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THE QUANTILE GARCH-DISTRIBUTED LAG (QGDL) FRAMEWORK: A UNIFIED MODEL FOR HIGH-VOLATILITY TIME SERIES

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Abstract. The essence of this article is that it has developed the Quantile GARCH-Distributed Lag (QGDL) model as a recent and superior model to the QARDL model. Although the QARDL model is only able to deal with asymmetric relationships in the quartiles and the assumption of homoscedasticity, our model is able to address this weakness using the simultaneous volatility clustering. The model that we propose is the most effective and correct in estimating the parameters in cases of changing conditional variance with time, which makes it the most appropriate measure to use in the analysis of extremely volatile financial time series.

Keywords: Quantile regression; GARCH; QARDL; Conditional heteroskedasticity; Volatility clustering

AMS (2020) Subject Classification: 62M10, 62P20, 62F12.

1. Introduction

Time series analysis has experienced fundamental changes throughout the last several decades, as the appearance of data with complicated statistical properties that do not conform to the classical assumptions of the linear model came in. In dynamic phenomena, the conditional mean is no longer held to be useful in explaining the entire distributional behavior of the series, particularly when conditional heteroskedasticity is present, as well as when there are asymmetric relationships between various quantiles of the data.

Traditionally, econometric literature has dealt with these complexities in two different directions:

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The First Path: It is based upon time-dependent conditional (heteroskedasticity) variance modeling, most prominently in the Autoregressive Conditional Heteroskedasticity (ARCH) models introduced by Engle (1982), and later generalized to GARCH models. The purpose of these models is to obtain the effect of volatility clustering, and guarantee efficiency of estimation in situations where the hypothesis of constant variance (homoskedasticity) is not met.
The Second Path: The focus has been changed into the analysis of time series in various sections of the distribution of the dependent variable through the Quantile Regression approach introduced by [7]. Such a framework was also applied to dynamic time series through the Quantile Autoregressive Distributed Lag (QARDL) framework proposed by Cho et al. (2015), where long-run and short-run relationships could be investigated on different quantile levels.

Although the analysis power of each of the paths separately is high, most time series in reality not only show the statistical properties of both paths but also have both asymmetric quantile relationships and time-varying conditional variance. Normal QARDL models normally assume that the error variance is constant whereas the traditional GARCH models are devoted to the modeling of the variance around the conditional mean. This has a methodological gap when addressing series that possess high volatility with different distributional dependencies. Additionally, although the traditional GARCH estimations are usually prone to non-normality and outlier effects, the implementation of the M-quantile estimations technologies have proven to be more robust in capturing the scale and location characteristics of a heteroscedastic process [13], which is a more resistant solution than the classical mean-based variance modeling.

The paper seeks to fill this gap by developing an elaborate theory framework of the hybrid model called as QGDL. This model consists of the flexibility of the QARDL structure to capture dynamic asymmetry coupled with the strength of GARCH models to deal with conditional heteroskedasticity. This integration is based on the changing trend on integrating GARCH-X errors in quantile regression, the fact that other covariates are allowed to affect the conditional variance means that a more exact estimation of risk and distributional behavior could be attained [24]

Since there is mathematical complexity associated with integrating these two frameworks; this paper concentrates on the statistical model building and evaluation of its performance using Monte Carlo simulations. The simulation exercise will be used to test the model with an objective of determining the model capabilities of generating correct parameter estimates in different distributional set-ups. This study aims to use a two-step Quasi-Maximum Likelihood Estimation (QMLE) methodology to achieve consistent and robust estimator, and give an efficient econometric model to be used in the analysis of time series with complex statistical characteristics.

The main contribution of the research is the creation of the model of QGDL as the developed model as an alternative to the QARDL model. Although the QARDL model handles the asymmetric relationship between the quartiles, under homoscedasticity only, our model has been able to overcome this drawback by simultaneously considering the volatility clustering. Our model comes out as a more efficient and precise estimation of parameters under time-varying conditional variance and hence the most appropriate in analyzing time series with high volatility in the financial market.

2. Literature Review and Theoretical Framework

In the next section, the review of recent empirical research that has applied the Quantile Autoregressive Distributed Lag (QARDL) framework is presented in a full scope. Although the QARDL model has become popular due to its capacity of capturing distributional asymmetry and nonlinear cointegration across various quantiles, an important gap in the literature has been present in dynamic modelling of conditional heteroscedasticity under the model.

As summarized in Table 1, a number of studies have tried to solve the market volatility either as a pre-processing step of GARCH models or doing parallel volatility analyses. Nevertheless, the majority of current studies still presuppose homogeneous variance in the quantile regressions, which can result in ineffective parameter estimation when volatility is concentrated in certain periods i.e. when the time series is subject to volatility clustering which is characteristic of the financial and macroeconomic time series.

Table 1. **Comparative Analysis of Literature Review and the Theoretical Justification for QGDL**

Study (Author & Year)	**Data Type**	**Research Problem**	**Key Conclusion & Research Gap**	**Feasibility/Necessity of GARCH Integration**
[9] Mensi et al. (2019)	Credit & Oil (Time Series)	Risk shocks & credit markets.	Focused on quantiles but neglected conditional volatility effects.	High Necessity; The intense volatility clustering in credit/oil markets makes the standard QARDL estimates biased without GARCH integration.
[11] Nor et al. (2020)	Macro (Somalia)	FX volatility determinants.	A unified QGARCHDL model is required to bridge the gap with the long run.	Basic; Used GARCH only without ARDL's long-run dynamics.
[17] Shahzad et al. (2021)	US Stocks (Time Series)	Macro fundamentals & stocks.	Proved quantile superiority but failed to model heteroscedasticity dynamically.	High; Essential for estimation efficiency during market crashes/shocks.
[21] Uche, Effiom (2021)	Capital Flight	Global uncertainties.	A gap exists in the structural integration of GARCH within the QARDL framework.	Technical; GARCH was used separately to extract variables only.
[23] Vartholomatou (2021)	Corporate Bonds	Long-run & short-run dynamics.	Focused on parametric flexibility but ignored variance dynamics.	Important; Bonds are affected by liquidity and volatility regimes.

[8] Lee et al. (2022)	Solar Stocks (China)	Investor sentiments & prices.	Neglected volatility clustering in error terms despite high sentiment shocks.	High; Sentiments create stochastic volatility that QARDL alone cannot capture.
[3] Apergis (2022)	Uncertainty (EPU)	Tail risk evaluation.	Noted GARCH's failure in tail risks, suggesting a need for a hybrid model.	The study highlights the deficiency of separate models, confirming that merging GARCH with quantiles is vital to capture tail risks and volatility simultaneously.
[15] Sahin, Sahin (2023)	Exchange Rates	Asymmetry in FX spreads.	Focused on sign asymmetry but ignored the conditional variance structure.	Essential; FX spreads follow explosive volatility patterns requiring GARCH.
[14] Ren et al. (2023)	Crude Oil Prices	Geopolitical risk & oil.	Found effects in high quantiles but without linking them to volatility states.	Crucial; Oil is highly sensitive to spikes; ignoring GARCH leads to biased forecasts.
[19] Sokhanvar, Bouri (2023)	Commodities & FX	Ukraine war shocks.	Neglecting volatility weakens the explanation of extreme crisis shocks.	Technical; Noted GARCH's forecasting power but didn't integrate it structurally.
[12] Olasehinde-Williams (2024)	Macro (Nigeria)	Rate volatility & growth.	Established quantile-specific impacts but lacked a unified GARCH structure.	Necessary; Interest rate data is known for persistent volatility clustering.
[1] Aldawsari et al. (2024)	Stocks and FX (India)	Quantile dependence.	Focused on diagnostic validity without modeling conditional variance.	Crucial; To ensure the stability of cointegration results under shocks.
[22]Umoru et al. (2024)	Energy Demand	Devaluation & growth.	Failed to provide a unified framework for both variance and quantiles.	Methodological Need; Bridging the gap between their disjointed steps through a unified framework to ensure more consistent long-run results.
[18] Singh, Dhariwal (2025)	Panel Data (G20)	Uncertainty & stock returns.	Developed Panel QARDL but assumed constant (homoscedastic) variance.	Strategic; Needed to control for volatility differences across 20 countries.

[2] Amagbo et al. (2025	Wheat Futures	Geopolitical risk.	Ignoring GARCH integration weakens the model's commodity risk management.	Crucial; Recognized GARCH behavior but applied standard QARDL.
[20] Tabash et al. (2025)	Carbon Market	Carbon emissions and stocks.	Needs a single framework combining quantiles, correlation, and volatility.	Strategic; Used DCC-GARCH separately from the QARDL model.

3. Model Specification

In this part, we suggest that the Quantile Autoregressive Distributed Lag (QARDL) model by [5] should be extended to include a GARCH error structure. The resulting Quantile GARCH-Distributed Lag (QGDL) framework offers a single framework to both examine long-run equilibrium relationships and conditional volatility clustering of the variances of various distributional quantiles.

Contrary to the conventional mean-based models, the QGDL model takes different approach when the structural relationship is defined at a given quantile (τ). In the case of each τ, the model estimates the distributional effect of regressors with internalization of time-varying conditional variance. The QGDL (p, q)-(m, s) model is formulated as follows: [5], Zhu(2021), [13]

3.1. The Quantile-Based Structural Construction.

Contrary to the conventional mean-based models, the QGDL model takes the analysis step when the structural relationship is defined at a given quantile (τ). In the case of each τ, the model estimates the distributional effect of regressors with internalization of time-varying conditional variance. The QGDL (p, q)-(m, s) model is formulated as follows:

$$Y_t = \alpha(\tau) + \sum_{j=1}^{p} \phi_j(\tau) Y_{t-j} + \sum_{j=0}^{q} \theta_j'(\tau) X_{t-j} + U_t(\tau) \tag{1}$$

Where:

- Y_t: refers to a dependent variable under time t.
- τ: The explicit quantile (e.g., 0.05 for the lower tail or 0.50 for the median).
- $\alpha(\tau)$: A quantile intercept.
- $\phi_j(\tau)$: A quantile autoregressive coefficients, on behalf of the impact for past values for Y.
- $\theta_j'(\tau)$: A quantile-dependent distributed lag coefficients, on behalf of the impact for independent variables X.
- $U_t(\tau)$: The quantile error term, defined as $U_t(\tau) = Y_t - Q_{Y_t}(\tau/\mathcal{F}_{t-1})$.

3.2. Integration of Conditional Heteroskedasticity (GARCH Structure).

For estimation efficiency and capturing the "volatility clustering" inherent in financial time series, a quantile error $U_t(\tau)$ has linked to the GARCH(m, s) process. A conditional variance equation at each quantile has expressed as

$$\sigma_t^2(\tau) = \omega(\tau) + \sum_{i=1}^{m} \nu_i(\tau) U_{t-i}^2(\tau) + \sum_{j=1}^{s} \eta_j(\tau) \sigma_{t-j}^2(\tau) \tag{2}$$

This integration allows a model for bridging the gap between asymmetric tail-risk analysis and variance modeling, overcoming a homoscedasticity limitation for standard QARDL models.

3.3. **Model Reformulation for Cointegration.** To facilitate cointegration testing, we reformulate the QGDL model to separate short-run dynamics from long-run relationships:

$$Y_t = \alpha(\tau) + \sum_{j=1}^{p} \phi_j(\tau) Y_{t-j} + \sum_{j=0}^{q-1} W_{t-j}' \delta_j(\tau) + X_t' \gamma(\tau) + U_t(\tau) \tag{3}$$

Where:

$$W_{t-j} := \Delta X_t,$$
$$\gamma(\tau) := \sum_{j=0}^{q} \theta_j(\tau),$$
$$\delta_j(\tau) := -\sum_{i=j+1}^{q} \theta_i(\tau),$$

All of the parameters in (3) measure the short-run dynamics, and the long-run relationship between Y_t and X_t can be captured by reformulating (3) into the following long-run quantile process as in equ (1,2,3) in Cho et al. (2015).

3.4. **Quantile Error Correction Model with GARCH (QGDL-ECM).** The model is further extended into an Error Correction Model (ECM) form to estimate the speed of adjustment toward equilibrium:

$$\Delta Y_t = \alpha(\tau) + \zeta(\tau)(Y_{t-1} - \beta(\tau)' X_{t-1}) + \sum_{j=1}^{p-1} \phi_j(\tau) \Delta Y_{t-j} + \sum_{j=0}^{q-1} \theta_j'(\tau) \Delta X_{t-j} + U_t(\tau) \tag{4}$$

where:

$$\zeta(\tau) = \sum_{i=1}^{p} \phi_i(\tau) - 1, \quad \text{Correction speed factor,}$$
$$\phi_j(\tau) = -\sum_{h=j+1}^{p} \phi_h(\tau) \quad \text{for } j = 1, \ldots, p-1,$$
$$\theta_j(\tau) := -\sum_{h=j+1}^{p} \theta_h(\tau) \quad \text{for } j = 1, \ldots, q-1$$

4. Theoretical Assumptions

The main objective of this section is to demonstrate the convergent properties of the proposed model's estimators. The importance of demonstrating the consistency property lies in ensuring that the estimators closely approximate the true values of the parameters, even with a large sample size, thus eliminating bias in the results. The asymptotic normality ensures the validity of hypothesis tests (such as the Wald and LM tests). The two-step independence of the estimation stages proves that errors in the first (quantile) stage do not invalidate or affect the validity of the second (GARCH) stage results. So we carry out the following assumptions for examining the regular properties of QGDL estimator.

Assumption A1 (Error Structure and Stationarity): (i) $\{\eta_t\}$ is i.i.d. with $\mathbb{E}[\eta_t] = 0$, $\mathbb{E}\left[\eta_t^2\right] = 1$, and $F_\eta(0) = \tau$.

(ii) $\{\sigma_t(\tau)\}$ is $\mathcal{F}_{t-1}$ measurable strictly stationary and ergodic, satisfying the $\mathrm{GARCH}(m, s)$ recursion (4.2).

(iii) $Y_t \sim I(1)$ and $X_{kt} \sim I(1)$ with cointegrating vector $\beta_0(\tau)$ such that $Y_t - \beta_0(\tau)^T X_t$ is $I(0)$.

(iv) For every $\tau \in (0,1)$, the AR polynomial $1 - \sum_{j=1}^{p} \alpha_{j,0}(\tau) z^j$ has all roots strictly outside the unit circle.

(v) The common integrated order is zero.

Assumption A2 (GARCH-Specific Conditions): (i) The fourth moment is finite: $\mathbb{E}\left[\varepsilon_t^4(\tau)\right] < \infty$, equivalently $\mathbb{E}\left[|\eta_t|^{4+\delta}\right] < \infty$ for some $\delta > 0$ [4].

(ii) Strict stationarity (geometric ergodicity): $\sum_{i=1}^{s} \gamma_{i,0}(\tau) + \sum_{j=1}^{m} \delta_{j,0}(\tau) < 1$ together with $\mathbb{E}\left[\ln\left(\sum_{j=1}^{m} \delta_{j,0}(\tau)\,\eta_{t-j}^2\right)\right] < 0$.

(iii) The unconditional variance exists and is finite: $\sigma_\infty^2(\tau) = \omega_0(\tau) / \left(1 - \sum \gamma_{i,0}(\tau) - \sum \delta_{j,0}(\tau)\right) < \infty$.

(iv) The GARCH parameter vector $\eta_0(\tau)$ lies in the interior of a compact set $\mathcal{H} \subset \mathbb{R}^{1+m+s}$.

Assumption A3 (Functional Convergence and Mixing): (i) $\{(X_t, Y_t)\}$ is strongly mixing (α-mixing) with mixing coefficients $\alpha(m) = O(m^{-a})$ for some $a > 2$.

(ii) The regressor vector $\psi_t(\tau)$ has finite $2+\delta$-th moment: $\mathbb{E}\left[\|\psi_t(\tau)\|^{2+\delta}\right] < \infty$.

(iii) Define the partial-sum process $S_T(r) = T^{-1/2} \sum_{t=1}^{\lfloor Tr \rfloor} g_t(\tau)$, where $g_t(\tau) = \psi_t(\tau)(\tau - \mathbb{I}\{\varepsilon_t(\tau) \le 0\})$. Then $S_T(\cdot) \rightsquigarrow B(\cdot)$ in $D([0,1])^{d_\theta}$, where $B(\cdot)$ is a d_θ-dimensional Brownian motion with long-run covariance matrix $\Omega(\tau) = \sum_h \mathbb{E}\left[g_t(\tau)\, g_{t-h}(\tau)^T\right]$.

Assumption A4 (Identification and Design): (i) The matrix $D(\tau) = \mathbb{E}\left[\psi_t(\tau)\,\psi_t(\tau)^T f_\eta(0)\right]$ is positive definite for every $\tau \in (0,1)$.

(ii) The mapping $\theta(\tau) \mapsto (\beta(\tau), \{\alpha_j(\tau)\}, \{\gamma_\ell(\tau)\})$ is injective on Θ.

(iii) The GARCH parameter $\eta_0(\tau)$ is identified through the autocovariance structure of $\left\{\varepsilon_t^2(\tau)\right\}$.

4.1. **Main Results.**

Theorem 4.1 (Consistency of the QGDL Estimator)**.** *Under Assumptions A1–A4, the QGDL estimator* $\hat{\theta}_T(\tau)$*, defined as the minimiser of the sample check-loss* $\hat{Q}_T(\theta) = \sum_{t=1}^{T} \rho_\tau\left(Y_t - \psi_t(\tau)^T \theta^{(R)}\right)$*, followed by a one-step QMLE for the GARCH parameters, satisfies* $\hat{\theta}_T(\tau) \to \theta_0(\tau)$ *almost surely.*

Theorem 4.2 (Asymptotic Normality of the QGDL Estimator)**.** *Under Assumptions A1–A4:*

$$\sqrt{T}\left(\hat{\theta}_T(\tau) - \theta_0(\tau)\right) \rightsquigarrow \mathcal{N}\left(0, H(\tau)^{-1}\Omega(\tau)\, H(\tau)^{-1}\right)$$

where $H(\tau) = \mathbb{E}\left[\psi_t(\tau)\,\psi_t(\tau)^T f_\eta(0)\right]$ *and* $\Omega(\tau)$ *is the long-run covariance matrix from Assumption A3.*

Corollary 4.1 (GARCH Parameters)**.** *Under Assumptions A1–A4:*

$$\sqrt{T}\left(\hat{\eta}_T(\tau) - \eta_0(\tau)\right) \rightsquigarrow \mathcal{N}\left(0, J(\tau)^{-1}\right)$$

where $J(\tau)$ is the Godambe (sandwich) information matrix of the GARCH recursion; see [4]

4.2. **Proofs.**

Proof of Theorem 4.1.

(i) **Convexity:** The check loss ρ_τ is convex, so $\hat{Q}_T(\theta)$ is convex and any minimiser is global.

(ii) **Uniform LLN:** By the strong mixing condition in A3(i) and moment bound A3(ii), a uniform law of large numbers [10] gives $\sup_{\theta\in\Theta}|\hat{Q}_T(\theta) - Q_0(\theta)| \to 0$ almost surely.

(iii) **Identification:** By the [6] identity and $\mathbb{E}[\eta_t] = 0$, $F_\eta(0) = \tau$, one obtains

$$Q_0(\theta) - Q_0(\theta_0) = \frac{1}{2}(\theta - \theta_0)^T H(\tau)(\theta - \theta_0) + o\left(\|\theta - \theta_0\|^2\right).$$

Positive-definiteness of $H(\tau)$ (A4(i)) implies that $\theta_0(\tau)$ is the unique minimiser of Q_0.

(iv) **Conclusion:** For any $\varepsilon > 0$, for θ with $\|\theta - \theta_0\| > \varepsilon$, $\hat{Q}_T(\theta) - \hat{Q}_T(\theta_0) \geq c_\varepsilon > 0$ eventually a.s. Hence $\hat{\theta}_T(\tau) \to \theta_0(\tau)$ a.s.

□

Proof of Theorem 4.2. Differentiating (4.8) gives the first-order condition:

$$\frac{1}{T}\sum_{t=1}^{T}\psi_t(\tau)\left(\tau - \mathbb{I}\left\{Y_t - \psi_t(\tau)^T\hat{\theta}_T(\tau) \leq 0\right\}\right) = 0$$

Subtracting the population condition $\mathbb{E}[\psi_t(\tau)(\tau - \mathbb{I}\{\eta_t \leq 0\})] = 0$ and applying the Bahadur representation [7] we obtain:

$$\hat{\theta}_T(\tau) - \theta_0(\tau) = H(\tau)^{-1}\cdot\frac{1}{T}\sum_{t=1}^{T}\psi_t(\tau)(\tau - \mathbb{I}\{\eta_t \leq 0\}) + o_p\left(T^{-\frac{1}{2}}\right) \tag{5}$$

The inner sum is exactly $S_T(1)$ from Assumption A3(iii). Hence:

$$\sqrt{T}\left(\hat{\theta}_T(\tau) - \theta_0(\tau)\right) = H(\tau)^{-1}S_T(1) + o_p(1) \rightsquigarrow H(\tau)^{-1}B(1) \sim \mathcal{N}\left(0, H(\tau)^{-1}\Omega(\tau)H(\tau)^{-1}\right)$$

by Cramér–Wold and the continuous-mapping theorem. □

Proof of Corollary 4.1. Since $\hat{\theta}_T(\tau) \to \theta_0(\tau)$ a.s. by Theorem 4.1, the residual process $\{\hat{\varepsilon}_t(\tau)\}$ inherits the GARCH (m, s) structure. Consistency of $\hat{\eta}_T(\tau)$ follows from [**?**] whose conditions are satisfied by A2(i)–(iv). Differentiating the QMLE objective twice gives the sandwich form; the score sum is asymptotically $\mathcal{N}(0, J(\tau))$ by the CLT under A2(i) and A3(i), yielding:

$$\sqrt{T}\left(\hat{\eta}_T(\tau) - \eta_0(\tau)\right) \rightsquigarrow \mathcal{N}\left(0, J(\tau)^{-1}\right)$$

□

“‘

5. Parameter Estimation

By reformulating the model in equation (4) as matrices, as follows:

$$Y_t = G_t'\lambda(\tau) + \widetilde{Y}_t{}'\phi(\tau) + U_t(\tau) = Z_t'\aleph(\tau) + U_t(\tau) \tag{5}$$

Where

$$Z_t := (G_t', \widetilde{Y}_t{}')' := (1, W_t', \ldots, W_{t-q+1}', X_t', Y_{t-1}', \ldots, Y_{t-P}')'$$
$$\aleph(\tau) := (\lambda(\tau)', \phi(\tau)')' := (\alpha(\tau), \delta(\tau)', \gamma(\tau)', \phi(\tau))'$$

To estimate the parameters of the QGDL framework, a model identifies the short-run dynamic coefficients, the long-run impact parameters and an error correction term within a unified quantile-based GARCH structure.

So to estimate parameters of QGDL the Short-run parameter estimator, Long-run impact coefficients from

$$\widetilde{\aleph}(\tau) := \arg\min_{\alpha(\tau)} \sum_{t=1}^{n} \rho(\tau)\{Y_t - Z_t'\aleph(\tau)\} \tag{6}$$

Where $\rho_\tau(u)$: is the check function with:

$$\rho_\tau(u) = u\psi_\tau(u) \tag{7}$$

$$\psi_\tau(u) = \tau - \mathbf{1}_{u \le 0} \tag{8}$$

and to estimate Long-run impact by plug in principle:

$$\widetilde{\beta}(\tau) := \widetilde{\gamma}(\tau)\left(1 - \sum_{j=1}^{p} \widetilde{\phi}_{j,n}(\tau)\right)^{-1} \tag{9}$$

After gaining a quantile regression residuals $U_t(\tau)$, we estimate the GARCH parameters by maximizing the Quantile Likelihood function.

$$\mathcal{L}_n^{\text{GARCH}}(\mathbb{Q}; \tau) = \sum_{t=1}^{n} \ell_t(\mathbb{Q}; \tau) \tag{10}$$

$$\mathbb{Q} = (\omega, \nu_1, \ldots, \nu_m, \eta_1, \ldots, \eta_s)'$$

$$\ell_t(\mathbb{Q}; \tau) = -\frac{1}{2}\log \sigma_t^2(\tau) - \frac{1}{2}\frac{U_t^2(\tau)}{\sigma_t^2(\tau)} \tag{11}$$

$$\sigma_t^2(\mathbb{Q}; \tau) = \omega + \sum_{i=1}^{m} \nu_i \widehat{U}_{t-i}^2(\tau) + \sum_{j=1}^{s} \eta_j \sigma_{t-j}^2(\tau; \theta) \tag{12}$$

The QMLE (Quasi-Maximum Likelihood Estimator) for GARCH parameters:

$$\widehat{\mathbb{Q}}_n(\tau) = \arg\max_{\mathbb{Q}\in\Theta} \mathcal{L}_n^{\text{GARCH}}(\theta; \tau) \tag{13}$$

and the distribution of Long and Short-run parameters as mention [5] is to converges Normal distribution and the distribution of parameter GARCH under Assumption (1-3) $\forall \tau \in (0, 1)$

$$\sqrt{n}(\widehat{\mathbb{Q}}_n(\tau) - \mathbb{Q}(\tau)) \sim N(0, V(\tau))$$

$$V(\tau) := J(\tau)^{-1} I(\tau) J(\tau)^{-1} \tag{14}$$

$$J(\tau) := \mathbb{E}\left[\frac{\partial^2 \ell_t(\mathbb{Q};\tau)}{\partial\mathbb{Q}\partial\mathbb{Q}'}\right]$$

$$I(\tau) = \mathbb{E}\left[\frac{\partial \ell_t(\mathbb{Q};\tau)}{\partial\mathbb{Q}}\frac{\partial \ell_t(\mathbb{Q};\tau)}{\partial\mathbb{Q}'}\right]$$

6. Hypothesis Testing

The study will follow the following procedures to test the statistical significance of the model parameters and test the structural efficiency of the tested model:

(1) Wald test is used to test the hypotheses concerning the parameters of long-term and short-term constraints. This test is expected to answer the hypothesis that these parameters are significantly different than zero at various quantitative levels, which will enable us to test the existence of quantitative cointegration and short-term adjustment process.

(2) Lagrange multiplier test (LM test) is used to check the existence of ARCH effect in the quantitative regression residuals then estimate the value of the variance parameters. This test is used to ensure that the conditional variance is varying and hence the usage of the GARCH structure in the model, with Hypothesis:

$$H_0^{\text{GARCH}} : \nu_i(\tau) = 0, \eta_j(\tau) = 0 \quad \forall i, j$$

$$LM_n^{\text{GARCH}}(\tau) = \frac{n}{2} g_n(\tau)' J_n(\tau)^{-1} g_n(\tau) \tag{15}$$

Where:

$$g_n(\tau) = \frac{1}{n}\sum_{t=1}^{n} \frac{\partial \ell_t}{\partial \mathbb{Q}}\bigg|_{\mathbb{Q}=\mathbb{Q}_0}$$

The distributions approximate the test under assumptions (1)-(3):

$$H_0^{\text{GARCH}} : LM_n^{\text{GARCH}}(\tau) \sim \chi^2_{m+s}$$

7. Monte Carlo Simulations

In order to assess the performance and strength of the proposed QGDL(p, q)-(m, s) model at various samples and varying quantities, a detailed simulation research was done.

a. Initialization: of set true parameters and determine sample sizes: $T \in \{100, 250, 500, 1000\}$ with quantiles: $\tau \in \{0.05, 0.25, 0.50, 0.75, 0.95\}$ and set number of replications: $R = 500$

b. Data generation, represented by: independent variable (X_t), which is a random walk calculation so as to make it a first-order integral I(1), $z_t \sim N(0, 1)$ and compute GARCH variance, quantile errors, and generate QGDL(p, q)-(m, s) based on equation (5).

c. Parameters estimation: once the design matrix Z_t is constructed, then minimize check function to get $\hat{\psi}(\tau)$, followed by computing of residuals.

d. Long-run coefficients (Plug-in approach): In the current step, the long-run impact parameters are calculated by using the 'Plug-in' method. This entails the replacement of the estimated short run dynamic coefficients into the steady state equation to find a consistent estimation of the equilibrium relationship at varying quantiles.

e. Maximize (GARCH through QMLE) of quasi-likelihood $\mathcal{L}(\eta)$, Stationarity constraints, $\hat{\eta} = [\hat{\omega}, \hat{\gamma}, \hat{\delta}]$, conditional variance must be computed.

7.1. **Initial Diagnostic Tests of Residuals.** In order to make sure of the correctness of the estimates as well as the sufficiency of the proposed model, a set of diagnostic tests was conducted in an entire set on the residues of the median quantile ($\tau = 0.50$). These tests confirm the methodological framework and make sure that the model is specified.

Table 2. Results of Residuals Diagnostic Tests

Test	**Statistic (χ^2)**	**P-value**
ARCH-LM Test	60.6079	9.12×10^{-12}
Ljung-Box Test	14.3841	0.1562
Jarque-Bera Test	9.7549	0.0076

The result of the ARCH-LM test displayed in Table 1 asserts the statistically significant impact of the instability of the conditional variance in the residues. This finding explains why GARCH model is used to cope with time variability of the series.

The p value of the Ljung Box test (0.1562) is less than 0.05 therefore we accept the null hypothesis that there is no autocorrelation. This implies that the model achieves the capturing of the underlying time dynamics of data.

Normality: The Jarque-Bra Test with P-value of 0.0076 rejected the hypothesis of normality, which states that the data has thick tails. This also highlights the value of Quantile regression which is more robust to abnormal distribution and outliers than the conventional techniques.

7.2. **Presentation of the results of estimating the parameters of the QGDL model.** Thus, the next step was the estimation of the parameters of the QGDL model, to measure the effectiveness of the proposed model, a Monte Carlo simulation of the QGDL$(1, 2)$-GARCH$(1, 1)$ model was conducted, where the ranks were assigned depending on the stability of the results and explaining the dynamics.

to determine the short-term and long-term parameters, and the conditional variance parameters, as illustrated in Table (3) below.

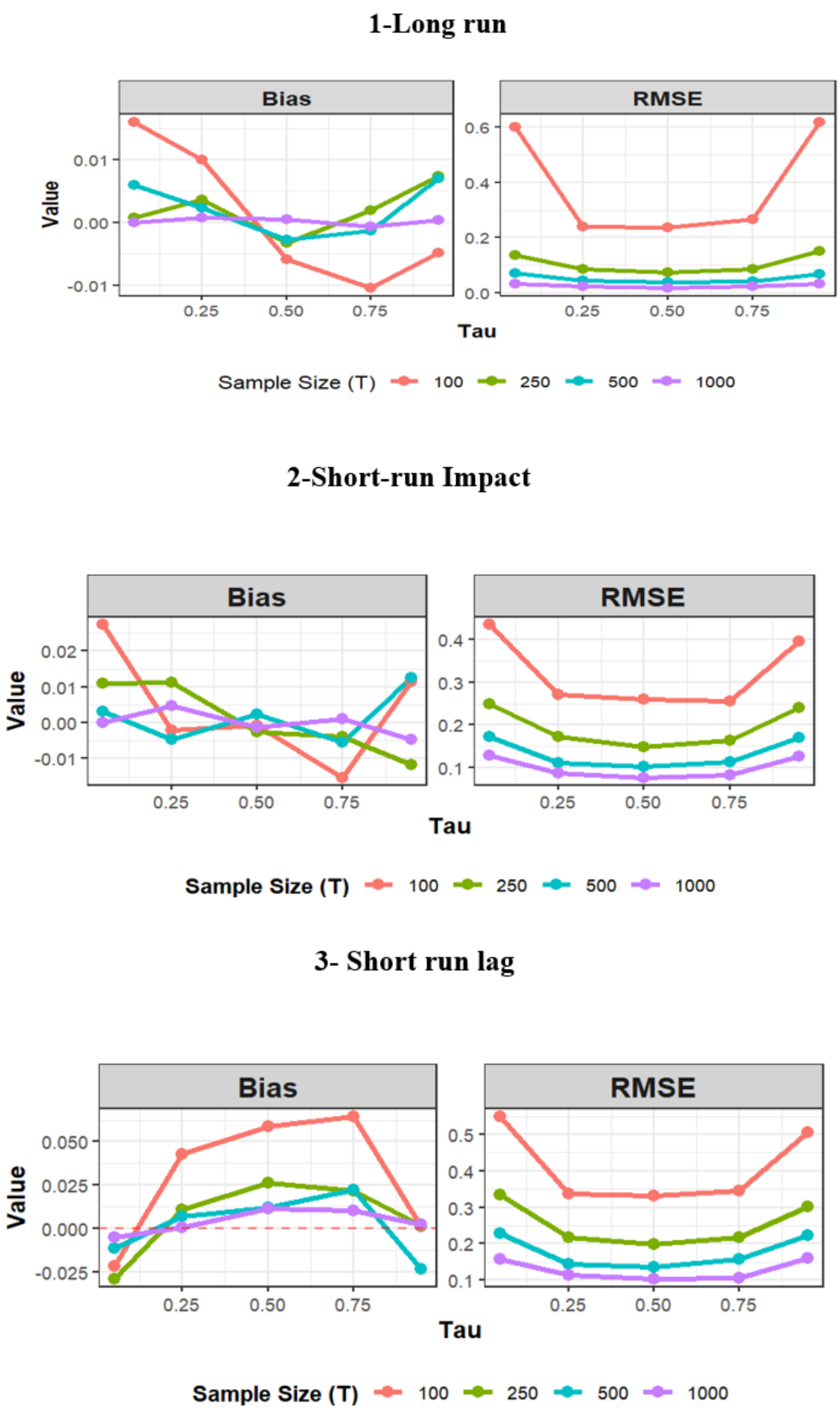
Table 3. Results of estimating the parameters of the QGDL model

Quantile (τ)	**Auto regressive**	**Error Correction**	**Short-run Impact**	**Short-run Lag**
0.05	0.5439	-0.4561	0.7606	0.4767
0.25	0.5957	-0.4043	0.8378	0.2784
0.50	0.6019	-0.3981	0.7869	0.3117
0.75	0.6044	-0.3956	0.6989	0.3862
0.95	0.6174	-0.3826	0.7909	0.2583

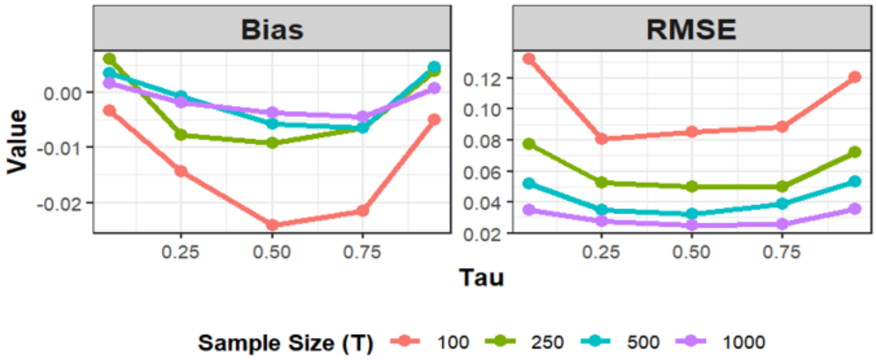
Long-run Coeff.	Constant W	ARCH Coeff.	GARCH Coeff.
2.7131	0.3751	0.0922	0.7878
2.7612	0.1789	0.0834	0.7639
2.7604	0.1239	0.1097	0.7408
2.7437	0.1115	0.0731	0.8365
2.7424	0.1048	0.0210	0.9434

Based on Table (3), it can be seen that estimated error correction coefficient is negative and statistically significant whereby all quantiles are considered. This establishes the presence of a fixed cointegration relationship, with the system correcting the system about 38 percent to 45 percent of its failures to reach long-term equilibrium at each time. The coefficients of the long-term effects are very constant at 2.74. This consistency shows that the relationship between the independent and dependent variables existing at equilibrium is strong at different market. The volatility dynamic indicates that the volatility coefficient peak is at the 0.95 percentile (0.9434) and hence there is always a high level of volatility when it is in a shock. This is the reason why the GARCH model should be included in order to reflect the so-called volatility clustering that cannot be determined by the typical QARDL models. The immediate and short-term parameters are immediate and short-term reaction of a single time period, which proves that the model is able to unravel complicated time dynamics.

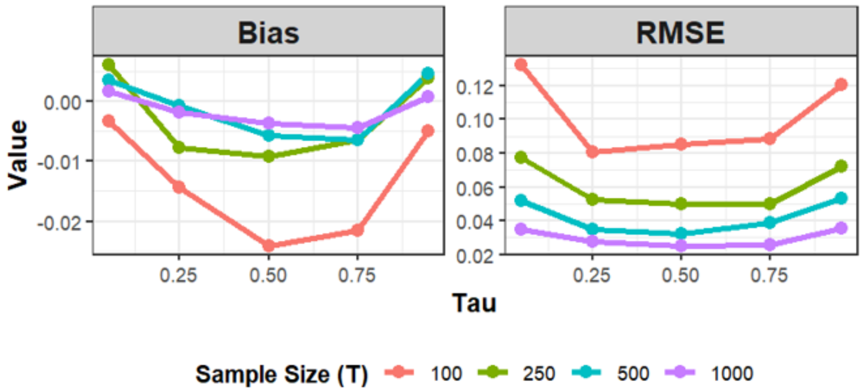
Figure 1. **Monte Carlo Simulation Results for Evaluating the Estimation Accuracy of the QGDL Model across Different Quantiles and Sample Sizes.**



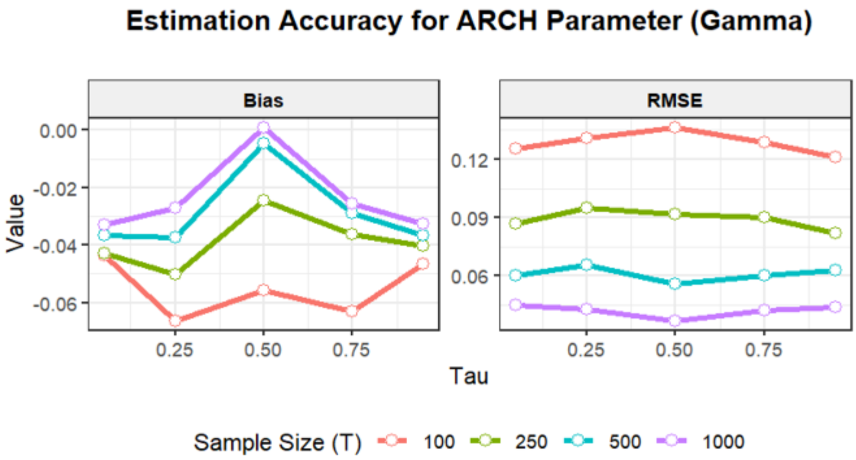
4- Error Correction



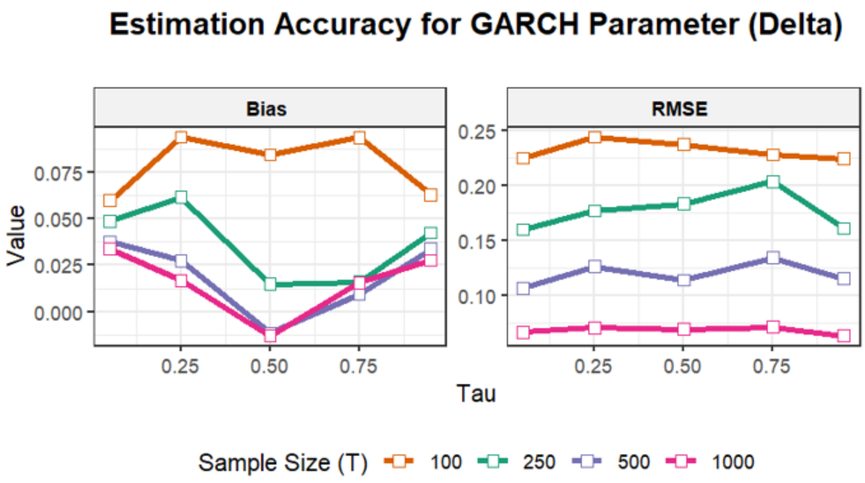
5- Autoregressive



6-ARCH Coeff.



7-GARCH -Coeff



The fact that the proposed QGDL model has the strong statistical properties is vivid, as seen in Figure (1). We can trace that as we increase the sample size ($T = 100$) to very large sample size ($T = 1000$), the bias curves approach zero and the RMSE values are reducing sharply and this shows that the estimators are approaching the assumed true values in data generation process (DGP). It is also reported that the estimation of the GARCH parameters is highly stable and this increases the ability of the model in the management of the time fluctuations even when long-term relationships are in equilibrium.

7.3. **Wald Test for Parameter.** The Wald test was carried out to statistically test the hypothesis of the parameter stability in the various quartiles. The value of this test is that it is a way to prove that the relationship between the variables is significant in the distribution, hence the effectiveness of the approach to use quantitative regression and not the conventional mean-based models.

TABLE 4. Results of the Wald test for parameter across quartiles for the QGDL model

Component	**Chi-Square**
Long-run Impact	49.12
Short-run Impact	45.98
ARCH Effect	38.64
GARCH Effect	36.41

Based on Table (4), we reject the null hypothesis, which holds the hypothesis of all components of the model having a constant parameter at a significance level of 1 percent because all the P-values were less than 0.001. This statistical rejection presents definite results on the finding of asymmetric effect, i.e. the volatility of the dependent variable with the values of the independent variables as well as the pattern of the volatility dynamic vary significantly between lower quartile, middle quartile and upper quartile. This proves the superiority of the suggested QGDL model in the representation of complexities of time series structure.

8. Conclusions

- It was observed that the GARCH structure when included in the QARDL framework can improve accuracy of long-term and short-term parameter estimates especially in situations where time volatility is high.
- Wald tests showed that the effect of independent variables and volatility changes differ significantly in the quartiles, which prove that the traditional linear models are not suitable in explaining such data.
- It was confirmed that the model is more accurate with larger sample sizes and that the stability of the RMSE curves is large with larger sample sizes in all the estimated parameters.
- The model was found to have a high ability to capture error correction speeds of between 38-45 percent which depicts its adaptability in the tracking of returns to the long term equilibrium in different market conditions.
- Diagnostic tests (Jarque-Bra test) were used to ensure that the proposed model is the most appropriate to work with the data that has distributional Non-normality.

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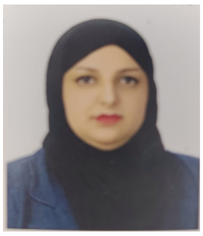

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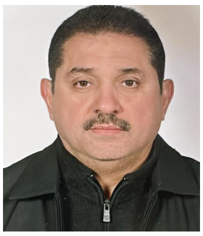

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