) : l \in V\}$, where $\omega_{MS}(l) \in [0,1]$ is the MMS of l in ω.*

Atanassov [4] defined IFS as a generalization of FS.

Definition 2.2. [4] *An IFS ω over a SU V is defined as: $\omega = \{(l, \omega_{MS}(l), \omega_{NMS}(l)) : l \in V\}$, where $\omega_{MS}(l) \in [0,1]$ is the MMS of l in ω and $\omega_{NMS}(l) \in [0,1]$ is the MNMS of l in ω with the condition $0 \leqslant \omega_{MS}(l) + \omega_{NMS}(l) \leqslant 1, \forall l \in V$.*

Based on the above concept, Banerjee and Basnet [7] studied IFSR.

Definition 2.3. [7] *Suppose, $\omega = (\omega_{MS}, \omega_{NMS})$ represents an IFS over a ring $(V, +, \cdot)$, then ω is termed as IFSR of V if the below specified conditions are fulfilled.*
(i) $\omega_{MS}(l - m) \geqslant \omega_{MS}(l) \wedge \omega_{MS}(m)$ and $\omega_{NMS}(l - m) \leqslant \omega_{NMS}(l) \vee \omega_{NMS}(m)$,
(ii) $\omega_{MS}(lm) \geqslant \omega_{MS}(l) \wedge \omega_{MS}(m)$ and $\omega_{NMS}(lm) \leqslant \omega_{NMS}(l) \vee \omega_{NMS}(m), \forall l, m \in V$.

Intuitionistic fuzzification of characteristic of a ring and divisor of zero in a ring was done by Sharma [15].

Definition 2.4. [15] *Let $\omega = \{(l, \omega_{MS}(l), \omega_{NMS}(l)) : l \in V\}$ be an IFSR over a ring V. If there exists a least positive integer r such that $\omega_{MS}(rl) = \omega_{MS}(0)$ and $\omega_{NMS}(rl) = \omega_{NMS}(0)$, $\forall l \in V$, then r is called intuitionistic fuzzy characteristic (IFCh) of V w.r.t. ω. If no such positive integer is found, then IFCh of V w.r.t. ω is zero.*

Definition 2.5. [15] *Assume that $\omega = (\omega_{MS}, \omega_{NMS})$ represents an IFSR of a ring $(V, +, \cdot)$. If $l, m \in V$ accompanied by*
(i) $\omega_{MS}(l) \neq \omega_{MS}(0), \omega_{NMS}(l) \neq \omega_{NMS}(0)$
(ii) $\omega_{MS}(m) \neq \omega_{MS}(0), \omega_{NMS}(m) \neq \omega_{NMS}(0)$
imply that $\omega_{MS}(lm) = \omega_{MS}(0)$, $\omega_{NMS}(lm) = \omega_{NMS}(0)$, then l and m are respectively called IFLD of zero and IFRD of zero.

More possible types of uncertainty handling tool was introduced by Cuong [10] in the form of PFS which is an extension of FS and IFS.

Definition 2.6. [10] *A PFS ω over a SU V is defined as $\omega = \{(l, \omega_{Pos}(l), \omega_{Neu}(l), \omega_{Neg}(l)) : l \in V\}$, where $\omega_{Pos}(l) \in [0, 1]$ is the MPMS of l in ω, $\omega_{Neu}(l) \in [0, 1]$ is the MNeuMS of l in ω and $\omega_{Neg}(l) \in [0, 1]$ is the MNegMS of l in ω with the condition $0 \leqslant \omega_{Pos}(l) + \omega_{Neu}(l) + \omega_{Neg}(l) \leqslant 1, \forall l \in V$. For all $l \in V$, $1 - (\omega_{Pos}(l) + \omega_{Neu}(l) + \omega_{Neg}(l))$ is called MRefMS l in ω.*

Some basic operations on PFSs are as follows.

Definition 2.7. [10] *Let $\omega = \{(l, \omega_{Pos}(l), \omega_{Neu}(l), \omega_{Neg}(l)) : l \in V\}$ and $\omega' = \{(l, \omega'_{Pos}(l), \omega'_{Neu}(l), \omega'_{Neg}(l)) : l \in V\}$ be two PFSs over the same SU V. Then*
(i) $\omega \subseteq \omega'$ iff $\omega_{Pos}(l) \leqslant \omega'_{Pos}(l)$, $\omega_{Neu}(l) \leqslant \omega'_{Neu}(l)$, $\omega_{Neg}(l) \geqslant \omega'_{Neg}(l), \forall l \in V$.
(ii) $\omega = \omega'$ iff $\omega_{Pos}(l) = \omega'_{Pos}(l)$, $\omega_{Neu}(l) = \omega'_{Neu}(l)$, $\omega_{Neg}(l) = \omega'_{Neg}(l), \forall l \in V$.
(iii) $\omega \cup \omega' = \{(l, \omega_{Pos}(l) \vee \omega'_{Pos}(l)), \omega_{Neu}(l) \wedge \omega'_{Neu}(l), \omega_{Neg}(l) \wedge \omega'_{Neg}(l))) : l \in V\}$.
(iv) $\omega \cap \omega' = \{(l, \omega_{Pos}(l) \wedge \omega'_{Pos}(l)), \omega_{Neu}(l) \wedge \omega'_{Neu}(l)), \omega_{Neg}(l) \vee \omega'_{Neg}(l))) : l \in V\}$.

Cartesian product (CP) of two PFSs is defined as follows.

Definition 2.8. [10] *Let $\omega = \{(l_1, \omega_{Pos}(l_1), \omega_{Neu}(l), \omega_{Neg}(l_1)) : l_1 \in V_1\}$ and $\omega' = \{(l_2, \omega'_{Pos}(l_2), \omega'_{Neu}(l_2), \omega'_{Neg}(l_2)) : l_2 \in V_2\}$ be two PFSs over the SsU V_1 and V_2 respectively. Then the CP of ω and ω' is the PFS $\omega \times \omega' = \{((l_1, l_2), \chi_{Pos}((l_1, l_2)), \chi_{Neu}((l_1, l_2)), \chi_{Neg}((l_1, l_2))) : (l_1, l_2) \in V_1 \times V_2\}$, where*
$\chi_{Pos}((l_1, l_2)) = \omega_{Pos}(l_1) \wedge \omega'_{Pos}(l_2)$,
$\chi_{Neu}((l_1, l_2)) = \omega_{Neu}(l_1) \wedge \omega'_{Neu}(l_2)$ and
$\chi_{Neg}((l_1, l_2)) = \omega_{Neg}(l_1) \vee \omega'_{Neg}(l_2), \forall (l_1, l_2) \in V_1 \times V_2$.

Dogra and Pal [11] introduced PFSR as an extension of IFSR.

Definition 2.9. [11] *Suppose, $\omega = (\omega_{Pos}, \omega_{Neu}, \omega_{Neg})$ represents a PFS over a ring $(V, +, \cdot)$, then ω is termed as PFSR of V if the below specified conditions are fulfilled.*
(i) $\omega_{Pos}(l - m) \geqslant \omega_{Pos}(l) \wedge \omega_{Pos}(m), \omega_{Neu}(l - m) \geqslant \omega_{Neu}(l) \wedge \omega_{Neu}(m)$ and $\omega_{Neg}(l - m) \leqslant \omega_{Neg}(l) \vee \omega_{Neg}(m)$,
(ii) $\omega_{Pos}(lm) \geqslant \omega_{Pos}(l) \wedge \omega_{Pos}(m), \omega_{Neu}(lm) \geqslant \omega_{Neu}(l) \wedge \omega_{Neu}(m)$ and $\omega_{Neg}(lm) \leqslant \omega_{Neg}(l) \vee \omega_{Neg}(m), \forall l, m \in V$.

3. Picture fuzzy characteristic (PFCh) and picture fuzzy divisor (PFD) of zero

Picture fuzzy ideal (PFI) is an important concept in algebra, like PFSR. This section introduces the concept of PFI of a ring. Also, this section explores the concepts of PFCh and PFD of zero in a ring w.r.t. some PFSR and investigates some important results on PFSR and PFI based on PFCh and PFD of zero.

At first, we define PFI over a ring.

Definition 3.1. *Suppose, $\omega = (\omega_{Pos}, \omega_{Neu}, \omega_{Neg})$ represents a PFS over a ring $(V, +, \cdot)$, then ω is termed as PFI of V if the below specified conditions are fulfilled.*
(i) $\omega_{Pos}(l-m) \geqslant \omega_{Pos}(l) \wedge \omega_{Pos}(m), \omega_{Neu}(l-m) \geqslant \omega_{Neu}(l) \wedge \omega_{Neu}(m)$ and $\omega_{Neg}(l-m) \leqslant \omega_{Neu}(l) \vee \omega_{Neu}(m)$,
(ii) $\omega_{Pos}(l \cdot m) \geqslant \omega_{Pos}(l) \vee \omega_{Pos}(m), \omega_{Neu}(l \cdot m) \geqslant \omega_{Neu}(l) \vee \omega_{Neu}(m)$ and $\omega_{Neg}(l \cdot m) \leqslant \omega_{Neg}(l) \wedge \omega_{Neg}(m)$, $\forall l, m \in V$.

Example 3.1. *Let us consider a PFSR $\omega = (\omega_{Pos}, \omega_{Neu}, \omega_{Neg})$ over the ring $V = (\mathbb{Z}, +, \cdot)$ as follows.*

for	$l \in 6\mathbb{Z}$	$l \in 2\mathbb{Z} - 6\mathbb{Z}$	*otherwise*
$\omega_{Pos}(l)$	0.48	0.3	0.1
$\omega_{Neu}(l)$	0.46	0.27	0.12
$\omega_{Neg}(l)$	0.04	0.41	0.55

By routine verification of all possible cases for $l, m \in V$, it can be shown that the conditions of Definition 3.1 are satisfied. Hence, ω is a PFI of V.

A relation between zero element and any other element in a ring (w.r.t. some PFI) is investigated by the following proposition.

Proposition 3.1. *Suppose, $\omega = (\omega_{Pos}, \omega_{Neu}, \omega_{Neg})$ represents a PFI over a ring $(V, +, \cdot)$, then $\omega_{Pos}(0) \geqslant \omega_{Pos}(l), \omega_{Neu}(0) \geqslant \omega_{Neu}(l)$ and $\omega_{Neg}(0) \leqslant \omega_{Neg}(l)$, $\forall l, m \in V$.*

Proof. Using PFI conditions we get
$\omega_{Pos}(0) = \omega_{Pos}(l-l) \geqslant \omega_{Pos}(l) \wedge \omega_{Pos}(l) = \omega_{Pos}(l)$,
$\omega_{Neu}(0) = \omega_{Neu}(l-l) \geqslant \omega_{Neu}(l) \wedge \omega_{Neu}(l) = \omega_{Neu}(l)$
and $\omega_{Neg}(0) = \omega_{Neg}(l-l) \leqslant \omega_{Neg}(l) \vee \omega_{Neg}(l) = \omega_{Neg}(l), \forall l \in V$.

Below we define PFCh of a ring w.r.t. some PFSR.

Definition 3.2. *Suppose, $\omega = (\omega_{Pos}, \omega_{Neu}, \omega_{Neg})$ represents a PFSR of a ring $(V, +, \cdot)$, then PFCh of V w.r.t. ω (which is denoted by $PFCh_\omega(V)$) is the least value of $r \in \mathbb{Z}^+$ s.t. $\omega_{Pos}(rl) = \omega_{Pos}(0)$, $\omega_{Neu}(rl) = \omega_{Neu}(0)$ and $\omega_{Neg}(rl) = \omega_{Neg}(0)$, $\forall l \in V$. If no such $r \in \mathbb{Z}^+$ is found, then $PFCh_\omega(V) = 0$.*

PFLD/PFRD of zero in a ring w.r.t. some PFSR is defined below.

Definition 3.3. *Assume that $\omega = (\omega_{Pos}, \omega_{Neu}, \omega_{Neg})$ represents a PFSR of a ring $(V, +, \cdot)$. If $l, m \in V$ accompanied by*
(i) $\omega_{Pos}(l) \neq \omega_{Pos}(0), \omega_{Neu}(l) \neq \omega_{Neu}(0), \omega_{Neg}(l) \neq \omega_{Neg}(0)$
(ii) $\omega_{Pos}(m) \neq \omega_{Pos}(0), \omega_{Neu}(m) \neq \omega_{Neu}(0), \omega_{Neg}(m) \neq \omega_{Neg}(0)$
imply that $\omega_{Pos}(lm) = \omega_{Pos}(0)$, $\omega_{Neu}(lm) = \omega_{Neu}(0)$, $\omega_{Neg}(lm) = \omega_{Neg}(0)$, then l and m are respectively called PFLD of zero and PFRD of zero.

Now, the question arises : when an element in a ring is not a PFLD/PFRD of zero? The answer of this question lies in the following two definitions.

Definition 3.4. *Assume that* $\omega = (\omega_{Pos}, \omega_{Neu}, \omega_{Neg})$ *represents a PFSR of a ring* $(V, +, \cdot)$. *If* $l, m \in V$ *with*
(i) $\omega_{Pos}(l) \neq \omega_{Pos}(0), \omega_{Neu}(l) \neq \omega_{Neu}(0), \omega_{Neg}(l) \neq \omega_{Neg}(0)$
(ii) $\omega_{Pos}(lm) = \omega_{Pos}(0)$, $\omega_{Neu}(lm) = \omega_{Neu}(0)$, $\omega_{Neg}(lm) = \omega_{Neg}(0)$
imply that $\omega_{Pos}(m) = \omega_{Pos}(0), \omega_{Neu}(m) = \omega_{Neu}(0), \omega_{Neg}(m) = \omega_{Neg}(0)$, *then* l *is not a PFLD of zero.*

Definition 3.5. *Assume that* $\omega = (\omega_{Pos}, \omega_{Neu}, \omega_{Neg})$ *represents a PFSR of a ring* $(V, +, \cdot)$. *If* $l, m \in V$ *with*
(i) $\omega_{Pos}(l) \neq \omega_{Pos}(0), \omega_{Neu}(l) \neq \omega_{Neu}(0), \omega_{Neg}(l) \neq \omega_{Neg}(0)$
(ii) $\omega_{Pos}(ml) = \omega_{Pos}(0)$, $\omega_{Neu}(ml) = \omega_{Neu}(0)$, $\omega_{Neg}(ml) = \omega_{Neg}(0)$
imply that $\omega_{Pos}(m) = \omega_{Pos}(0), \omega_{Neu}(m) = \omega_{Neu}(0), \omega_{Neg}(m) = \omega_{Neg}(0)$, *then* l *is not a PFRD of zero.*

If $\{l \in V : \omega_{Pos}(l) = \omega_{Pos}(0), \omega_{Neu}(l) = \omega_{Neu}(0), \omega_{Neg}(l) = \omega_{Neg}(0)\} = \{0\}$, then $PFCh_\omega(V)$ reduces to ordinary characteristic of V. Thus, the concept of PFCh of a ring is the generalized concept of ordinary characteristic of a ring or in other words, ordinary characteristic of a ring is a special case of PFCh of a ring. Under the same prescribed condition, PFD of zero in a ring reduces to the ordinary divisor of zero in a ring. So, PFD of zero is the extended concept of ordinary divisor of zero or in other words, it can be said that ordinary divisor of zero is a special case of PFD of zero.

An important proposition related to PFCh of ring is provided below w.r.t. some PFI.

Proposition 3.2. *Assume that* $\omega = (\omega_{Pos}, \omega_{Neu}, \omega_{Neg})$ *represents a PFI of a ring* V *with unity* e. *If for least value of* $r \in \mathbb{Z}^+$, $\omega_{Pos}(re) = \omega_{Pos}(0)$, $\omega_{Neu}(re) = \omega_{Neu}(0)$ *and* $\omega_{Neg}(re) = \omega_{Neg}(0)$, *then* $PFCh_\omega(V) = r$. *If no* $r \in \mathbb{Z}^+$ *is found, then* $PFCh_\omega(V) = 0$.

Proof. We prove this proposition in two cases.

Case 1: Given that ω is a PFI of V. Let for least value of $r \in \mathbb{Z}^+$, $\omega_{Pos}(re) = \omega_{Pos}(0)$, $\omega_{Neu}(re) = \omega_{Neu}(0)$ and $\omega_{Neg}(re) = \omega_{Neg}(0)$. Now, using PFI properties of ω, one can write
$\omega_{Pos}(lr) = \omega_{Pos}(l + l + ... + l \text{ } (r \text{ times}))$
$= \omega_{Pos}(l(e + e + ... + e) \text{ } (r \text{ times})))$
$= \omega_{Pos}(l(re)) \geqslant \omega_{Pos}(l) \vee \omega_{Pos}(re) = \omega_{Pos}(l) \vee \omega_{Pos}(0)$
$= \omega_{Pos}(0)$ (using Proposition 3.1),
$\omega_{Neu}(lr) = \omega_{Neu}(l + l + ... + l \text{ } (r \text{ times}))$
$= \omega_{Neu}(l(e + e + ... + e) \text{ } (r \text{ times})))$
$= \omega_{Neu}(l(re)) \geqslant \omega_{Neu}(l) \vee \omega_{Neu}(re) = \omega_{Neu}(l) \vee \omega_{Neu}(0)$
$= \omega_{Neu}(0)$ (using Proposition 3.1)
and $\omega_{Neg}(lr) = \omega_{Neg}(l + l + ... + l \text{ } (r \text{ times}))$
$= \omega_{Neg}(l(e + e + ... + e) \text{ } (r \text{ times})))$
$= \omega_{Neg}(l(re)) \leqslant \omega_{Neg}(l) \wedge \omega_{Neg}(re)$
$= \omega_{Neg}(l) \wedge \omega_{Neg}(0) = \omega_{Neg}(0)$ (using Proposition 3.1), $\forall l \in V$.
Also, in case of PFI, we are aware of the result that
$\omega_{Pos}(0) \geqslant \omega_{Pos}(lr)$, $\omega_{Neu}(0) \geqslant \omega_{Neu}(lr)$ and $\omega_{Neg}(0) \leqslant \omega_{Neg}(lr)$, $\forall l \in V$ (using Proposition 3.1).
As a result, $\omega_{Pos}(lr) = \omega_{Pos}(0)$, $\omega_{Neu}(lr) = \omega_{Neu}(0)$ and $\omega_{Neg}(lr) = \omega_{Neg}(0)$, $\forall l \in V$.
If possible, let for a positive integer $s < r$, $\omega_{Pos}(sl) = \omega_{Pos}(0)$, $\omega_{Neu}(sl) = \omega_{Neu}(0)$ and $\omega_{Neg}(sl) = \omega_{Neg}(0)$, $\forall l \in V$. Then taking $l = e$ in particular, one can obtain that

$\omega_{Pos}(se) = \omega_{Pos}(0)$, $\omega_{Neu}(se) = \omega_{Neu}(0)$ and $\omega_{Neg}(se) = \omega_{Neg}(0)$. This contradicts the given condition. Hence, for least value of $r \in \mathbb{Z}^+$, $\omega_{Pos}(lr) = \omega_{Pos}(0)$, $\omega_{Neu}(lr) = \omega_{Neu}(0)$ and $\omega_{Neg}(lr) = \omega_{Neg}(0)$, $\forall l \in V$. Hence, $PFCh_\omega(V) = r$.

Case 2: The PFCh measures the smallest positive integer r (if it exists) for which the conditions $\omega_{Pos}(re) = \omega_{Pos}(0)$, $\omega_{Neu}(re) = \omega_{Neu}(0)$ and $\omega_{Neg}(re) = \omega_{Neg}(0)$ hold. If no such r exists, the natural conclusion is to assign the characteristic a value of 0, indicating the absence of any r that satisfies the given membership criteria. In this case, $PFCh_\omega(V) = r$.

PFCh of a non trivial ring with unity is either zero or a prime number when the ring has no PFD of zero. This important property is investigated here through the following proposition.

Proposition 3.3. *Suppose, $\omega = (\omega_{Pos}, \omega_{Neu}, \omega_{Neg})$ represents a PFSR over a non-trivial ring V with unity e s.t. V has no PFD of zero, then $PFCh_\omega(V)$ is either zero or a prime number.*

Proof. Suppose, $PFCh_\omega(V) = r$, then for least value of $r \in \mathbb{Z}^+$;
$\omega_{Pos}(rl) = \omega_{Pos}(0)$, $\omega_{Neu}(rl) = \omega_{Neu}(0)$ and $\omega_{Neg}(rl) = \omega_{Neg}(0)$, $\forall l \in V$.
Taking $l = e$ in particular, one can obtain that for least value of $r \in \mathbb{Z}^+$;
$\omega_{Pos}(re) = \omega_{Pos}(0)$, $\omega_{Neu}(re) = \omega_{Neu}(0)$ and $\omega_{Neg}(re) = \omega_{Neg}(0)$.
Here $r \neq 1$, as V is non-trivial.
If possible, let r be a composite number. Then, $r = r_1 r_2$, where $r_1, r_2 \in (1, r)$.
Now, one can write
$\omega_{Pos}(0) = \omega_{Pos}(re) = \omega_{Pos}(r_1 r_2 e) = \omega_{Pos}((r_1 e)(r_2 e))$, $\omega_{Neu}(0) = \omega_{Neu}(re) = \omega_{Neu}(r_1 r_2 e)$ $= \omega_{Neu}((r_1 e)(r_2 e))$ and $\omega_{Neg}(0) = \omega_{Neg}(re) = \omega_{Neg}(r_1 r_2 e) = \omega_{Neg}((r_1 e)(r_2 e))$. Given that V has no PFD of zero. Then, either $[\omega_{Pos}(r_1 e) = \omega_{Pos}(0)$, $\omega_{Neu}(r_1 e) = \omega_{Neu}(0)$, $\omega_{Neg}(r_1 e) = \omega_{Neg}(0)]$ or $[\omega_{Pos}(r_2 e) = \omega_{Pos}(0), \omega_{Neu}(r_2 e) = \omega_{Neu}(0), \omega_{Neg}(r_2 e) = \omega_{Neg}(0)]$.
So, either $PFCh_\omega(V) \leqslant r_1$ or $PFCh_\omega(V) \leqslant r_2$.
In each of the cases, the fact $PFCh_\omega(V) = r$ is contradicted. Hence, r is not composite. As a result, r must be a prime.

Unit element in a PFI under some certain conditions is not a PFD of zero. This important property is investigated here in terms of the following proposition.

Proposition 3.4. *Let l be a unit in a ring V with unity e and $\omega = (\omega_{Pos}, \omega_{Neu}, \omega_{Neg})$ be a PFI of V. If $\omega_{Pos}(l) \neq \omega_{Pos}(0)$, $\omega_{Neu}(l) \neq \omega_{Neu}(0)$ and $\omega_{Neg}(l) \neq \omega_{Neg}(0)$, then l is not a PFD of zero in V.*

Proof. Given that l is a unit in V. So, l^{-1} exists.
Assume that $\omega_{Pos}(lu) = \omega_{Pos}(0)$, $\omega_{Neu}(lu) = \omega_{Neu}(0)$ and $\omega_{Neg}(lu) = \omega_{Neg}(0)$.
Also, given that $\omega_{Pos}(l) \neq \omega_{Pos}(0)$, $\omega_{Neu}(l) \neq \omega_{Neu}(0)$ and $\omega_{Neg}(l) \neq \omega_{Neg}(0)$.
Using PFI properties of ω, one can write that
$\omega_{Pos}(u) = \omega_{Pos}(l^{-1}lu) \geqslant \omega_{Pos}(l^{-1}) \vee \omega_{Pos}(lu) = \omega_{Pos}(l^{-1}) \vee \omega_{Pos}(0) = \omega_{Pos}(0)$,
$\omega_{Neu}(u) = \omega_{Neu}(l^{-1}lu) \geqslant \omega_{Neu}(l^{-1}) \vee \omega_{Neu}(lu) = \omega_{Neu}(l^{-1}) \vee \omega_{Neu}(0) = \omega_{Neu}(0)$
and $\omega_{Neg}(u) = \omega_{Neg}(l^{-1}lu) \leqslant \omega_{Neg}(l^{-1}) \wedge \omega_{Neg}(lu) = \omega_{Neg}(l^{-1}) \wedge \omega_{Neg}(0) = \omega_{Neg}(0)$.
Also, we are aware of the result that
$\omega_{Pos}(0) \geqslant \omega_{Pos}(u)$, $\omega_{Neu}(0) \geqslant \omega_{Neu}(u)$ and $\omega_{Neg}(0) \leqslant \omega_{Neg}(u)$ (using Proposition 3.1).
In this way, $\omega_{Pos}(u) = \omega_{Pos}(0)$, $\omega_{Neu}(u) = \omega_{Neu}(0)$ and $\omega_{Neg}(u) = \omega_{Neg}(0)$. So l is not a PFLD of zero.
Now, let us suppose that $\omega_{Pos}(ul) = \omega_{Pos}(0)$, $\omega_{Neu}(ul) = \omega_{Neu}(0)$ and $\omega_{Neg}(ul) = \omega_{Neg}(0)$.

Also, given that $\omega_{Pos}(l) \neq \omega_{Pos}(0)$, $\omega_{Neu}(l) \neq \omega_{Neu}(0)$ and $\omega_{Neg}(l) \neq \omega_{Neg}(0)$.
Using PFI properties of ω, one can write
$\omega_{Pos}(u) = \omega_{Pos}(ull^{-1}) \geqslant \omega_{Pos}(ul) \vee \omega_{Pos}(l^{-1}) = \omega_{Pos}(0) \vee \omega_{Pos}(l^{-1}) = \omega_{Pos}(0)$,
$\omega_{Neu}(u) = \omega_{Neu}(ull^{-1}) \geqslant \omega_{Neu}(ul) \vee \omega_{Neu}(l^{-1}) = \omega_{Neu}(0) \vee \omega_{Neu}(l^{-1}) = \omega_{Neu}(0)$
and $\omega_{Neg}(u) = \omega_{Neg}(ull^{-1}) \leqslant \omega_{Neg}(ul) \wedge \omega_{Neg}(l^{-1}) = \omega_{Neg}(0) \wedge \omega_{Neg}(l^{-1}) = \omega_{Neg}(0)$.
Also, we are aware of the result
$\omega_{Pos}(0) \geqslant \omega_{Pos}(u)$, $\omega_{Neu}(0) \geqslant \omega_{Neu}(u)$ and $\omega_{Neg}(0) \leqslant \omega_{Neg}(u)$ (using Proposition 3.1).
In this way, $\omega_{Pos}(u) = \omega_{Pos}(0)$, $\omega_{Neu}(u) = \omega_{Neu}(0)$ and $\omega_{Neg}(u) = \omega_{Neg}(0)$. So l is not a PFRD of zero. Hence, l is not a PFD of zero.

An important formula for calculating PFCh of CP of PFSRs w.r.t. CP of two rings is developed here and this formula is given in terms of the following proposition.

Proposition 3.5. *Assume that $\omega = (\omega_{Pos}, \omega_{Neu}, \omega_{Neg})$ and $\omega' = (\omega'_{Pos}, \omega'_{Neu}, \omega'_{Neg})$ represents two PFIs over the rings V_1 and V_2 respectively. If $PFCh_{\omega}(V_1) = r_1$ and $PFCh_{\omega'}(V_2) = r_2$, then $PFCh_{\omega\times\omega'}(V_1 \times V_2) = lcm\{r_1, r_2\}$.*

Proof. Since $PFCh_{\omega}(V_1) = r_1$ and $PFCh_{\omega'}(V_2) = r_2$, therefore for least values of r_1, $r_2 \in \mathbb{Z}^+$; $\omega_{Pos}(r_1l_1)=\omega_{Pos}(0)$, $\omega_{Neu}(r_1l_1) = \omega_{Neu}(0)$, $\omega_{Neg}(r_1l_1) = \omega_{Neg}(0)$, $\forall l_1 \in V_1$ and $\omega'_{Pos}(r_2l_2) = \omega'_{Pos}(0)$, $\omega'_{Neu}(r_2l_2) = \omega'_{Neu}(0)$, $\omega'_{Neg}(r_2l_2) = \omega'_{Neg}(0)$, $\forall l_2 \in V_2$. Say $\omega \times \omega'$ $= \chi = (\chi_{Pos}, \chi_{Neu}, \chi_{Neg})$. Now, for another $r_3 \in \mathbb{Z}^+$, we have
$\chi_{Pos}(r_3(l_1, l_2)) = \omega_{Pos}(r_3l_1) \wedge \omega'_{Pos}(r_3l_2)$
$= \omega_{Pos}(\frac{r_3}{r_1}r_1l_1) \wedge \omega'_{Pos}(\frac{r_3}{r_2}r_2l_2)$ [when each of r_1 and r_2 divides r_3]
$\geqslant [\omega_{Pos}(\frac{r_3}{r_1}) \vee \omega_{Pos}(r_1l_1)] \wedge [\omega'_{Pos}(\frac{r_3}{r_2}) \vee \omega'_{Pos}(r_2l_2)]$ [using PFI properties of ω and ω']
$= [\omega_{Pos}(\frac{r_3}{r_1}) \vee \omega_{Pos}(0)] \wedge [\omega'_{Pos}(\frac{r_3}{r_2}) \vee \omega'_{Pos}(0)]$
$= \omega_{Pos}(0) \wedge \omega'_{Pos}(0)$
$= \chi_{Pos}((0, 0))$ (using Proposition 3.1)

$\chi_{Neu}(r_3(l_1, l_2)) = \omega_{Neu}(r_3l_1) \wedge \omega'_{Neu}(r_3l_2)$
$= \omega_{Neu}(\frac{r_3}{r_1}r_1l_1) \wedge \omega'_{Neu}(\frac{r_3}{r_2}r_2l_2)$ [when each of r_1 and r_2 divides r_3]
$\geqslant [\omega_{Neu}(\frac{r_3}{r_1}) \vee \omega_{Neu}(r_1l_1)] \wedge [\omega'_{Pos}(\frac{r_3}{r_2}) \vee \omega'_{Pos}(r_2l_2)]$ [using PFI properties of ω and ω']
$= [\omega_{Neu}(\frac{r_3}{r_1}) \vee \omega_{Neu}(0)] \wedge [\omega'_{Neu}(\frac{r_3}{r_2}) \vee \omega'_{Neu}(0)]$
$= \omega_{Neu}(0) \wedge \omega'_{Neu}(0)$
$= \chi_{Neu}((0, 0))$ (using Proposition 3.1)

and $\chi_{Neg}(r_3(l_1, l_2)) = \omega_{Neg}(r_3l_1) \vee \omega'_{Neg}(r_3l_2)$
$= \omega_{Neg}(\frac{r_3}{r_1}r_1l_1) \vee \omega'_{Neg}(\frac{r_3}{r_2}r_2l_2)$ [when each of r_1 and r_2 divides r_3]
$\leqslant [\omega_{Neg}(\frac{r_3}{r_1}) \wedge \omega_{Neg}(r_1l_1)] \vee [\omega'_{Neg}(\frac{r_3}{r_2}) \wedge \omega'_{Neg}(r_2l_2)]$ [using PFI properties of ω and ω']
$= [\omega_{Neg}(\frac{r_3}{r_1}) \wedge \omega_{Neg}(0)] \vee [\omega'_{Neg}(\frac{r_3}{r_2}) \wedge \omega'_{Neg}(0)]$
$= \omega_{Neg}(0) \vee \omega'_{Neg}(0)$
$= \chi_{Neg}((0, 0))$ (using Proposition 3.1)
Since each of r_1 and r_2 divides r_3, therefore r_3 is the multiple of both r_1 and r_2. Since it requires to find PFCh of $V_1 \times V_2$ w.r.t. $\chi = \omega \times \omega'$, therefore r_3 should be taken as the lcm of r_1 and r_2. Hence, $r_3 = lcm\{r_1, r_2\}$ acts as the least positive integer for which $\chi_{Pos}(r_3(l_1, l_2)) = \chi_{Pos}((0, 0)), \chi_{Neu}(r_3(l_1, l_2)) = \chi_{Neu}((0, 0))$ and $\chi_{Neg}(r_3(l_1, l_2))$ $= \chi_{Neg}((0, 0))$. In this way, $PFCh_{\omega\times\omega'}(V_1 \times V_2) = r_3 = lcm\{r_1, r_2\}$.

4. Applications

Prior to discussing the applications, it should be noted that customer feedback and sentiment data often incorporate both positive, negative, and neutral dimensions together.

The picture fuzzy approach successfully reflects the multi-faceted nature of this kind of uncertainty. From this point of view, PFCh allows examining the stability and consistency of customer feedback, and PFD of zero reveals hidden structural characteristics in the area of customer sentiment analysis.

4.1. **Application of PFCh in Customer Feedback Analysis.** Suppose a company engaged in providing services collects customer feedback on a monthly basis. In this case, assume that time periods form a ring V and customer sentiments (positive, neutral, negative) can be expressed as a PFSR ω.

In this example, the meaning of PFCh is the number of iterations required for the sentiment to reach stability.

(i) If $PFCh_{\omega}(V) = r$, then the sentiment reaches stability after r months i.e. the service improvements are effective.

(ii) If $PFCh_{\omega}(V) = 0$, the sentiment never reaches stability i.e. the changes are not effective.

Hence, PFCh can assist the company in determining whether its service improvements start producing consistent results or not.

4.2. **Application of PFD of zero in Customer Sentiment Analysis.** In this case, the PFD of zero will be applied to the PFS instead of the PFSR. The reason behind this is to have more flexibility in the assignment of PFVs.

Let us consider an SU denoted by V, containing all possible services factors of a business including product quality and customer service. Assume a PFS called ω is defined on V in which customers' sentiments towards the service factors are defined through their positive, neutral, and negative memberships.

Consider $l, m \in V$ where l denotes product quality and m denotes customer service.

Assume the corresponding PFVs are:

$$(\omega_{Pos}(l), \omega_{Neu}(l), \omega_{Neg}(l)) = (0.8, 0.1, 0.1),$$
$$(\omega_{Pos}(m), \omega_{Neu}(m), \omega_{Neg}(m)) = (0.75, 0.2, 0.05).$$

Assume the base sentiment is given by:

$$(\omega_{Pos}(0), \omega_{Neu}(0), \omega_{Neg}(0)) = (0.6, 0.02, 0.143).$$

Using the operational rules,

$$\omega_{Pos}(lm) = \omega_{Pos}(l) \cdot \omega_{Pos}(m),$$
$$\omega_{Neu}(lm) = \omega_{Neu}(l) \cdot \omega_{Neu}(m),$$
$$\omega_{Neg}(lm) = \omega_{Neg}(l) + \omega_{Neg}(m) - \omega_{Neg}(l) \cdot \omega_{Neg}(m),$$

we have

$$(\omega_{Pos}(lm), \omega_{Neu}(lm), \omega_{Neg}(lm)) = (0.6, 0.02, 0.145) \approx (0.6, 0.02, 0.143).$$

This value is approximately the same as the baseline PFV.

Even though the sentiments of l and m are not at par with the baseline sentiments, their cumulative effect results in the achievement of the baseline level of sentiment. Therefore, we can consider these sentiments as *critical sentiments*.

If the cumulative sentiment is not close to the baseline sentiment, then there is a need to improve these sentiments, that is, improve product quality and customer service.

Limitation of the Proposed Method: The current study deals with the application of the abstract concept in the PFS environment. The primary limitation of the proposed method is that it can be used only when the data shows the picture fuzziness, i.e., there should be positive, neutral, and negative memberships.

5. Conclusion and future scopes

In this paper, we have defined PFCh and PFD of zero in a ring, w.r.t. some PFSR. We have studied their important properties and clearly explained the conditions under which PFCh and PFD of zero become the ordinary characteristic and divisor of zero. We have shown that, under some certain conditions, the unit element of a ring is not a PFD of zero. We also calculated the PFCh of the CP of two rings, using the CP of two PFSRs. Finally, we have demonstrated real-life applications of PFCh and PFD of zero, highlighting their practical use in solving problems.

The study of PFSs and their applications in algebraic structures offers many opportunities for future research. One possible direction is the development of complex picture fuzzy sets, which add extra features like phase information to traditional PFS. This makes them useful for applications in fields like signal processing, control systems, and quantum mechanics, where patterns often repeat or cycle. Another area to explore is bipolar fuzzy sets, which help handle situations with both positive and negative opinions, such as in decision-making, sentiment analysis, or solving problems with conflicting criteria. Building on this, complex bipolar fuzzy sets can represent both positive-negative evaluations and complex values at the same time. These could be applied to social networks, where relationships may have both good and bad aspects, or in areas where patterns change over time. Similarly, spherical fuzzy sets and T-spherical fuzzy sets provide a better way to handle multi-dimensional uncertainty. These sets are particularly useful in robotics, medical diagnosis, or any field where decisions depend on several uncertain factors at once.

By combining spherical fuzzy concepts with complex numbers, complex spherical fuzzy sets can handle even more complicated data. These are helpful in advanced technologies like machine learning, control systems, and multi-agent robotics, where multiple variables interact.

In addition to theoretical work, these advanced fuzzy sets have many real-life applications. They can be used in supply chain management, financial analysis, risk assessment, and environmental studies, where uncertainty plays a big role. They can also be applied to big data, optimization problems, and intelligent systems to solve practical challenges.

Future research can also explore how these new fuzzy sets fit into broader mathematical structures like semirings and lattices. Finally, developing easy-to-use software tools to work with these fuzzy systems will help researchers and industries apply them effectively. In summary, the possibilities for future work in this area are vast, covering both theoretical advancements and practical solutions to real-world problems.

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Madhumangal Pal for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.13, N.2.