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ON A SUBCLASS OF TILTED STARLIKE FUNCTIONS ASSOCIATED WITH EPICYCLOID DOMAIN

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Abstract. This paper investigates a subclass of analytic and univalent functions defined on the open unit disk. A new subclass of tilted starlike functions associated with an epicycloid-shaped domain is defined. These functions are characterized by a subordination condition involving a tilted factor. The study focuses on the analytic characterization of the class, particularly the estimation of initial coefficients. Furthermore, upper bounds are obtained for the Fekete–Szegö functional and the second Hankel determinant. The results not only extend several known findings in the literature but also demonstrate sharpness, since they reduce to known subclasses in special cases, offering new perspectives on the behavior of tilted starlike functions associated with epicycloidal image domains.

Keywords: univalent function, tilted starlike functions, coefficient estimates, Hankel determinant, Fekete–Szegö inequality.

AMS Subject Classification: 30C45, 30C80.

1. Introduction

Consider the family $H(E)$ consisting of analytical functions within the open unit disk E, where $E = \{z \in \mathbb{C} : |z| < 1\}$. Let A consist of functions $f \in H(E)$, normalize by conditions $f(0) = 0$ and $f^{'}(0) = 1$ and having the following series form

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$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in E. \tag{1}$$

We define S to represent the class of functions $f \in A$ that are univalent in E. Starlike and convex functions are well-known examples within this class [7].

Let P represent the set of Carathéodory functions comprising analytic functions p which hold under the condition $\mathrm{Re}p(z) > 0$ and analytic in E. Each function $p \in P$ takes the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n. \tag{2}$$

According to Miller and Mocanu (see [19]), given two analytic functions, $f, g \in H(E)$, f is considered subordinate to g (denoted $f \prec g$) given the existence of an analytic Schwarz function, $\omega \in H(E)$, with $\omega(0) = 0$ and $|\omega(z)| < 1$ for which $f(z) = g(\omega(z))$. In particular, $f \prec g$ implies that $f(0) = g(0)$ and $f(E) \subset g(E)$.

Ma and Minda [18] employed the idea of subordination to devise an effective method for defining a subclass of starlike functions, initially presented the general expression of this class as follows:

$$S^*(\varphi) := \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}. \tag{3}$$

Assume that φ is analytic within E and its real part is positive and satisfies $\varphi(0) = 1$ and $\varphi'(0) > 0$. Such functions map the unit disk E into domains that are starlike with respect to 1 and exhibit symmetry across the real axis.

In recent years, several subclasses have been studied as particular cases of the class $S^*(\varphi)$. In this context, some authors have replaced φ with functions such as Fibonacci numbers, Bell numbers, and the modified sigmoid function, as well as with mappings related to shell-like curves and conic domains (see [16]). For instance, Sharma et al. [28] introduced the class $S^*_C(z) := S^*\left(1 + \frac{4}{3}z + \frac{2}{3}z^2\right)$ associated with the cardioid domain. Similarly, Wani and Swaminathan [37] studied the Ma–Minda subclass related to the nephroid domain, defined by $\varphi_{Ne}(z) = 1+z-\frac{1}{3}z^3$. More recently, Cho et al. [4] investigated starlike functions associated with the sine function, given by $S^*_S(z) := S^*(1 + \sin z)$.

Several researchers have investigated starlike functions associated with epicycloids. An epicycloid is a plane curve generated by a point on the circumference of a circle of radius b, rolling along the circumference of a stationary circle with radius a [12]. In 2018, subclass of starlike functions defined on a domain with three cusps was introduced by Gandhi [11], given as follows:

$$S^*_{3,\mathrm{L}} := \left\{ f \in S : \frac{zf'(z)}{f(z)} \prec 1 + \frac{4}{5}z + \frac{1}{5}z^4 \right\}, z \in E. \tag{4}$$

Later on, Gandhi et al. [12] introduced a generalized expression involving an $n-1$ cusps of epicycloid in the form

$$\varphi_{n-1,\mathrm{L}} = 1 + \frac{n}{n+1}z + \frac{1}{n+1}z^n, z \in E.$$

This function satisfies all the criteria provided by Ma and Minda [18] regarding the properties of the unified class of starlike functions. Notably, for $n \geq 2$, the unit disk E is mapped onto a region enclosed by an epicycloid possessing $n-1$ cusps. Utilizing the function $\varphi_{n-1,\mathrm{L}}$, Gandhi et al. [12] presented a subclass of starlike functions, written in the following:

$$S^*_{n-1,\mathrm{L}} := \left\{ f \in S : \frac{zf'(z)}{f(z)} \prec 1 + \frac{n}{n+1}z + \frac{1}{n+1}z^n \right\}, z \in E. \tag{5}$$

Subsequently, Shi et al. [31] studied the class $S^*_{n-1,\mathrm{L}}$ with $n = 5$, which led to the subclass $S^*_{4,\mathrm{L}}$ associated with a four-leaf shaped domain, represented as follows:

$$S^*_{4,\mathrm{L}} := \left\{ f \in S : \frac{zf'(z)}{f(z)} \prec 1 + \frac{5}{6}z + \frac{1}{6}z^5 \right\}, z \in E. \tag{6}$$

In addition, significant attention has been directed toward the generalization of existing subclasses to obtain broader perspectives within univalent function theory. For instance, Yunus et al. [38] extended the class of starlike functions associated with the cardioid originally introduced by Sharma et al. [28], denoted by $S^*_{GC}(k) := S^*(1 + 2kz + kz^2)$. Similarly, Kargar and Trojnar-Spelina [15] proposed a generalization of the classical Koebe function by introducing a new subclass of starlike functions satisfying $\frac{zf'(z)}{f(z)} - 1 \prec k_{p,q}(z)$. More recently, Tang et al. [34] further expanded the three-leaf-shaped domains initially introduced by Gandhi [11], defining the class $S^*_{3\mathrm{L,s}} := \left\{ g \in S : \frac{2zg'(z)}{g(z)-g(-z)} \prec 1 + \frac{4}{5}z + \frac{1}{5}z^4 \right\}$.

Separately, another direction in generalizing starlike functions involves introducing rotational parameters, leading to the concept of tilted functions. The idea of incorporating a rotation factor into the geometric behavior of analytic functions has been studied by several authors. For instance, Mohamad [20] examined a class of analytic functions denoted by $G(\alpha, \delta)$, characterized by the condition

$$p(z) = \frac{e^{i\alpha} f'(z) - i \sin\alpha - \delta}{t_{\alpha\delta}},$$

where $\mathrm{Re} e^{i\alpha} f'(z) > \delta$, $|\alpha| < \pi$, $0 \le \delta < 1$, and $t_{\alpha\delta} = \cos\alpha - \delta > 0$.

Meanwhile, inspired by the research conducted by El-Ashwah and Thomas [9], Halim [13], Dahhar and Janteng [6], a new subclass of tilted starlike functions with respect to conjugate points, $S^*_c(\alpha, \delta, A, B)$ was introduced by Wahid et al. [35] of order and satisfying the condition

$$\left\{ e^{i\alpha} \frac{zf'(z)}{f(z)} - \delta - i \sin\alpha \right\} \frac{1}{t_{\alpha\delta}} \prec \frac{1 + Az}{1 + Bz}, \; -1 \le B < A \le 1.$$

In the context of univalent functions, a significant area of study involves the Hankel determinant. In [22] and [23], Pommerenke presented the concept of the Hankel determinant of order q, $H_q(n)$ given that $n \ge 1$ and $q \ge 1$ in the form

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{7}$$

Various orders are obtained for each value of q and n. From (7), $H_2(1) = a_3 - a_2^2$ can be obtained with $a_1 = 1$. The sharp inequality $|H_2(1)| = \left|a_3 - a_2^2\right| \le 1$ is valid for functions that are analytic and univalent within E (see [7]). Then, Fekete and Szegö [10] further generalized the estimate $|H_2(1)| = \left|a_3 - \mu a_2^2\right|$ where $\mu \in \mathbb{C}$ and $f \in S$. The well-known problem solved by Fekete-Szegö concerns finding the maximum value of a specific coefficient functional. Next, when both q and n are set to 2, the second Hankel determinant is obtained, where $|H_2(2)| = |a_2a_4 - a_3^2|$. Efforts to establish upper bounds for this determinant have been ongoing since 2000. For instance, Janteng et al. [14] calculated the sharp bound of $H_2(2)$ in the context of starlike and convex functions.

Earlier, Shi et al. [29] obtained bounds for $|H_3(1)|$ in the cardioid-associated subclasses S^*_{car}, C_{car}, and R_{car}. Shi et al. [30] established sharp upper bounds for the second Hankel determinant and derived the sharp Fekete–Szegö inequality for the class $S^*_{3,\text{L}}$ (see equation (4)). However, they did not address the sharp bounds of the third Hankel determinant. To fill this gap, Arif et al. [1] employed a multivariable optimization technique over a cuboidal domain, which yielded sharp bounds for the third Hankel determinant and thus extended the existing results. Shi et al. [31] established the Fekete–Szegö inequality and upper bounds for the second and third Hankel determinants for class $S^*_{n-1,\text{L}}$, including sharp boundaries of $|H_3(1)|$ in class $S^*_{4,\text{L}}$. Shi et al. [32] studied the bounded turning functions $\mathcal{BT}_{\exp}$ linked to the exponential function, while Barukab et al. [3] considered the subclass $\mathcal{BT}_s$ associated with the petal-shaped domain; in both cases, sharp results were established for the Fekete–Szegö inequality and Hankel determinants. Additional contributions on Hankel determinants can be found in [5, 21, 25, 26, 27, 33].

Motivated by the work of Wahid et al. [35] and Shi et al. [31], this study aims to generalize the class $S^*_{n-1,\text{L}}$ introduced by Gandhi et al. [12], by incorporating a tilted factor to further extend this class functions. Thus, we define the following class:

Definition 1.1. *The function $f \in S^*_{T,n-1}(\alpha)$ if and only if*

$$\left(e^{i\alpha}\frac{zf'(z)}{f(z)} - i\sin\alpha\right)\frac{1}{\cos\alpha} \prec 1 + \frac{n}{n+1}z + \frac{1}{n+1}z^n, z \in E,$$

where $n \geq 2$ and $|\alpha| < \frac{\pi}{2}$. From the concept of subordination, $f \in S^*_{T,n-1}(\alpha)$ holds if and only if

$$\left(e^{i\alpha}\frac{zf'(z)}{f(z)} - i\sin\alpha\right)\frac{1}{\cos\alpha} = 1 + \frac{n}{n+1}\,\omega(z) + \frac{1}{n+1}\,[\omega(z)]^n = p(z), \quad \omega \in H(E). \tag{8}$$

If $\alpha = 0$, the class $S^*_{T,n-1}(\alpha)$ reduces to $S^*_{n-1,\text{L}}$, as defined by Gandhi et al. [12] (see (5)). Furthermore, when $\alpha = 0$ and $n = 5$, the class coincides with that introduced by Shi et al. [31], denoted by $S^*_{4,\text{L}}$ (see (6)).

In this study, we aim to establish upper bounds for the Fekete–Szegö functional and the second Hankel determinant within the class $S^*_{T,n-1}(\alpha)$.

2. Preliminary Results

The main results are supported by the lemmas presented in this section.

Lemma 2.1. ([24], [17]) *For a function $p \in P$ expressed as in (2), there exists x, δ satisfying $|x| \leq 1$, $|\delta| \leq 1$ with $c_1 \in [0, 2]$, such that*

$$2c_2 = c_1^2 + \left(4 - c_1^2\right)x, \tag{9}$$

$$4c_3 = c_1^3 + 2c_1x(4 - c_1^2) - x^2c_1(4 - c_1^2) + 2(1 - |x|^2)(4 - c_1^2)\delta. \tag{10}$$

Lemma 2.2. ([8]) *Let $p \in P$ be a function of the form (2) and $\mu \in \mathbb{C}$. Then*

$$|c_n - \mu c_k c_{n-k}| \leq 2\max\{1, |2\mu - 1|\}, 1 \leq k \leq n - 1. \tag{11}$$

Lemma 2.3. ([7]) *For a function $p \in P$ given by (2), the sharp inequality $|c_n| \leq 2$ holds for each $n \geq 1$.*

3. Main Results

To commence, we derive the Fekete–Szegö inequality corresponding to the class $S^*_{T,n-1}(\alpha)$.

Theorem 3.1. *If $f \in S^*_{T,n-1}(\alpha)$ be given by the form (1). Then for any $\lambda \in \mathbb{C}$,*

$$\left|a_3 - \lambda a_2^2\right| \leq \frac{n\cos\alpha}{2(n+1)} \max\left\{1, \left|\frac{(2\lambda-1)\, n\cos\alpha}{(n+1)}\right|\right\}.$$

Proof. Assumed that $f \in S^*_{T,n-1}(\alpha)$. Then, using the concept of subordination, it follows that a Schwarz function ω exists, fulfilling the condition $\omega(0) = 0$ and $\omega(z) < 1$, leading to

$$\left(e^{i\alpha}\frac{zf'(z)}{f(z)} - i\sin\alpha\right)\frac{1}{\cos\alpha} = 1 + \frac{n}{n+1}\omega(z) + \frac{1}{n+1}[\omega(z)]^n = \delta(z). \tag{12}$$

Let the function be defined as

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots. \tag{13}$$

It is evident that $p \in P$. It follows that

$$\begin{aligned}\omega(z) = \frac{p(z)-1}{p(z)+1} &= \frac{c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots} \\ &= \tfrac{1}{2}c_1 z + \left(\tfrac{1}{2}c_2 - \tfrac{1}{4}c_1^2\right) z^2 + \left(\tfrac{1}{8}c_1^3 - \tfrac{1}{2}c_1 c_2 + \tfrac{1}{2}c_3\right) z^3 \\ &\quad + \left(\tfrac{1}{2}c_4 - \tfrac{1}{2}c_1 c_3 - \tfrac{1}{4}c_2^2 - \tfrac{1}{16}c_1^4 + \tfrac{3}{8}c_1^2 c_2\right) z^4 + \cdots.\end{aligned} \tag{14}$$

Now, by using (1), we get

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= 1 + a_2 z + \left(2a_3 - a_2^2\right) z^2 + \left(a_2^3 - 3a_2 a_3 + 3a_4\right) z^3 \\ &\quad + \left(4a_5 - a_2^4 + 4a_2^2 a_3 - 4a_2 a_4 - 2a_3^2\right) z^4 + \cdots.\end{aligned} \tag{15}$$

Next, the left-hand expression in equation (12) can be expressed as

$$\begin{aligned}\left(e^{i\alpha}\frac{zf'(z)}{f(z)} - i\sin\alpha\right)\frac{1}{\cos\alpha} &= 1 + \frac{e^{i\alpha}a_2}{\cos\alpha}z + \frac{e^{i\alpha}(2a_3 - a_2^2)}{\cos\alpha}z^2 + \frac{e^{i\alpha}(a_2^3 - 3a_2 a_3 + 3a_4)}{\cos\alpha}z^3 \\ &\quad + \frac{e^{i\alpha}(4a_5 - a_2^4 + 4a_2^2 a_3 - 4a_2 a_4 - 2a_3^2)}{\cos\alpha}z^4 + \cdots.\end{aligned} \tag{16}$$

By substituting the series expansion of (14) into right-hand side of (12), we obtain

$$\begin{aligned}\delta(z) &= 1 + \frac{n}{2(n+1)}c_1 z + \left(\frac{n}{2(n+1)}c_2 - \frac{n}{4(n+1)}c_1^2\right) z^2 \\ &\quad + \left(\frac{n}{2(n+1)}c_3 - \frac{n}{2(n+1)}c_1 c_2 + \frac{n}{8(n+1)}c_1^3\right) z^3 + \cdots.\end{aligned} \tag{17}$$

From the comparison of (16) and (17), we derive

$$a_2 = \frac{\cos\alpha}{e^{i\alpha}} \cdot \frac{n}{2(n+1)} c_1, \tag{18}$$

$$a_3 = \frac{\cos\alpha}{e^{i\alpha}} \cdot \frac{n}{8(n+1)^2} \left[2(n+1)c_2 - \left(n + 1 - \frac{n\cos\alpha}{e^{i\alpha}} \right) c_1^2 \right], \tag{19}$$

and

$$\begin{aligned} a_4 = \frac{\cos\alpha}{e^{i\alpha}} \cdot \frac{n}{48(n+1)^3} \Bigg[& 8(n+1)^2 c_3 - \left((n+1) \left(8(n+1) - \frac{6n\cos\alpha}{e^{i\alpha}} \right) \right) c_1 c_2 \\ & + \left(2(n+1)^2 - 2n^2 \frac{\cos^2\alpha}{e^{i2\alpha}} - 3n \frac{\cos\alpha}{e^{i\alpha}} \left(n + 1 - \frac{n\cos\alpha}{e^{i\alpha}} \right) \right) c_1^3 \Bigg]. \end{aligned} \tag{20}$$

Thus,

$$\left| a_3 - \lambda a_2^2 \right| = \left| \frac{\cos\alpha}{e^{i\alpha}} \cdot \frac{n}{4(n+1)} c_2 - \frac{e^{i\alpha}(n+1) + n\cos\alpha\,(2\lambda - 1)}{2e^{i\alpha}(n+1)} c_1^2 \right|.$$

By applying **Lemma 2.2** and since $\left| e^{i\alpha} \right| = 1$, we obtain

$$\begin{aligned} \left| a_3 - \lambda a_2^2 \right| &\le 2 \left| \frac{\cos\alpha}{e^{i\alpha}} \cdot \frac{n}{4(n+1)} \right| \cdot \max\left\{ 1, \left| \frac{(2\lambda-1)n\cos\alpha}{n+1} \right| \right\} \\ &= \frac{n\cos\alpha}{2(n+1)} \cdot \max\left\{ 1, \left| \frac{(2\lambda-1)n\cos\alpha}{n+1} \right| \right\}. \end{aligned} \tag{21}$$

Proof of **Theorem 3.1** completes. □

For $\alpha = 0$ and $\lambda = 1$, we get the subsequent corollary:

Corollary 3.1. *Let $f \in S^*_{n-1,\mathrm{L}}$. Then,*

$$\left| a_3 - a_2^2 \right| \le \frac{n}{2(n+1)}. \tag{22}$$

The inequality of (22) *from* ***Corollary 3.1*** *coincides with the sharp results obtained by Shi et al.* [31] *for the class of $S^*_{n-1,\mathrm{L}}$.*

Next, we prove for the second Hankel determinant within the class $f \in S^*_{T,n-1}(\alpha)$.

Theorem 3.2. *If $f \in S^*_{T,n-1}(\alpha)$ and take the form outlined in* (1)*, then*

$$|H_2(2)| = \left| a_2 a_4 - a_3^2 \right| \le \frac{n^2 \cos^2\alpha}{4(n+1)^2}.$$

Proof. Using (18)-(20) yields

$$
\begin{aligned}
H_2(2) &= a_2a_4 - a_3^2 \\
&= \frac{1}{192(n+1)^4}\frac{\cos\alpha}{e^{i\alpha}}\Bigg[(4n(n+1))^2 c_1c_3 - \frac{12n^2(n+1)^2\cos\alpha}{e^{i\alpha}}c_2^2 \\
&+ \left(\begin{array}{l} 4n^2(n+1)^2 - 4n^4\frac{\cos^2\alpha}{e^{i2\alpha}} - 6n^3\frac{\cos\alpha}{e^{i\alpha}}\left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right) \\ -\frac{3n^2(n+1)^2\cos\alpha}{e^{i\alpha}} + \frac{6n^3(n+1)\cos^2\alpha}{e^{i2\alpha}} - \frac{3n^4\cos^3\alpha}{e^{i3\alpha}} \end{array}\right) c_1^4 \\
&- \left(\begin{array}{l} (4n(n+1))^2 - 12n^3(n+1)\frac{\cos\alpha}{e^{i\alpha}} \\ -\frac{12n^2(n+1)\cos\alpha}{e^{i\alpha}}\left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right) \end{array}\right) c_1^2c_2\Bigg].
\end{aligned}
$$

By **Lemma 2.1**, we write c_2 and c_3 in terms of c_1 and by **Lemma 2.3**, assuming that $c_1 = c \in [0,2]$, then we have

$$
\begin{aligned}
H_2(2) = \frac{\cos\alpha}{192(n+1)^4e^{i\alpha}}\Bigg[&-\frac{\cos\alpha}{e^{i\alpha}}(3n^4+6n^3+3n^2)x^2(4-c^2)^2 + \frac{\cos^2\alpha}{e^{i2\alpha}}\left(2-3\frac{\cos\alpha}{e^{i\alpha}}\right)n^4c^4 \\
&+ 6\frac{\cos\alpha}{e^{i\alpha}}\left(n^4 - n^4\frac{\cos\alpha}{e^{i\alpha}} + n^3 - n^3\frac{\cos\alpha}{e^{i\alpha}}\right)(4-c^2)c^2x + (8n^4+16n^3+8n^2)(4-c^2)(1-|x|^2)c\delta \\
&- (4n^4+8n^3+4n^2)x^2c^2(4-c^2)\Bigg]. \qquad (23)
\end{aligned}
$$

By applying the modulus to both sides of (23) and using the triangle inequality, it follows that

$$
\begin{aligned}
|H_2(2)| \le \frac{\cos\alpha}{192(n+1)^4\,|e^{i\alpha}|}\Bigg[&\left|-\frac{\cos\alpha}{e^{i\alpha}}(3n^4+6n^3+3n^2)\right||x|^2|4-c^2|^2 + \left|2\frac{\cos^2\alpha}{e^{i2\alpha}}n^4c^4\right| + \left|-3\frac{\cos^3\alpha}{e^{i3\alpha}}n^4c^4\right| \\
&+ \left|6\frac{\cos\alpha}{e^{i\alpha}}n^4(4-c^2)c^2x\right| + \left|-6\frac{\cos^2\alpha}{e^{i2\alpha}}n^4(4-c^2)c^2x\right| + \left|6\frac{\cos\alpha}{e^{i\alpha}}n^3(4-c^2)c^2x\right| \\
&+ \left|-6\frac{\cos^2\alpha}{e^{i2\alpha}}n^3(4-c^2)c^2x\right| + \left|(8n^4+16n^3+8n^2)(4-c^2)(1-|x|^2)c\delta\right| \\
&+ \left|-(4n^4+8n^3+4n^2)x^2c^2(4-c^2)\right|\Bigg].
\end{aligned}
$$

Since $\left|e^{i\alpha}\right| = 1$, $\left|e^{i2\alpha}\right| = 1$, and $\left|e^{i3\alpha}\right| = 1$, we get

$$
\begin{aligned}
|H_2(2)| \le \frac{\cos\alpha}{192(n+1)^4}\{&\cos\alpha(3n^4+6n^3+3n^2)|x|^2(4-c^2)^2 \\
&+ 2n^4c^4\cos^2\alpha + 3n^4c^4\cos^3\alpha \\
&+ 6n^4(4-c^2)c^2|x|\cos\alpha + 6n^4(4-c^2)c^2|x|\cos^2\alpha \\
&+ 6n^3(4-c^2)c^2|x|\cos\alpha + 6n^3(4-c^2)c^2|x|\cos^2\alpha \\
&+ (8n^4+16n^3+8n^2)(4-c^2)(1-|x|^2)c|\delta| \\
&+ (4n^4+8n^3+4n^2)|x|^2c^2(4-c^2)\}.
\end{aligned}
$$

Let $|x| = y$ with y ≤ 1 and by taking $|\delta| \le 1$, we derive $G\,(c,y)$ as

$$
\begin{aligned}
|H_2(2)| \leq \frac{\cos\alpha}{192(n+1)^4}\{ &\cos\alpha(3n^4+6n^3+3n^2)y^2(4-c^2)^2 \\
&+ 2n^4c^4\cos^2\alpha + 3n^4c^4\cos^3\alpha \\
&+ 6n^4(4-c^2)c^2y\cos\alpha + 6n^4(4-c^2)c^2y\cos^2\alpha \\
&+ 6n^3(4-c^2)c^2y\cos\alpha + 6n^3(4-c^2)c^2y\cos^2\alpha \\
&+ (8n^4+16n^3+8n^2)(4-c^2)(1-y^2)c \\
&+ (4n^4+8n^3+4n^2)y^2c^2(4-c^2)\,\}. \qquad (24)
\end{aligned}
$$

For every fixed $c \in (0,2)$ and $y \in (0,1)$, from (24) confirms that $G'(c,y) > 0$. Consequently, $G(c,y)$ is increasing in y which implies $G(c,y) \leq G(c,1) = G(c)$. Thus, we have

$$
\begin{aligned}
G(c) = \frac{\cos\alpha}{192(n+1)^4}\{ &\cos\alpha(3n^4+6n^3+3n^2)(4-c^2)^2 \\
&+ 2n^4c^4\cos^2\alpha + 3n^4c^4\cos^3\alpha \\
&+ 6n^4(4-c^2)c^2\cos\alpha + 6n^4(4-c^2)c^2\cos^2\alpha \\
&+ 6n^3(4-c^2)c^2\cos\alpha + 6n^3(4-c^2)c^2\cos^2\alpha \\
&+ (4n^4+8n^3+4n^2)c^2(4-c^2)\}. \qquad (25)
\end{aligned}
$$

Then, by differentiating the function (25), we get

$$
\begin{aligned}
G'(c) = \frac{\cos\alpha}{48(n+1)^4}\{ &(3n^4+6n^3+3n^2)(c^3-4c)\cos\alpha \\
&+ \left(2n^4c^3\cos^2\alpha + 3n^4c^3\cos^3\alpha\right) \\
&+ \Big[6(n^4\cos\alpha + n^4\cos^2\alpha + n^3\cos\alpha + n^3\cos^2\alpha) \\
&+ (4n^4+8n^3+4n^2)\Big](2c-c^3)\}.
\end{aligned}
$$

Now, we want to find the solution for c by setting $G'(c) = 0$. By a simple calculation, the solutions for c are

$$
c = 0, \quad c = \pm 2\sqrt{\frac{\begin{aligned}&(3n^4+6n^3+3n^2)\cos\alpha \\ &\quad - \Big[3(n^4\cos\alpha + n^4\cos^2\alpha + n^3\cos\alpha + n^3\cos^2\alpha) \\ &\qquad + 2(n^4+2n^3+n^2)\Big]\end{aligned}}{\begin{aligned}&(3n^4+6n^3+3n^2)\cos\alpha + 2n^4\cos^2\alpha + 3n^4\cos^3\alpha \\ &\quad - \Big[6(n^4\cos\alpha + n^4\cos^2\alpha + n^3\cos\alpha + n^3\cos^2\alpha) \\ &\qquad + (4n^4+8n^3+4n^2)\Big]\end{aligned}}}.
$$

By our assumptions of $c \in [0,2]$, it is evident that $G(c)$ reaches its maximum at $c = 0$. Thus, from (24), the upper bound of $G(c,y)$ occurs at $c = 0$ and $y = 1$, is given by

$$\begin{aligned}\left|a_2a_4 - a_3^2\right| &\leq \frac{48n^4 + 96n^3 + 48n^2}{192(n+1)^4}\cos^2\alpha \\ &= \frac{n^2\cos^2\alpha}{4(n+1)^2}.\end{aligned}$$

Proof of **Theorem 3.2** completes. □

Setting $\alpha = 0$ leads to the corollary stated below:

Corollary 3.2. *Let $f \in S^*_{n-1,\mathrm{L}}$. Then,*

$$|H_2(2)| \leq \frac{n^2}{4(n+1)^2}. \tag{26}$$

The inequality presented in (26) *of **Corollary 3.2** aligns with the sharp bound established by Shi et al.* [31] *for the class of $S^*_{n-1,\mathrm{L}}$.*

4. Conclusion

In this paper, we introduced the subclass of tilted starlike functions $S^*_{T,n-1}(\alpha)$ associated with the epicycloid domain. The principal contributions include coefficient estimates and sharp inequalities, particularly the Fekete–Szegö inequality and the second Hankel determinant. The sharpness of the obtained results was discussed through Corollaries 3.1 and 3.2, which reduce to the known sharp estimates established by Shi et al. [31]. This confirms the validity of our approach and shows that the present results not only generalize but also unify earlier findings within Geometric Function Theory. Furthermore, this study highlights potential directions for future research. In particular, the framework developed here can be extended to study other functionals such as the Toeplitz determinant (see [36]), or adapted to subclasses of bi-univalent functions (see [2]). Moreover, these findings may be extended to more general subclasses of analytic functions or applied to related problems in complex analysis, offering promising directions for future research.

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