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SYMBOLIC DIFFERENTIATION ALGORITHMS FOR HYPERBOLIC PERTURBATION PROBLEMS WITH NEUMANN CONDITIONS USING ASYMPTOTIC FORMULAS AND UNIFORM DIFFERENCE SCHEMES

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Abstract. In this paper, we investigate asymptotic formulas and ε-uniform difference schemes for the numerical solution of perturbation problems. Unfortunately, both of these numerical methods involving sine and cosine fitting operator functions, which depend on ρ ($\rho = \tau/\varepsilon$, ε small parameter and τ stepsize in t), are highly challenging with the realization to use symbolic differential algorithms. We present a symbolic differentiation algorithm developed for the numerical solution of hyperbolic perturbation problems with Neumann boundary conditions, employing asymptotic formulas and ε-uniform difference schemes. The symbolic differential algorithms are implemented by Matlab, and the results of numerical experiments are presented.

Keywords: hyperbolic perturbation problems, asymptotic formulas, uniform difference schemes.

AMS Subject Classification: 35L90, 47D09, 34E10, 65M06.

1. Asymptotic Solution Formulas and the Uniform Difference Scheme

The theory of singular perturbation problems is a developing research field. These types of problems arise in various areas such as quantum mechanics, fluid dynamics, solid mechanics, reaction-diffusion processes, aerodynamics, etc (see, e.g., [1]-[11]). The role of asymptotic methods and uniform difference schemes in solving these problems are significant and they have been investigated by many researchers (see, e.g., [9]-[15]). Note

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that all classical difference schemes for differential equations with a small parameter in the higher order derivative are not uniform in ε convergence.

In the present paper, we consider the hyperbolic perturbation problem

$$\begin{cases} \varepsilon^2 u_{tt}(t,x) - \sum\limits_{r=1}^{n} (a_r(x) u_{x_r})x_r + \delta u(t,x) = f(t,x), \\ 0 < t < T, x = (x_1, ..., x_n) \in \Omega, \\ u(0,x) = \varphi(x), u_t(0,x) = \psi(x), \quad x \in \overline{\Omega}, \\ \frac{\partial u(t,x)}{\partial \vec{n}} = 0, \quad x \in S, 0 \le t \le T, \end{cases} \tag{1}$$

where $\Omega \subset R^n$ be a bounded open domain with smooth boundary S, $\vec{n}$ is the outward unit normal vector on S, $\overline{\Omega} = \Omega \cup S$, $[0,T] \times \Omega$, $a_r(x), \varphi(x), \psi(x)$, and $f(t,x)$ are given sufficiently smooth functions, $\delta > 0$ is the sufficiently large number, $\varepsilon > 0$ is the arbitrary positive number. We assume that $a_r(x) \ge a > 0$.

A significant improvement in efficiency is observed when the proposed ε-uniform difference scheme is compared with the classical first-order difference scheme. To illustrate the limitations of classical difference schemes that are not in ε-uniform in solving perturbation problems, we consider the following Cauchy perturbation problem subject to Neumann boundary conditions

$$\begin{cases} \varepsilon^2 u_{tt}(t,x) - u_{xx}(t,x) = f(t,x), t \in (0,1), x \in (0,\pi), \\ u(0,x) = \varphi(x), u_t(0,x) = \psi(x), x \in [0,\pi], \\ u_x(t,0) = u_x(t,\pi) = 0, t \in [0,1]. \end{cases} \tag{2}$$

One can easily verify that for $f(t,x) = e^{-t}\cos(x), \varphi(x) = \cos(x)$, and $\psi(x) = \cos(x)/\varepsilon$ the exact solution is

$$u(t,x) = \frac{1}{\varepsilon^2+1}\left(\varepsilon^2 \cos\left(\frac{t}{\varepsilon}\right) + \left(\varepsilon^2+\varepsilon+1\right)\sin\left(\frac{t}{\varepsilon}\right) + e^{-t}\right)\cos(x). \tag{3}$$

The set of a family of grid points $[0,1]_\tau \times [0,\pi]_h = \{(t_k, x_n) : t_k = k\tau, 0 \le k \le N, N\tau = 1, x_n = nh, 0 \le n \le M, Mh = \pi\}$ depending on the small parameters τ and h is considered. For the approximate solution of the Cauchy problem (2), let us use the classical first-order of accuracy difference scheme

$$\begin{cases} \varepsilon^2 \frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} - \frac{u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1}}{h^2} = f(t_k, x_n), \\ f(t_k, x_n) = e^{-t_k}\cos(x_n), 1 \le k \le N-1, 1 \le n \le M-1, \\ u_n^0 = \cos(x_n), \tau^{-1}(u_n^1 - u_n^0) = \cos(x_n)/\varepsilon, \ 0 \le n \le M, \\ u_1^{k+1} - u_0^{k+1} = 0, u_{M+1}^{k+1} - u_M^{k+1} = 0, 0 \le k \le N-1 \end{cases} \tag{4}$$

which is not uniform in ε. Numerical experiments were performed using MATLAB software, employing the modified Gauss elimination method. Approximate maximum pointwise ε-uniform errors were computed using the formula

$$E_\varepsilon^N = \max_{\substack{1 \le k \le N-1, \\ 0 \le n \le M}} \left\| u(t_k, x_n, \varepsilon) - u_{n,\varepsilon}^k \right\|_{L_2(\overline{\Omega})}. \tag{5}$$

Here, $u(t_k, x, \varepsilon)$ and $u_\varepsilon^k(x)$ are exact and numerical solutions for problem (2) at (t_k, x_n, ε), respectively.

The following table presents the error results, E_ε, associated with the approximate solution to problem (2) for various arbitrary values of ε.

In Table 1 the errors E_ε for problem (2) computed with the classical first-order difference scheme on three uniform meshes ($N = M = 20, 40, 80$) and for several values of the perturbation parameter ε. For fixed ε, the error decreases roughly by a factor of 2 when

TABLE 1. Errors, E_ε, for the approximate solution of problem (2).

Errors for the first order ε-uniform difference scheme			
ε	$N=M=20$	$N=M=40$	$N=M=80$
1	0.1495	0.0745	0.0371
10^{-1}	0.9499	0.7129	0.4578
10^{-2}	6.0173	1.9204	1.0882
10^{-3}	50.3114	25.1572	12.5788

the mesh is refined from $N=20$ to $N=40$ and then to $N=80$, which is consistent with a first-order method in the mesh size h.

However, for a fixed mesh, the error E_ε grows dramatically as ε decreases. For instance, on the finest grid $N=M=80$ we have

$$E_1 \approx 3.71\times 10^{-2}, \quad E_{10^{-2}} \approx 1.08, \quad E_{10^{-3}} \approx 12.57,$$

so the error increases by more than two orders of magnitude when ε is reduced from 1 to 10^{-3}. This clearly shows that the classical first-order difference scheme is *not* uniform with respect to ε. Hence the classical scheme does not provide an ε-uniform approximation for the singularly perturbed problem (2).

This highlights the limitations of classical difference schemes when applied to problems with small ε-values. These findings emphasize the importance of employing more advanced, $\varepsilon-$uniform difference schemes for solving singularly perturbed problems, particularly in the presence of small parameters.

Firstly, we consider asymptotic formulas for the solution of problem (1). Subsequently, we present the ε-uniform difference schemes. The following theorem provides the asymptotic formulas.

Theorem 1.1. *Suppose that the function $f(t,x)$ has $2m+3$ derivatives and*

$$\left\| f^{(2m+3)}(t,\cdot)\right\|_{L_2(\overline{\Omega})} \leq M, 0\leq t\leq T.$$

Then for small ε and even number m the following $(m+2)$-nd order asymptotic formula

$$u(t,x)=\sum_{i=0}^{m}\varepsilon^i\left[u_i(t,x)+v_i\left(\frac{t}{\varepsilon},x\right)\right]+o\left(\varepsilon^{m+2}\right) \tag{6}$$

for the solution of problem (1) holds and for small ε and odd number m the following $(m+1)$-th order asymptotic formula

$$u(t,x)=\sum_{i=0}^{m}\varepsilon^i\left[u_i(t,x)+v_i\left(\frac{t}{\varepsilon},x\right)\right]+o\left(\varepsilon^{m+1}\right) \tag{7}$$

for the solution of problem (1) holds, where the functions $u_i(t,x)$ and $v_i\left(\frac{t}{\varepsilon},x\right)$ for $i\in\{1,2,...,m\}, x\in[0,l]$ are defined by the following formulas

$$u_0(t,x)=(A^x)^{-1}f(t,x), u_1(t,x)=0, \tag{8}$$

$$u_i(t,x)=-(A^x)^{-1}u''_{i-2}(t,x), i=2,...,m,$$

$$(v_i)_{\xi\xi}(\xi,x)+A^x v_i(\xi,x)=0, i=0,1,...,m, \tag{9}$$

$$v_i(0,x)=-u_i(0,x), 1\leq i\leq m, \tag{10}$$

$$v_0(0,x)=\varphi(x)-u_0(0,x), \tag{11}$$

$$(v_i)_\xi(0,x)=u'_{i-1}(0,x), 2\leq i\leq m, \tag{12}$$

$$(v_0)_\xi (0,x) = 0, (v_1)_\xi (0,x) = \psi(x) - u_0'(0,x). \tag{13}$$

Here, the self-adjoint positive definite differential operator A^x in $L_2(\overline{\Omega})$ defined by the formula

$$A^x u(x) = -\sum_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} + \delta u(x) \tag{14}$$

with domain

$$D(A^x) = \Big\{u(x) : u(x), u_{x_r}(x), (a_r(x) u_{x_r})_{x_r} \in L_2(\overline{\Omega}),$$
$$1 \le r \le n, \quad \frac{\partial u(x)}{\partial \vec{n}} = 0, x \in S\Big\}.$$

The proof of Theorem 1.1 is based on Theorem 2.5 in [9] for the abstract perturbation problem in a Banach space and on estimates

$$\|c(t)\|_{L_2(\overline{\Omega})\to L_2(\overline{\Omega})} \le 1, \ \|(A^x)^{\frac{1}{2}} s(t)\|_{L_2(\overline{\Omega})\to L_2(\overline{\Omega})} \le 1, \ t \ge 0 \tag{15}$$

for the operator A^x defined by the formula (14). Here , $\{c(t), t \ge 0\}$ is a strongly continuous cosine operator function defined by the formula

$$c(t)\varphi(x) = \frac{e^{i(A^x)^{1/2}t} + e^{-i(A^x)^{1/2}t}}{2}\varphi(x), \tag{16}$$

and $\{s(t), t \ge 0\}$ is a strongly continuous sine operator-function defined by the formula

$$s(t)\varphi(x) = (A^x)^{-1/2}\frac{e^{i(A^x)^{1/2}t} - e^{-i(A^x)^{1/2}t}}{2i}\varphi(x). \tag{17}$$

Second, we consider the $\varepsilon-$uniform difference schemes. Many studies in the literature have presented the numerical verification of the ε-uniform difference schemes on perturbation problems for specific types of hyperbolic PDEs. However, to our knowledge, in the general case, no classic finite-difference approximation leads to uniform convergence for ε. In the present paper, the numerical realization of high-order uniform difference schemes for hyperbolic problems, in the the general case is presented for the first time. We study the high-order of approximation of single-step uniform in ε difference scheme. Let the uniform grid space on the segment $[0,T]$ be

$$[0,T]_\tau = \{t_k = k\tau, k = 0, 1, \dots, N, N\tau = T\},$$

with step τ. We denote $\rho = \tau/\varepsilon$. The high-order of accuracy two-step uniform difference scheme

$$\rho^{-2}(\omega_{k+1}(x) - 2\omega_k(x) + \omega_{k-1}(x)) = 2\rho^{-2}(c(\rho) - I)\omega_k(x) + \tau^{-1}\varepsilon^2 \widetilde{f}_k(x), \tag{18}$$

$$\widetilde{f}_k(x) = \widetilde{f}_{1,k+1}(x) + s(\rho)\widetilde{f}_{2,k}(x) - c(\rho)\widetilde{f}_{1,k}(x), 1 \le k \le N-1,$$
$$\omega_0(x) = \varphi(x), \omega_1(x) = c(\rho)\varphi(x) + s(\rho)\psi(x) + \tau f_{1,1}(x) \tag{19}$$

is considered. Here,

$$\widetilde{f}_{1,k}(x) = (\tau A^x)^{-1}[f(t_k,x) - c(\rho) f(t_{k-1},x)]$$
$$-(\tau A^x)^{-1}\sum_{j=0}^{m-1}\widetilde{B}_j f^{(j+1)}(t_{k-1},x), \tag{20}$$

$$\widetilde{f}_{2,k}(x) = -\left(\varepsilon^2\tau\right)^{-1} s(\rho) f(t_{k-1},x)$$
$$+\left(\varepsilon^2\tau\right)^{-1}\sum_{j=0}^{m-1}\widetilde{C}_j f^{(j+1)}(t_{k-1},x), 1 \le k \le N, \tag{21}$$

$$\widetilde{B}_0 = s\left(\rho\right), \widetilde{B}_1 = \varepsilon^2 (A^x)^{-1} \left(I - c\left(\rho\right)\right), \tag{22}$$

$$\widetilde{B}_{2j} = \left(\varepsilon^2 (A^x)^{-1}\right)^j \widetilde{B}_0 - \sum_{i=1}^{j} \left(\varepsilon^2 (A^x)^{-1}\right)^{j-i+1} \frac{\tau^{2i-1}}{(2i-1)!}, 2 \leq 2j \leq m-1, \tag{23}$$

$$\widetilde{B}_{2j+1} = \left(\varepsilon^2 (A^x)^{-1}\right)^j \widetilde{B}_1 - \sum_{i=1}^{j} \left(\varepsilon^2 (A^x)^{-1}\right)^{j-i+1} \frac{\tau^{2i}}{(2i)!}, 3 \leq 2j+1 \leq m-1, \tag{24}$$

$$\widetilde{C}_0 = \varepsilon^2 (A^x)^{-1} \left(I - c\left(\rho\right)\right), \tag{25}$$

$$\widetilde{C}_j = -\left(\varepsilon^2 A^x\right)^{-1} \widetilde{B}_{j-1} + \left(\varepsilon^2 A^x\right)^{-1} \frac{\tau^j}{j!}, 1 \leq j \leq m-1 \tag{26}$$

for the approximate solution of problem (1). The uniform convergence of the difference schemes in ε is given by the next theorem.

Theorem 1.2. *Suppose that the function $f\left(t,x\right)$ has $(m+1)$ derivatives and*

$$\left\| f^{(m+1)}(t,.) \right\|_{L_2(\overline{\Omega})} \leq M, 0 \leq t \leq T.$$

Then for the solution of difference problem (18)-(19) the following convergence estimate is valid:

$$\max_{0 \leq k \leq N} \left\| u_k - \omega_k \right\|_{L_2(\overline{\Omega})} \leq M\tau^m, \tag{27}$$

where M does not depend on τ and ε.

The proof of Theorem 1.2 is based on Theorem 3.3 in [9], which is for the solution of the abstract problem in a Banach space with the positive operator, and on the estimates (15).

2. Symbolic Differentiation Algorithms for the Perturbation Problem

In the present section we verify the efficiency of the proposed ε-uniform difference scheme (18)-(26). In the last section the classical first-order difference scheme (4) which is not ε-uniform, is used to solve problem (2). The error results showed that the classical difference schemes that are not uniform in ε are not useful in solving perturbation problems. Now let us use the proposed first order of accuracy ε-uniform difference scheme (18)-(26) to solve the mixed boundary value perturbation problem (2). For the realization of asymptotic formulas (6), (7), and uniform difference scheme (18)-(26) we use the following cosine operator function

$$c\left(\frac{t}{\varepsilon}\right)\varphi = \sum_{k=0}^{\infty} \frac{2}{\pi} \cos\left(\frac{kt}{\varepsilon}\right) \cos\left(kx\right) \int_0^{\pi} \varphi\left(y\right) \cos\left(ky\right) dy, \tag{28}$$

and the sine operator function

$$s\left(\frac{t}{\varepsilon}\right)\psi = \sum_{k=1}^{\infty} \frac{2}{\pi} \cos\left(kx\right) \frac{\varepsilon}{k} \sin\left(\frac{kt}{\varepsilon}\right) \int_0^{\pi} \psi\left(y\right) \cos kydy. \tag{29}$$

2.1. **Symbolic Differential Algorithm and Numerical Experiments.** We now present the symbolic differential algorithm associated with the ε-uniform difference scheme defined by equations (18)–(26). As part of the numerical experiments, various initial conditions are considered to evaluate the performance and robustness of the scheme. This section introduces two new key algorithms that form the basis of the computational approach.

Algorithm 1: ε–Uniform Difference Scheme

Input: Symbolic variables: x, y, xi, m, s, k, kk, mm, t, b, Independent variables: xx, tt, ϵ, τ.

Output: Approximate and exact solutions together with error evaluations for $j = 1, \ldots, N$.

(1) Define the initial conditions and express the exact solution symbolically.
(2) Construct the approximation formula and evaluate it at all mesh points.
(3) Apply the cosine operator (Eq. 28) using symbolic integration.
(4) Apply the sine operator (Eq. 29) to the necessary symbolic expressions.
(5) Compute absolute and relative errors for various choices of ϵ, τ, x, and N.

Algorithm 2: Asymptotic Solution Procedure

Input: Symbolic expressions for $u_0(t,x)$, $u_1(t,x)$, and the operator form of the differential equation.

Output: Asymptotic approximation and associated error estimates.

(1) The initial conditions and the exact solution defined symbolically.
(2) Compute the higher-order time derivatives as described in Theorem 1.1.
(3) Construct the asymptotic expansion from these derivatives.
(4) Apply the cosine and sine operators to v_0 and v_t.
(5) Form the asymptotic approximation by summing the resulting series elements.
(6) Evaluate the exact solution and compute pointwise errors for $j = 1, \ldots, N$.

All symbolic computations are carried out by MATLAB R2024b with symbolic tools. While the source scripts are not included due to journal policy, the algorithms above provide a comprehensive, reproducible framework for reconstructing the numerical experiments.

We present the errors in the tables below at different ε, τ, x, and N values. Note that our computations are restricted to a finite range of parameter values. The errors for asymptotic solutions are computed by the formula

$$E_\varepsilon(t,x) = \max_{t\in[0,1]} \left\| u(t,x,\varepsilon) - \widetilde{u}(t,x,\varepsilon) \right\|_{L_2(\overline{\Omega})}. \tag{30}$$

Here, $u(t,x,\varepsilon)$ is the exact solution and $\widetilde{u}(t,x,\varepsilon)$ is the asymptotic solution of problem (2) at the point (t,x,ε). The corresponding numerical values are reported in Table 2 for ε ranging from 10^{-1} to 10^{-5}.

From Table 2 we observe that the error E_ε decreases very rapidly as ε becomes smaller. Even for the relatively moderate value $\varepsilon = 10^{-1}$, the error is already small ($E_{10^{-1}} \approx 6.3 \times 10^{-2}$), and when ε is reduced to 10^{-3} the error drops below 10^{-3}. For $\varepsilon = 10^{-4}$ and $\varepsilon = 10^{-5}$, the errors are of orders 10^{-5} and 10^{-7}, respectively, which indicates a very high level of accuracy in the singular perturbation regime. The errors consistently decrease as ε decreases, verifying the reliability of the asymptotic method for small ε values. The numerical results indicate a good convergence for the asymptotic solution.

TABLE 2. Errors, E_ε, for the asymptotic solutions of problem (2).

Errors for asymptotic solutions	
ε	$N=10,\ x=1,\ t=1$
10^{-1}	0.0630
10^{-2}	0.0054
10^{-3}	8.9383e-04
10^{-4}	3.3030e-05
10^{-5}	3.8625e-07

The approximate maximum pointwise ε-uniform errors are obtained with the formula (5) and the exact maximum point-wise errors are computed with

$$E_\varepsilon^N = \max_{1\le k\le N-1} \left\| u\left(t_k, x, \varepsilon\right) - u_\varepsilon^k \right\|_{L_2(\overline{\Omega})}$$

for $\Omega = (0,\pi)$. Here, $u\left(t_k, x, \varepsilon\right)$ and $u_\varepsilon^k\left(x\right)$ are exact and numerical solutions for problem (2) at (t_k, x_n, ε), respectively.

The maximum pointwise errors are computed by

$$E^N = \max_{\varepsilon=1,10^{-1},\ldots,10^{-20}} E_\varepsilon^N$$

for the ε-uniform high order of accuracy difference scheme (18)-(26) corresponding to problem (2) for a variety of ε, τ, x values.

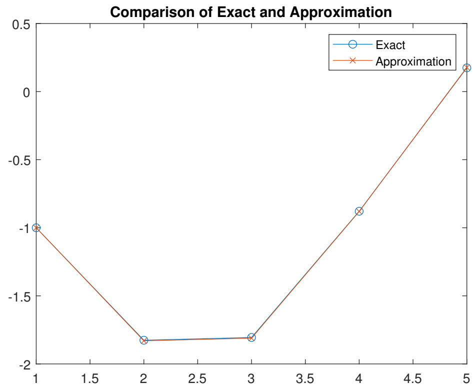
TABLE 3. Errors, E_ε^N, for the ε-uniform difference scheme (18)-(26).

Errors for ε-Uniform DS		
ε	$\tau=10^{-1}$, $N=6$	$\tau=10^{-1}$, $N=8$
10^{-1}	9.992e-16	9.992e-16
10^{-2}	3.8858e-15	3.8858e-15
10^{-3}	6.1562e-14	6.1562e-14

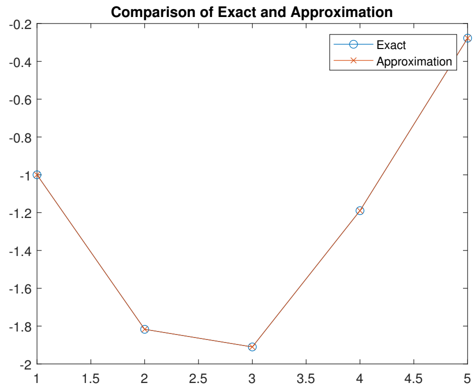
In Table 3 the errors E_ε^N obtained by the ε-uniform difference scheme (18)–(26) for problem (2) are presented. We observe that, for ε ranging from 10^{-1} down to 10^{-3}, the errors remain extremely small, essentially at the level of machine precision (between 10^{-15} and 10^{-14}), even on very coarse grids ($N=6$ and $N=8$ with $\tau=10^{-1}$). Moreover, the errors are practically independent of both ε and the number of spatial nodes N. This behaviour clearly demonstrates the ε-uniform convergence and high efficiency of the proposed scheme.

These findings confirm that the ε-uniform difference scheme is not only accurate but also consistent across a wide range of parameters, making it an effective tool for solving such problems.

The line graphs in the following figures illustrate the comparison between the ε-uniform high order of accuracy difference scheme solution for (18)-(26) and the exact solution for problem (2). In the Figures, the curves are parallel and overlapped as ε goes small, and this indicates that the proposed scheme converges independently of the values of the perturbation parameters.



(A) $N = 8$, $\varepsilon = 10^{-1}$



(B) $N = 8$, $\varepsilon = 10^{-3}$

FIGURE 1. Comparison of figures with different ε values.

3. Conclusions

In this paper, symbolic differentiation algorithms for the asymptotic and uniform difference schemes solutions of the initial value perturbation problem (1) are presented. The high order of accuracy two-step uniform difference schemes for the solution to this problem are presented. In the future, we are interested in the study of asymptotic formulas and high-order of accuracy uniform difference schemes for the solution of hyperbolic nonlocal perturbation problem

$$\begin{cases} \varepsilon^2 \frac{d^2u(t)}{dt^2} + Au(t) = f(t), 0 < t < T, \\ u(0) = \alpha u(T) + \varphi, \; u^{'}(0) = \beta u^{'}(T) + \psi. \end{cases} \tag{31}$$

Absolutely stable two-step difference schemes for the numerical solution of problem (31), when $\varepsilon = 1$, were investigated for local and nonlocal hyperbolic problems (see, e.g., [11], [12]).

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