TWMS J. App. Eng. Math. V.2, N.1, 2012, pp. 75-93

COERCIVE SOLVABILITY OF PARABOLIC DIFFERENTIAL EQUATIONS WITH DEPENDENT OPERATORS

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Abstract. In the present paper the nonlocal-boundary value problem for the differential equation of parabolic type

$$v'(t)+A(t)v(t)=f(t)\quad (0\le t\le T), v(0)=v(\lambda)+\varphi, 0<\lambda\le T$$

in an arbitrary Banach space with the linear positive operators $A(t)$ is considered. The well-posedness of this problem is established in Banach spaces $C_0^{\beta,\gamma}(E)$ of all continuous functions E-valued functions $\varphi(t)$ on $[0,T]$ satisfying a Hölder condition with a weight $(t+\tau)^{\gamma}$. New exact estimates in Holder norms for the solution of three nonlocal-boundary value problems for parabolic equations are obtained.

Keywords: Parabolic equations, NBV problems, Banach spaces, positive operators.

AMS Subject Classification: 47D06, 35K20, 35M10

1. Introduction. A Cauchy Problem

It is known that (see, e.g., [1]- [5] and the references given therein) many applied problems in fluid mechanics, other areas of physics and mathematical biology were formulated into nonlocal mathematical models. However, such problems were not well investigated in general.

In the paper [6] the well-posedness in the spaces of smooth functions of the nonlocal boundary value problem

$$v^{'}(t)+Av(t)=f(t)(0\le t\le 1), v(0)=v(\lambda)+\mu\ (0<\lambda\le 1)$$

for the differential equation in an arbitrary Banach space E with the strongly positive operator A was established. The importance of coercive (well-posedness) inequalities is well-known [10] and [32].

Finally, methods for numerical solutions of the evolution differential equations have been studied extensively by many researchers (see [7]-[9], [11]- [32] and the references therein).

Before going to discuss well-posedness of nonlocal-boundary value problem, let us consider the abstract Cauchy problem for the differential equation

$$v'(t)+A(t)v(t)=f(t)\quad (0\le t\le T), v(0)=v_0 \tag{1}$$

in an arbitrary Banach space E with the linear (unbounded) operators $A(t)$. Here $v(t)$ and $f(t)$ are the unknown and the given functions, respectively, defined on $[0,T]$ with values

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§ Manuscript received 24 January 2012.

in E. The derivative $v'(t)$ is understood as the limit in the norm of E of the corresponding ratio of differences. $A(t)$ is a given, closed, linear operator in E with domain $D(A(t)) = D$, independent of t and dense in E. Finally, v_0 is a given element of E.

A function $v(t)$ is called a solution of the problem (1.1) if the following conditions are satisfied:

i. $v(t)$ is continuously differentiable on the segment $[0, T]$. The derivative at the endpoints of the segment are understood as the appropriate unilateral derivatives.
ii. The element $v(t)$ belongs to $D = D(A(t))$ for all $t \in [0, T]$, and the function $A(t)v(t)$ is continuous on $[0, T]$.
iii. $v(t)$ satisfies the equation and the initial condition (1.1).

A solution of problem (1.1) defined in this manner will from now on be referred to as a solution of problem (1.1) in the space $C(E) = C([0, T], E)$. Here $C(E)$ stands for the Banach space of all continuous functions $\varphi(t)$ defined on $[0, T]$ with values in E equipped with the norm

$$||\varphi||_{C(E)} = \max_{0 \le t \le T} ||\varphi(t)||_E.$$

From the existence of the such solutions evidently follows that $f(t) \in C(E)$ and $v_0 \in D$.

We say that the problem (1.1) is well posed in $C(E)$ if the following conditions are satisfied:

1. Problem (1.1) is uniquely solvable for any $f(t) \in C(E)$ and any $v_0 \in D$. This means that an additive and homogeneous operator $v(t) = v(t; f(t), v_0)$ is defined which acts from $C(E) \times D$ to $C(E)$ and gives the solution of problem (1.1) in $C(E)$.
2. $v(t; f(t), v_0)$, regarded as an operator from $C(E) \times D$ to $C(E)$, is continuous. Here $C(E) \times D$ is understood as the normed space of the pairs $(f(t), v_0)$, $f(t) \in C(E)$ and $v_0 \in D$, equipped with the norm

$$||(f(t), v_0)||_{C(E)\times D} = ||f||_{C(E)} + ||v_0||_D.$$

By Banach's theorem in $C(E)$ and these properties one has coercive inequality

$$\left\|v'\right\|_{C(E)} + \|A(.)v\|_{C(E)} \le M_C[\|f\|_{C(E)} + \ ||v_0||_D], \tag{2}$$

where M_C $(1 \le M_C < +\infty)$ does not depend on v_0 and $f(t)$.

The inequality (2) is called the coercivity inequality in $C(E)$ for (1.1). If $A(t) = A$, then the coercivity inequality implies the analyticity of the semigroup $\exp\{-sA\}(s \ge 0)$, i.e. the following estimates

$$\|\exp\ (-sA)\|_{E\to E}, \|sA\exp(-sA)\|_{E\to E} \le M(s > 0)$$

hold for some $M \in [1, +\infty)$. Thus, the analyticity of the semigroup $\exp\{-sA\}(s \ge 0)$ is a necessary for the well-posedness of problem (1.1) in $C(E)$. Unfortunately, the analyticity of the semigroup $\exp\{-sA\}$ $(s \ge 0)$ is not a sufficient for the well-posedness of problem (1.1) in $C(E)$.

Suppose that for each $t\ \epsilon[0, T]$ the operator $-A(t)$ generates an analytic semigroup $\exp\{-sA(t)\}$ $(s \ge 0)$ with exponentially decreasing norm, when $s \longrightarrow +\infty$, i.e. the following estimates

$$\|\exp\ (-sA(t))\|_{E\to E}, \|sA(t)\exp(-sA(t))\|_{E\to E} \le Me^{-\delta s}(s > 0) \tag{3}$$

hold for some $M \in [1, +\infty)$, $\delta \in (0, +\infty)$. From this inequality it follows the operator $A^{-1}(t)$ exists and bounded, and hence $A(t)$ is closed in $C(E)$.

Suppose that the operator $A(t)A^{-1}(s)$ is Hölder continuous in t in the uniform operator topology for each fixed s, that is,

$$||[A(t) - A(\tau)]A^{-1}(s)||_{E\to E} \leq M|t-\tau|^{\varepsilon}, 0 < \varepsilon \leq 1, \tag{4}$$

where M and ε are positive constants independent of t, s and τ for $0 \leq t, s, \tau \leq T$.

If the function $f(t)$ is not only continuous, but also continuously differentiable on $[0, T]$, and $v_0 \in D$, it is easy to show that the formula

$$v(t) = v(t,0)v_0 + \int_0^t v(t,s)f(s)ds \tag{5}$$

gives a solution of problem (1.1). Here $v(t,s)$ is the fundamental solution of (1.1).

Now we will give lemmas and estimates from [13] concerning the semigroup $\exp\{-sA(t)\}$ $(s \geq 0)$ and the fundamental solution $v(t,s)$ of (1.1) and theorem on well-posedness of (1.1) which will be useful in the sequel.

Lemma 1.1. *For any $0 < s < s+\tau < T$, $0 \leq t \leq T$ and $0 \leq \alpha \leq 1$ one has the inequality*

$$\|exp\ (-sA(t)) - exp\ \ (-(s+\tau)A(t))\|_{E\to E} \leq M\frac{\tau^{\alpha}}{(s+\tau)^{\alpha}}, \tag{6}$$

where M does not depend on α, t, s, and τ.

Lemma 1.2. *For any $0 \leq s, \tau, t \leq T$ and $0 \leq \varepsilon \leq 1$ the following estimates hold:*

$$\left\|[exp\ (-tA(\tau)) - exp\ \ (-sA(\tau))]A^{-1}(\tau)\right\|_{E\to E} \leq M|t-s|e^{-\delta\ min\ \{t,s\}}, \tag{7}$$

$$\left\|A(t)[exp\ (-tA(\tau)) - exp\ \ (-sA(\tau))]A^{-2}(\tau)\right\|_{E\to E} \leq M|t-s|e^{-\delta\ min\ \{t,s\}}, \tag{8}$$

where $M \geq 0$ and $\delta > 0$ do not depend on ε, t, s, and τ.

Lemma 1.3. *For any $0 \leq s < t \leq T$ and $u\ \epsilon D$ the following identities hold:*

$$v(t,s)u = exp\{-(t-s)A(s)\}u \tag{9}$$

$$+\int_s^t v(t,z)[A(s) - A(z)]A^{-1}(s)exp\ \{-(z-s)A(s)\}A(s)udz,$$

$$v(t,s)u = exp\{-(t-s)A(t)\}u \tag{10}$$

$$+\int_s^t exp\ \{-(t-z)A(t)\}[A(z) - A(t)]v(z,s)udz.$$

Lemma 1.4. *For any $0 \leq s < t \leq t+r \leq T$, $0 \leq \alpha \leq 1$ and $0 \leq \varepsilon \leq 1$ the following estimates hold:*

$$\|v(t,s)\|_{E\to E} \leq M, \tag{11}$$

$$\left\|A(t)v(t,s)A^{-1}(s)\right\|_{E\to E} \leq M, \tag{12}$$

$$\|A(t)v(t,s)\|_{E\to E} \leq \frac{M}{t-s}, \tag{13}$$

where $M \geq 0$ does not depend on ε, t and s.

With the help of $A(t)$ we introduce the fractional spaces $E_\alpha(E,A(t))$, $0<\alpha<1$, consisting of all $v\ \epsilon E$ for which the following norms are finite:

$$\| v \|_{E_\alpha}= \sup_{z>0} z^{1-\alpha} \| A(t)exp\{-zA(t)\}v \|_E .$$

From (1.3) and (1.4) it follows that $E_\alpha(E,A(t))=E_\alpha(E,A(0))$ for all $0<\alpha<1$ and $0\le t\le T$.

A function $v(t)$ is said to be a solution of problem (1.1) in $F(E)$ if it is a solution of this problem in $C(E)$ and the function $v'(t)$ and $A(t)v(t)$ belong to $F(E)$.

As in the case of the space $C(E)$, we say that the problem (1.1) is well-posed in $F(E)$, if the following two conditions are satisfied:

1. For any $f\in F(E)$ and $v_0\in D$ there exists the unique solution $v(t)=v(t;f(t),v_0)$ in $F(E)$ of problem (1.1).This means that an additive and homogeneous operator $v(t;f(t),v_0)$ is defined which acts from $F(E)\times D$ to $F(E)$ and gives the solution of (1.1) in $F(E)$.
2. $v(t;f(t),v_0)$, regarded as an operator from $F(E)\times D$ to $F(E)$, is continuous.Here $F(E)\times D$ is understood as the normed space of the pairs $(f(t),v_0)$, $f(t)\in F(E)$ and $v_0\in D$, equipped with the norm

$$||(f(t),v_0)||_{F(E)\times D}=||f||_{F(E)}+||v_0||_D.$$

We set $F(E)$ equal to $C_0^{\beta,\gamma}(E),(0\le\gamma\le\beta,0<\beta<1)$ space, obtained by completion of the set of all smooth E-valued functions $\varphi(t)$ on $[0,T]$ with respect to the norm

$$\| \varphi \|_{C_0^{\beta,\gamma}(E)}= \max_{0\le t\le T}||\varphi(t)||_E+\sup_{0\le t<t+\tau\le T}\frac{(t+\tau)^\gamma \| \varphi(t+\tau)-\varphi(t) \|_E}{\tau^\beta}.$$

Let us give, the following theorem on well-posedness of (1.1) in $C_0^{\beta,\gamma}(E)$ from [13].

Theorem 1.1. *Suppose $v_0'\in E_{\beta-\gamma}$, $f(t)\in C_0^{\beta,\gamma}(E)(0\le\gamma\le\beta,0<\beta<1)$. Suppose that the assumptions (1.3) and (1.4) hold and $0<\beta\le\varepsilon<1$. Then for the solution $v(t)$ in $C_0^{\beta,\gamma}(E)$ of the Cauchy problem (1.1) the coercive inequalities*

$$\| v' \|_{C(E_{\beta-\gamma})}\le M[\| v_0' \|_{E_{\beta-\gamma}}+\beta^{-1}(1-\beta)^{-1}\| f \|_{C_0^{\beta,\gamma}(E)}],$$

$$\| v' \|_{C_0^{\beta,\gamma}(E)}+\| A(.)v \|_{C_0^{\beta,\gamma}(E)}$$

$$\le M[|v_0'|_0^{\beta,\gamma}+\beta^{-1}(1-\beta)^{-1}\| f \|_{C_0^{\beta,\gamma}(E)}]$$

hold, where M does not depend on β,γ,v_0' and $f(t)$. Here, $|w|_0^{\beta,\gamma}$ denotes norm of the Banach space $E_0^{\beta,\gamma}$ consists of those $w\in E$ for which the norm

$$|w|_0^{\beta,\gamma}=\max_{0\le z\le T}||e^{-zA(t)}w||_E+\sup_{0\le z<z+\tau\le T}\tau^{-\beta}(z+\tau)^\gamma||(e^{-(z+\tau)A(t)}-e^{-zA(t)})w||_E$$

is finite.

In the present paper the nonlocal-boundary value problem for differential equation of parabolic type

$$v'(t)+A(t)v(t)=f(t)\quad(0\le t\le T),v(0)=v(\lambda)+\varphi,0<\lambda\le T \tag{14}$$

in an arbitrary Banach space with the linear positive operators $A(t)$ is considered. The well-posedness of problem (14) in $C_0^{\beta,\gamma}(E)$ spaces is established. New exact estimates in Holder norms for the solution of three nonlocal-boundary value problems for parabolic equations are obtained.

2. Nonlocal Boundary Value Problem. Well-Posedness

Now we will give lemmas on the fundamental solution $v(t,s)$ of (1).

Lemma 2.1. *Assume that* $A(t)A(p)^{-1} = A(t+\lambda)A(p)^{-1}, p \in [0,T]$ *for any* $0 \leq t \leq t+\lambda$ *Then, for any* $0 \leq s < t \leq t+\lambda$ *and* $u \ \epsilon D$ *the following identity holds*

$$v(t,s)u = v(t+\lambda,s)u. \tag{15}$$

The proof of Lemma 2.1 is based on identities (9) and (10).

Lemma 2.2. *Under the assumption of Lemma 2.1 there exists the inverse of the operator* $I - v(\lambda,0)$ *in* E *and the following estimate holds*

$$\left\|(I - v(\lambda,0))^{-1}\right\|_{E\to E} \leq M(\lambda), \tag{16}$$

$$\left\|A(0)\left(I - v(\lambda,0)\right)^{-1}A(\lambda)^{-1}\right\|_{E\to E} \leq M(\lambda). \tag{17}$$

The proof of Lemma 2.2 is based on identity (15).

A function $v(t)$ is called a solution of the problem (14) if the following conditions are satisfied:

i. $v(t)$ is continuously differentiable on the segment $[0,T]$.
ii. The element $v(t)$ belongs to D for all $t \in [0,T]$, and the function $A(t)v(t)$ is continuous on $[0,T]$.
iii. $v(t)$ satisfies the equation and the nonlocal boundary condition (14).

We say that the problem (14) is well posed in $C(E)$ if the following conditions are satisfied:

1. Problem (14) is uniquely solvable for any $f(t) \in C(E)$ and any $\varphi \in D$. This means that an additive and homogeneous operator $v(t) = v(t; f(t), \varphi)$ is defined which acts from $C(E) \times D$ to $C(E)$ and gives the solution of problem (1.1) in $C(E)$.
2. $v(t; f(t), \varphi)$, regarded as an operator from $C(E) \times D$ to $C(E)$, is continuous. Here $C(E) \times D$ is understood as the normed space of the pairs $(f(t), \varphi)$, $f(t) \in C(E)$ and $\varphi \in D$, equipped with the norm

$$||(f(t),\varphi)||_{C(E)\times D} = ||f||_{C(E)} + ||\varphi||_D.$$

By Banach's theorem in $C(E)$ and these properties one has coercive inequality

$$\left\|v'\right\|_{C(E)} + \|A(.)v\|_{C(E)} \leq M_C[\|f\|_{C(E)} + \ ||\varphi||_D], \tag{18}$$

where M_C $(1 \leq M_C < +\infty)$ does not depend on φ and $f(t)$.

The inequality (18) is called the coercivity inequality in $C(E)$ for (14). If $A(t) = A$, then the coercivity inequality implies the analyticity of the semigroup $\exp\{-sA\}(s \geq 0)$. Thus, the analyticity of the semigroup $\exp\{-sA\}(s \geq 0)$ is a necessary for the well-posedness of problem (14) in $C(E)$. Unfortunately, the analyticity of the semigroup $\exp\{-sA\}$ $(s \geq 0)$ is not a sufficient for the well-posedness of problem (14) in $C(E)$.

A function $v(t)$ is said to be a solution of problem (14) in $F(E)$ if it is a solution of this problem in $C(E)$ and the function $v'(t)$ and $A(t)v(t)$ belong to $F(E)$.

As in the case of the space $C(E)$, we say that the problem (14) is well-posed in $F(E)$, if the following two conditions are satisfied:

1. For any $f \in F(E)$ and $\varphi \in D$ there exists the unique solution $v(t) = v(t; f(t), \varphi)$ in $F(E)$ of problem (14).This means that an additive and homogeneous operator $v(t; f(t), \varphi)$ is defined which acts from $F(E) \times D$ to $F(E)$ and gives the solution of (14) in $F(E)$.
2. $v(t; f(t), \varphi)$, regarded as an operator from $F(E) \times D$ to $F(E)$, is continuous.Here $F(E) \times D$ is understood as the normed space of the pairs $(f(t), \varphi)$, $f(t) \in F(E)$ and $\varphi \in D$, equipped with the norm

$$||(f(t), \varphi)||_{F(E)\times D} = ||f||_{F(E)} + ||\varphi||_D.$$

The main result of present paper is the following theorem on well-posedness of (14) in $C_0^{\beta,\gamma}(E)$.

Theorem 2.1. *Suppose $A(0)\varphi + f(\lambda) - f(0) \in E_{\beta-\gamma}$, $f(t) \in C_0^{\beta,\gamma}(E)(0 \le \gamma \le \beta, 0 < \beta < 1)$. Suppose that the assumptions (1.3), (1.4) and (15) hold and $0 < \beta \le \varepsilon < 1$. Then for the solution $v(t)$ in $C_0^{\beta,\gamma}(E)$ of the nonlocal boundary value problem (14) the coercive inequalities*

$$\| v' \|_{C(E_{\beta-\gamma})} \le M(\lambda)[\| A(0)\varphi + f(\lambda) - f(0) \|_{E_{\beta-\gamma}} + \beta^{-1}(1-\beta)^{-1} \| f \|_{C_0^{\beta,\gamma}(E)}],$$

$$\| v' \|_{C_0^{\beta,\gamma}(E)} + \| A(.)v \|_{C_0^{\beta,\gamma}(E)}$$

$$\le M(\lambda)[|A(0)\varphi + f(\lambda) - f(0)|_0^{\beta,\gamma} + \beta^{-1}(1-\beta)^{-1} \| f \|_{C_0^{\beta,\gamma}(E)}]$$

hold, where $M(\lambda)$ does not depend on β, γ, φ and $f(t)$.

Proof. If $v(t)$ is a solution in $C_0^{\beta,\gamma}(E)$ of problem (14), then it is a solution in $C(E)$ of this problem. Hence, by (5), we get the following representation for the solution of problem (14)

$$\begin{aligned} v(t) &= v(t,0)v(0) + \int_0^t v(t,s)f(s)ds, \\ v(0) &= (I - v(\lambda,0))^{-1} \left(\int_0^\lambda v(\lambda,s)f(s)ds + \varphi \right). \end{aligned} \tag{19}$$

Using equation (14) and formula (19), we get

$$v_0' = v'(0) = -A(0)v(0) + f(0) = -A(0)\left(I - v(\lambda,0)\right)^{-1} \left(\int_0^\lambda v(\lambda,s)f(s)ds + \varphi \right) + f(0) \tag{20}$$

$$= A(0)(I - v(\lambda,0))^{-1} \int_0^\lambda v(\lambda,s)(f(\lambda) - f(s))ds$$

$$-A(0)(I - v(\lambda,0))^{-1} \int_0^\lambda v(\lambda,s)[A(\lambda) - A(s)]A^{-1}(\lambda)f(\lambda)ds$$

$$-A(0)(I - v(\lambda,0))^{-1} \left((I - v(\lambda,0))A^{-1}(\lambda)f(\lambda) + \varphi\right) + f(0)$$

$$= A(0)(I - v(\lambda,0))^{-1} \int_0^\lambda v(\lambda,s)(f(\lambda) - f(s))ds$$

$$-A(0)(I-v(\lambda,0))^{-1}\int\limits_0^\lambda v(\lambda,s)[A(\lambda)-A(s)]A^{-1}(\lambda)f(\lambda)ds$$

$$-A(0)A^{-1}(\lambda)f(\lambda)-A(0)(I-v(\lambda,0))^{-1}\varphi+f(0)$$

$$=A(0)(I-v(\lambda,0))^{-1}\int\limits_0^\lambda v(\lambda,s)(f(\lambda)-f(s))ds$$

$$-A(0)(I-v(\lambda,0))^{-1}\int\limits_0^\lambda v(\lambda,s)[A(\lambda)-A(s)]A^{-1}(\lambda)f(\lambda)ds$$

$$+A(0)(I-v(\lambda,0))^{-1}A^{-1}(0)\ \ (-A(0)\varphi-f(\lambda)+f(0))$$

$$+A(0)(I-v(\lambda,0))^{-1}\ v(\lambda,0)\ \ \left(A^{-1}(\lambda)f(\lambda)-A^{-1}(0)f(0)\right)$$
$$+A(0)(I-v(\lambda,0))^{-1}v(\lambda,0)\ A^{-1}(\lambda)\left(A(\lambda)-A(0)\right)\ A^{-1}(0)\ f(\lambda)$$
$$=K_1+K_2+K_3+K_4,$$

where

$$K_1=A(0)(I-v(\lambda,0))^{-1}\int\limits_0^\lambda v(\lambda,s)(f(\lambda)-f(s))ds,$$

$$K_2=-A(0)(I-v(\lambda,0))^{-1}\int\limits_0^\lambda v(\lambda,s)[A(\lambda)-A(s)]A^{-1}(\lambda)f(\lambda)ds,$$

$$K_3=A(0)(I-v(\lambda,0))^{-1}A^{-1}(0)\ \ (-A(0)\varphi-f(\lambda)+f(0)),$$

$$\begin{aligned}K_4 \ &= \ A(0)(I-v(\lambda,0))^{-1}\ v(\lambda,0)\ \ \left(A^{-1}(\lambda)f(\lambda)-A^{-1}(0)f(0)\right)\\ &\quad +A(0)(I-v(\lambda,0))^{-1}v(\lambda,0)\ A^{-1}(\lambda)\left(A(\lambda)-A(0)\right)\ A^{-1}(0)\ f(\lambda).\end{aligned}$$

Then the proof of Theorem 2.1 is based on the Theorem 1.1 and the following estimates

$$\| v_0' \|_{E_{\beta-\gamma}}\leq M(\lambda)\left[\| -A(0)\varphi-f(\lambda)+f(0)\|_{E_{\beta-\gamma}}+\beta^{-1}(1-\beta)^{-1}\| f \|_{C_0^{\beta,\gamma}(E)}\right], \quad (21)$$

$$|v_0'|_0^{\beta,\gamma}\leq M(\lambda)\left[|-A(0)\varphi-f(\lambda)+f(0)|_0^{\beta,\gamma}+\beta^{-1}(1-\beta)^{-1}\| f \|_{C_0^{\beta,\gamma}(E)}\right]. \quad (22)$$

Let us estimate K_m for any $m=1,2,3,4$ in $E_{\beta-\gamma}$ and $E_0^{\beta,\gamma}$, separately. We start with K_1. Applying the inequality (17), we get

$$\|K_1\|_{E_{\beta-\gamma}}\leq M(\lambda)\left\|A(\lambda)\int\limits_0^\lambda v(\lambda,s)(f(\lambda)-f(s))ds\right\|_{E_{\beta-\gamma}}, \quad (23)$$

$$|K_1|_0^{\beta,\gamma}\leq M(\lambda)\left|A(\lambda)\int\limits_0^\lambda v(\lambda,s)(f(\lambda)-f(s))ds\right|_0^{\beta,\gamma}. \quad (24)$$

Using estimates (3), (11), (12) and (15), we obtain

$$z^{1-\beta+\gamma} \parallel A(\lambda)exp\{-zA(\lambda)\}A(\lambda)\int_0^\lambda v(\lambda,s)(f(\lambda)-f(s))ds||_E \tag{25}$$

$$\leq z^{1-\beta+\gamma}\int_0^\lambda ||A^2(\lambda)\exp\{-zA(\lambda)\}v(t,s)||_{E\to E}||f(\lambda)-f(s)||_E ds$$

$$\leq Mz^{1-\beta+\gamma}\int_0^\lambda \min\left[\frac{1}{z^2},\frac{1}{(\lambda-s)^2}\right](\lambda-s)^\beta\lambda^{-\gamma}ds||f||_{C_0^{\beta,\gamma}(E)}$$

$$\leq M_1 z^{1-\beta+\gamma}\int_0^\lambda \frac{(\lambda-s)^\beta ds}{(z+\lambda-s)^2\lambda^\gamma}||f||_{C_0^{\beta,\gamma}(E)}$$

for all $z>0$. We will prove that

$$z^{1-\beta+\gamma}\int_0^\lambda \frac{(\lambda-s)^\beta ds}{(z+\lambda-s)^2\lambda^\gamma}\leq\frac{1}{\beta(1-\beta)} \tag{26}$$

for any $z>0$.If $z\leq\lambda$, then

$$z^{1-\beta+\gamma}\int_0^\lambda \frac{(\lambda-s)^\beta ds}{(z+\lambda-s)^2\lambda^\gamma}\leq z^{1-\beta}\int_0^\lambda\frac{ds}{(z+\lambda-s)^{2-\beta}}\leq\frac{1}{1-\beta}.$$

If $\lambda\leq z$, then

$$z^{1-\beta+\gamma}\int_0^\lambda \frac{(\lambda-s)^\beta ds}{(z+\lambda-s)^2\lambda^\gamma}\leq\frac{1}{z^{\beta-\gamma}\lambda^\gamma}\int_0^\lambda\frac{ds}{(\lambda-s)^{1-\beta}}=\frac{\lambda^{\beta-\gamma}}{\beta z^{\beta-\gamma}}<\frac{1}{\beta}.$$

From these estimates it follows (26) . Applying (26) , (23), (25), we get

$$\|K_1\|_{E_{\beta-\gamma}}\leq\frac{M(\lambda)}{\beta(1-\beta)}||f||_{C_0^{\beta,\gamma}(E)}. \tag{27}$$

Using estimates (3), (11), (12) and (15), we obtain

$$\parallel exp\{-zA(\lambda)\}A(\lambda)\int_0^\lambda v(\lambda,s)(f(\lambda)-f(s))ds||_E \tag{28}$$

$$\leq\int_0^\lambda ||A(\lambda)\exp\{-zA(\lambda)\}v(t,s)||_{E\to E}||f(\lambda)-f(s)||_E ds$$

$$\leq M\int_0^\lambda \min\left[\frac{1}{z},\frac{1}{(\lambda-s)}\right](\lambda-s)^\beta\lambda^{-\gamma}ds||f||_{C_0^{\beta,\gamma}(E)}$$

$$\leq M_1 \int\limits_0^\lambda \frac{1}{z+\lambda-s}(\lambda-s)^\beta \lambda^{-\gamma} ds ||f||_{C_0^{\beta,\gamma}(E)} \leq M_1 \int\limits_0^\lambda \frac{ds}{(\lambda-s)^{1-\beta}} \lambda^{-\gamma} ds ||f||_{C_0^{\beta,\gamma}(E)}$$

$$\leq \frac{M_1}{\beta} \lambda^{\beta-\gamma} ||f||_{C_0^{\beta,\gamma}(E)} \leq \frac{M_1}{\beta} T^{\beta-\gamma} ||f||_{C_0^{\beta,\gamma}(E)}$$

for any $z>0$. If $\lambda \leq \tau+z$, then Using estimates (6) for $\alpha=\beta$, (11), (12) and (15), we obtain

$$\tau^{-\beta}(z+\tau)^\gamma \left\| (e^{-(z+\tau)A(\lambda)} - e^{-zA(\lambda)}) A(\lambda) \int\limits_0^\lambda v(\lambda,s)(f(\lambda)-f(s))ds \right\|_E \tag{29}$$

$$\leq M\tau^{-\beta}(z+\tau)^\gamma \frac{\tau^\beta}{(z+\tau)^\beta} \left\| A(\lambda) \int\limits_0^\lambda v(\lambda,s)(f(\lambda)-f(s))ds \right\|_E$$

$$\leq \frac{M_1}{(z+\tau)^{\beta-\gamma}} \int\limits_0^\lambda \frac{ds}{(\lambda-s)^{1-\beta}} \lambda^{-\gamma} ||f||_{C_0^{\beta,\gamma}(E)} \leq \frac{M_1}{(z+\tau)^{\beta-\gamma}} \frac{\lambda^{\beta-\gamma}}{\beta} ||f||_{C_0^{\beta,\gamma}(E)} \leq \frac{M_2}{\beta} ||f||_{C_0^{\beta,\gamma}(E)}$$

for any $0 \leq z < z+\tau \leq T$. If $\tau+z \leq \lambda$ and $\tau \leq z$, then using estimates (12) and (3), we obtain

$$\tau^{-\beta}(z+\tau)^\gamma \left\| (e^{-(z+\tau)A(\lambda)} - e^{-zA(\lambda)}) A(\lambda) \int\limits_0^\lambda v(\lambda,s)(f(\lambda)-f(s))ds \right\|_E \tag{30}$$

$$\leq M\tau^{-\beta}(z+\tau)^\gamma \int\limits_0^\lambda \left\| A(\lambda)(e^{-(z+\tau)A(\lambda)} - e^{-zA(\lambda)}) v(\lambda,s)(f(\lambda)-f(s)) \right\|_E ds$$

$$\leq \frac{M_1}{(z+\tau)^{-\gamma}} \int\limits_0^\lambda \frac{\tau^{1-\beta} ds}{(z+\lambda-s)^{2-\beta}} \lambda^{-\gamma} ||f||_{C_0^{\beta,\gamma}(E)} \leq \frac{M_2}{z^{1-\beta}} \frac{\tau^{1-\beta}}{1-\beta} ||f||_{C_0^{\beta,\gamma}(E)} \leq \frac{M_3}{1-\beta} ||f||_{C_0^{\beta,\gamma}(E)}$$

for any $0 \leq z < z+\tau \leq T$. If $\tau+z \leq \lambda$ and $\tau \geq z$, then, using estimates (3), (11), (12) and (8), we obtain

$$\tau^{-\beta}(z+\tau)^\gamma \left\| (e^{-(z+\tau)A(\lambda)} - e^{-zA(\lambda)}) A(\lambda) \int\limits_0^\lambda v(\lambda,s)(f(\lambda)-f(s))ds \right\|_E \tag{31}$$

$$\leq M\tau^{-\beta}(z+\tau)^\gamma \int\limits_0^{\lambda-\tau} \left\| A(\lambda)(e^{-(z+\tau)A(\lambda)} - e^{-zA(\lambda)}) v(\lambda,s)(f(\lambda)-f(s)) \right\|_E ds$$

$$+M\tau^{-\beta}(z+\tau)^\gamma \int\limits_{\lambda-\tau}^\lambda \left\| A(\lambda)(e^{-(z+\tau)A(\lambda)} - e^{-zA(\lambda)}) v(\lambda,s)(f(\lambda)-f(s)) \right\|_E ds$$

$$\leq \frac{\tau^{-\beta} M_1}{(z+\tau)^{-\gamma}} \int\limits_0^{\lambda-\tau} \frac{\tau ds}{(z+\lambda-s)^{2-\beta}} \lambda^{-\gamma} ||f||_{C_0^{\beta,\gamma}(E)}$$

$$+\frac{\tau^{-\beta}M_1}{(z+\tau)^{-\gamma}}\int\limits_{\lambda-\tau}^{\lambda}\frac{ds}{(\lambda-s)^{1-\beta}}\lambda^{-\gamma}||f||_{C_0^{\beta,\gamma}(E)}\leq\frac{M_2}{(z+\tau)^{\beta}}\frac{\tau}{(1-\beta)(z+\tau)^{1-\beta}}||f||_{C_0^{\beta,\gamma}(E)}$$

$$+\frac{M_2}{(z+\tau)^{\beta}}\frac{\tau^{\beta}}{\beta}||f||_{C_0^{\beta,\gamma}(E)}\leq\frac{M_3}{\beta(1-\beta)}||f||_{C_0^{\beta,\gamma}(E)}$$

for any $0\leq z<z+\tau\leq T$. Applying (24), (28), (29), (30), (31), we get

$$|K_1|_0^{\beta,\gamma}\leq\frac{M(\lambda)}{\beta(1-\beta)}||f||_{C_0^{\beta,\gamma}(E)}. \tag{32}$$

Now, we estimate K_2. Applying the inequality (17), we get

$$\|K_2\|_{E_{\beta-\gamma}}\leq M(\lambda)\left\|A(\lambda)\int\limits_0^{\lambda}v(\lambda,s)[A(\lambda)-A(s)]A^{-1}(\lambda)f(\lambda)ds\right\|_{E_{\beta-\gamma}}, \tag{33}$$

$$|K_2|_0^{\beta,\gamma}\leq M(\lambda)\left|A(\lambda)\int\limits_0^{\lambda}v(\lambda,s)[A(\lambda)-A(s)]A^{-1}(\lambda)f(\lambda)ds\right|_0^{\beta,\gamma}. \tag{34}$$

Using estimates (3), (11), (12) and (15), we obtain

$$z^{1-(\beta-\gamma)}\left\|A(\lambda)\exp\{-zA(\lambda)\}\int\limits_0^{\lambda}A(\lambda)v(\lambda,s)[A(\lambda)-A(s)]A^{-1}(\lambda)f(\lambda)ds\right\|_E \tag{35}$$

$$\leq z^{1-\beta+\gamma}\int\limits_0^{\lambda}||A^2(\lambda)\exp\{-zA(\lambda)\}v(\lambda,s)||_{E\to E}||[A(\lambda)-A(s)]A^{-1}(\lambda)||_{E\to E}||f(\lambda)||_E ds$$

$$\leq z^{1-\beta+\gamma}\int\limits_0^{\lambda}\min\left[\frac{1}{z^2},\frac{1}{(\lambda-s)^2}\right](\lambda-s)^{\varepsilon}ds||f||_{C_0^{\beta,\gamma}(E)}\leq$$

$$\leq M_1 z^{1-\beta+\gamma}\int\limits_0^{\lambda}\frac{(\lambda-s)^{\varepsilon}ds}{(z+\lambda-s)^2}||f||_{C_0^{\beta,\gamma}(E)}$$

for all $z>0$. We will prove that

$$z^{1-\beta+\gamma}\int\limits_0^{\lambda}\frac{(\lambda-s)^{\varepsilon}ds}{(z+\lambda-s)^2}\leq\frac{M}{\beta(1-\beta)} \tag{36}$$

for any $z>0$. If $z\leq\lambda$, then

$$z^{1-\beta+\gamma}\int\limits_0^{\lambda}\frac{(\lambda-s)^{\varepsilon}ds}{(z+\lambda-s)^2}\leq z^{1-\beta}\lambda^{\gamma}\int\limits_0^{\lambda}\frac{(z+\lambda)^{\varepsilon-\beta}\,ds}{(z+\lambda-s)^{2-\beta}}\leq\frac{2\lambda^{\varepsilon-\beta+\gamma}}{1-\beta}\leq\frac{2T^2}{1-\beta}.$$

If $\lambda\leq z$, then

$$z^{1-\beta+\gamma}\int\limits_0^\lambda \frac{(\lambda-s)^\varepsilon ds}{(z+\lambda-s)^2}\leq \frac{1}{z^{\beta-\gamma}}\int\limits_0^\lambda \frac{ds}{(\lambda-s)^{1-\varepsilon}}=\frac{\lambda^\varepsilon}{\varepsilon z^{\beta-\gamma}}<\frac{\lambda^\varepsilon}{\varepsilon\lambda^{\beta-\gamma}}\leq\frac{T^2}{\beta}.$$

From these estimates it follows (36) . Applying (36) , (33), (35), we get

$$\|K_2\|_{E_{\beta-\gamma}}\leq \frac{M(\lambda)}{\beta(1-\beta)}||f||_{C_0^{\beta,\gamma}(E)}. \tag{37}$$

Using estimates (3), (11), (12) and (15), we obtain

$$\|\ exp\{-zA(\lambda)\}A(\lambda)\int\limits_0^\lambda v(\lambda,s)[A(\lambda)-A(s)]A^{-1}(\lambda)f(\lambda)ds||_E \tag{38}$$

$$\leq\int\limits_0^\lambda ||A(\lambda)\exp\{-zA(\lambda)\}v(t,s)||_{E\to E}\left\|[A(\lambda)-A(s)]A^{-1}(\lambda)\right\|_{E\to E}||f(\lambda)||_E ds$$

$$\leq M\int\limits_0^\lambda \min\left[\frac{1}{z},\frac{1}{(\lambda-s)}\right](\lambda-s)^\varepsilon ds||f||_{C_0^{\beta,\gamma}(E)}$$

$$\leq M_1\int\limits_0^\lambda \frac{1}{z+\lambda-s}(\lambda-s)^\varepsilon ds||f||_{C_0^{\beta,\gamma}(E)}\leq M_1\int\limits_0^\lambda \frac{ds}{(\lambda-s)^{1-\varepsilon}}ds||f||_{C_0^{\beta,\gamma}(E)}$$

$$\leq \frac{M_1}{\varepsilon}\lambda^\varepsilon||f||_{C_0^{\beta,\gamma}(E)}\leq\frac{M_1}{\varepsilon}T^\varepsilon||f||_{C_0^{\beta,\gamma}(E)}\leq\frac{M_1}{\beta}T||f||_{C_0^{\beta,\gamma}(E)}$$

for any $z>0$. If $\lambda\leq\tau+z$, then Using estimates (3), (6) for $\alpha=\beta$, (11), (12) and (15), we obtain

$$\tau^{-\beta}(z+\tau)^\gamma\left\|(e^{-(z+\tau)A(\lambda)}-e^{-zA(\lambda)})A(\lambda)\int\limits_0^\lambda v(\lambda,s)[A(\lambda)-A(s)]A^{-1}(\lambda)f(\lambda)ds\right\|_E \tag{39}$$

$$\leq M\tau^{-\beta}(z+\tau)^\gamma\frac{\tau^\beta}{(z+\tau)^\beta}\left\|A(\lambda)\int\limits_0^\lambda v(\lambda,s)[A(\lambda)-A(s)]A^{-1}(\lambda)f(\lambda)ds\right\|_E$$

$$\leq\frac{M_1}{(z+\tau)^{\beta-\gamma}}\int\limits_0^\lambda\frac{ds}{(\lambda-s)^{1-\varepsilon}}||f||_{C_0^{\beta,\gamma}(E)}\leq\frac{M_1}{(z+\tau)^{\beta-\gamma}}\frac{\lambda^\varepsilon}{\varepsilon}||f||_{C_0^{\beta,\gamma}(E)}\leq\frac{M_2T^2}{\beta}||f||_{C_0^{\beta,\gamma}(E)}$$

for any $0\leq z<z+\tau\leq T$. If $\tau+z\leq\lambda$ and $\tau\leq z$, then using estimates (3), (11) and (13), we obtain

$$\tau^{-\beta}(z+\tau)^\gamma\left\|(e^{-(z+\tau)A(\lambda)}-e^{-zA(\lambda)})A(\lambda)\int\limits_0^\lambda v(\lambda,s)[A(\lambda)-A(s)]A^{-1}(\lambda)f(\lambda)ds\right\|_E \tag{40}$$

$$\leq M\tau^{-\beta}(z+\tau)^\gamma\int\limits_0^\lambda\left\|A(\lambda)(e^{-(z+\tau)A(\lambda)}-e^{-zA(\lambda)})v(\lambda,s)[A(\lambda)-A(s)]A^{-1}(\lambda)f(\lambda)\right\|_E ds$$

$$\leq \frac{M_1}{(z+\tau)^{-\gamma}} \int_0^{\lambda} \frac{(z+\lambda)^{\varepsilon-\beta}\,\tau^{1-\beta} ds}{(z+\lambda-s)^{2-\beta}} ||f||_{C_0^{\beta,\gamma}(E)} \leq \frac{M_2}{z^{1-\beta}} \frac{\tau^{1-\beta} T^{\varepsilon-\beta}}{1-\beta} ||f||_{C_0^{\beta,\gamma}(E)} \leq \frac{M_3}{1-\beta} ||f||_{C_0^{\beta,\gamma}(E)}$$

for any $0 \leq z < z+\tau \leq T$. If $\tau + z \leq \lambda$ and $\tau \geq z$, then using estimates (12) and (3), we obtain

$$\tau^{-\beta}(z+\tau)^{\gamma} \left\| (e^{-(z+\tau)A(\lambda)} - e^{-zA(\lambda)}) A(\lambda) \int_0^{\lambda} v(\lambda,s)[A(\lambda)-A(s)]A^{-1}(\lambda) f(\lambda) ds \right\|_E \tag{41}$$

$$\leq M\tau^{-\beta}(z+\tau)^{\gamma} \int_0^{\lambda-\tau} \left\| A(\lambda)(e^{-(z+\tau)A(\lambda)} - e^{-zA(\lambda)}) v(\lambda,s)[A(\lambda)-A(s)]A^{-1}(\lambda) f(\lambda) \right\|_E ds$$

$$+M\tau^{-\beta}(z+\tau)^{\gamma} \int_{\lambda-\tau}^{\lambda} \left\| A(\lambda)(e^{-(z+\tau)A(\lambda)} - e^{-zA(\lambda)}) v(\lambda,s)[A(\lambda)-A(s)]A^{-1}(\lambda) f(\lambda) \right\|_E ds$$

$$\leq \frac{\tau^{-\beta} M_1}{(z+\tau)^{-\gamma}} \int_0^{\lambda-\tau} \frac{\lambda^{\varepsilon-\beta}\tau ds}{(z+\lambda-s)^{2-\beta}} ||f||_{C_0^{\beta,\gamma}(E)}$$

$$+\frac{\tau^{-\beta} M_1}{(z+\tau)^{-\gamma}} \int_{\lambda-\tau}^{\lambda} \frac{ds}{(\lambda-s)^{1-\varepsilon}} ||f||_{C_0^{\beta,\gamma}(E)} \leq \frac{M_2 T^{\varepsilon-\beta+\gamma}\tau^{1-\beta}}{(1-\beta)\,(z+\tau)^{1-\beta}} ||f||_{C_0^{\beta,\gamma}(E)}$$

$$+\frac{M_2}{(z+\tau)^{\beta-\gamma}} \frac{\tau^{\varepsilon}}{\varepsilon} ||f||_{C_0^{\beta,\gamma}(E)} \leq \frac{M_3}{\beta\,(1-\beta)} ||f||_{C_0^{\beta,\gamma}(E)}$$

for any $0 \leq z < z+\tau \leq T$. Applying (38), (39), (40), (41),we get

$$|K_2|_0^{\beta,\gamma} \leq \frac{M(\lambda)}{\beta(1-\beta)} ||f||_{C_0^{\beta,\gamma}(E)}. \tag{42}$$

Applying the inequality (17), we get

$$\|K_3\|_{E_{\beta-\gamma}} \leq M(\lambda)\,\|A(0)\varphi + f(\lambda) - f(0)\|_{E_{\beta-\gamma}}\,, \tag{43}$$

$$|K_3|_0^{\beta,\gamma} \leq M(\lambda)\,|A(0)\varphi + f(\lambda) - f(0)|_0^{\beta,\gamma}\,. \tag{44}$$

Finally, we estimate K_4. Applying the inequality (17), we get

$$\|K_4\|_{E_{\beta-\gamma}} \leq M(\lambda)\,\|A(\lambda)v(\lambda,0)$$

$$\times \left(\left(A^{-1}(\lambda)f(\lambda) - A^{-1}(0)f(0)\right) + A^{-1}(\lambda)\,(A(\lambda)-A(0))\;\; A^{-1}(0)\;\; f(\lambda)\right)\|_{E_{\beta-\gamma}}\,, \tag{45}$$

$$|K_4|_0^{\beta,\gamma} \leq M(\lambda)\,|A(\lambda)v(\lambda,0)$$

$$\times \left(\left(A^{-1}(\lambda)f(\lambda) - A^{-1}(0)f(0)\right) + A^{-1}(\lambda)\,(A(\lambda)-A(0))\;\; A^{-1}(0)\;\; f(\lambda)\right)|_0^{\beta,\gamma}\,. \tag{46}$$

Applying the inequality (17), we get

$$z^{1-(\beta-\gamma)}\left\|A^2(\lambda)\exp\{-zA(\lambda)\}v(\lambda,0)\right.$$

$$\times \left(\left(A^{-1}(\lambda)f(\lambda) - A^{-1}(0)f(0)\right) + A^{-1}(\lambda)\,(A(\lambda)-A(0))\;\; A^{-1}(0)\;\; f(\lambda)\right)\|_E \tag{47}$$

$$\leq z^{1-\beta+\gamma}||A^2(\lambda)\exp\{-zA(\lambda)\}v(\lambda,0)A^{-1}(\lambda)||_{E\to E}\left(||f(\lambda)||_E + ||A(\lambda)A^{-1}(0)||_{E\to E}||f(0)||_E\right.$$

$$+||[A(\lambda)-A(0)]A^{-1}(0)||_{E\to E}||f(\lambda)||_E\big)$$

$$\leq Mz^{1-\beta+\gamma}\min\left[\frac{1}{z},\frac{1}{\lambda}\right]||f||_{C_0^{\beta,\gamma}(E)}\leq M_1\frac{z^{1-\beta+\gamma}}{z+\lambda}||f||_{C_0^{\beta,\gamma}(E)}\leq M_1\lambda^{-\beta+\gamma}||f||_{C_0^{\beta,\gamma}(E)}$$

for all $z>0$. Applying (47) , (45), we get

$$\|K_4\|_{E_{\beta-\gamma}}\leq M_1(\lambda)||f||_{C_0^{\beta,\gamma}(E)}. \tag{48}$$

Using estimates (3), (11), (12) and (15), we obtain

$$\|exp\{-zA(\lambda)\}A(\lambda)v(\lambda,0)$$
$$\times\big(\ \big(A^{-1}(\lambda)f(\lambda)-A^{-1}(0)f(0)\big)+A^{-1}(\lambda)\left(A(\lambda)-A(0)\right)\ A^{-1}(0)\ f(\lambda)\big)\big\|_E \tag{49}$$
$$\leq||A(\lambda)\exp\{-zA(\lambda)\}v(\lambda,0)A^{-1}(\lambda)||_{E\to E}\left(||f(\lambda)||_E+||A(\lambda)A^{-1}(0)||_{E\to E}||f(0)||_E\right.$$
$$\left.+||[A(\lambda)-A(0)]A^{-1}(0)||_{E\to E}||f(\lambda)||_E\right)$$
$$\leq M\min\left[\frac{1}{z},\frac{1}{\lambda}\right]||f||_{C_0^{\beta,\gamma}(E)}\leq M_1\frac{1}{\lambda}||f||_{C_0^{\beta,\gamma}(E)}$$

for any $z>0$. If $\lambda\leq\tau+z$, then Using estimates (6) for $\alpha=\beta$, (11), (12) and (15), we obtain

$$\tau^{-\beta}(z+\tau)^{\gamma}\Big\|(e^{-(z+\tau)A(\lambda)}-e^{-zA(\lambda)})A(\lambda)v(\lambda,0) \tag{50}$$
$$\times\big(\ \big(A^{-1}(\lambda)f(\lambda)-A^{-1}(0)f(0)\big)+A^{-1}(\lambda)\left(A(\lambda)-A(0)\right)\ A^{-1}(0)\ f(\lambda)\big)\big\|_E$$
$$\leq M\tau^{-\beta}(z+\tau)^{\gamma}\frac{\tau^{\beta}}{\lambda\left(z+\tau\right)^{\beta}}\big\|\big(\ \big(A^{-1}(\lambda)f(\lambda)-A^{-1}(0)f(0)\big)+A^{-1}(\lambda)\left(A(\lambda)-A(0)\right)\ A^{-1}(0)\ f(\lambda)\big)\big\|_E$$
$$\leq\frac{M_1}{\lambda(z+\tau)^{\beta-\gamma}}\left(\lambda^{\beta-\gamma}+\lambda^{\varepsilon}\right)||f||_{C_0^{\beta,\gamma}(E)}\leq M_2(\lambda)||f||_{C_0^{\beta,\gamma}(E)}$$

for any $0\leq z<z+\tau\leq T$. If $\tau+z\leq\lambda$ and $\tau\leq z$, then using estimates (12) and (3), we obtain

$$\tau^{-\beta}(z+\tau)^{\gamma}\Big\|(e^{-(z+\tau)A(\lambda)}-e^{-zA(\lambda)})A(\lambda)v(\lambda,0) \tag{51}$$
$$\times\big(\ \big(A^{-1}(\lambda)f(\lambda)-A^{-1}(0)f(0)\big)+A^{-1}(\lambda)\left(A(\lambda)-A(0)\right)\ A^{-1}(0)\ f(\lambda)\big)\big\|_E$$
$$\leq M(z+\tau)^{\gamma}\left(\frac{\tau^{1-\beta}}{(z+\lambda)^{2+\gamma-\beta}}+\frac{\tau^{1-\beta}}{(z+\lambda)^{2-\varepsilon}}\right)||f||_{C_0^{\beta,\gamma}(E)}\leq M_1(\lambda)||f||_{C_0^{\beta,\gamma}(E)}$$

for any $0\leq z<z+\tau\leq T$. If $\tau+z\leq\lambda$ and $\tau\geq z$, then using estimates (12) and (3), we obtain

$$\tau^{-\beta}(z+\tau)^{\gamma}\Big\|(e^{-(z+\tau)A(\lambda)}-e^{-zA(\lambda)})A(\lambda)v(\lambda,0) \tag{52}$$
$$\times\big(\ \big(A^{-1}(\lambda)f(\lambda)-A^{-1}(0)f(0)\big)+A^{-1}(\lambda)\left(A(\lambda)-A(0)\right)\ A^{-1}(0)\ f(\lambda)\big)\big\|_E$$

$$\leq\frac{M_1}{(z+\tau)^{-\gamma}}\left(\frac{\tau^{1-\beta}}{(z+\lambda)^{2+\gamma-\beta}}+\frac{\tau^{1-\beta}}{(z+\lambda)^{2-\varepsilon}}\right)||f||_{C_0^{\beta,\gamma}(E)}\leq M_1(\lambda)||f||_{C_0^{\beta,\gamma}(E)}$$

for any $0\leq z<z+\tau\leq T$. Applying (46), (49), (50), (51), (52),we get

$$|K_4|_0^{\beta,\gamma}\leq M_1(\lambda)||f||_{C_0^{\beta,\gamma}(E)}. \tag{53}$$

Combining the estimates (27), (37), (43), (48) and (32), (42), (44), (53), we get estimates (21), (22). Theorem 2.1 is proved. □

It is easy to show that

$$|u|_0^{\beta,\gamma} \leq \frac{M}{\beta-\gamma}||u||_{E_{\beta-\gamma}} \, (u \in E_{\beta-\gamma}) \, . \tag{54}$$

Theorem 2.1 admit the following corollary.

Theorem 2.2. *Suppose* $A(0)\varphi + f(\lambda) - f(0) \in E_{\beta-\gamma}$, $f(t) \in C_0^{\beta,\gamma}(E)(0 \leq \gamma \leq \beta, 0 < \beta < 1)$. *Suppose that the assumptions (1.3), (1.4) and (15) hold and* $0 < \beta \leq \varepsilon < 1$. *Then for the solution* $v(t)$ *in* $C_0^{\beta,\gamma}(E)$ *of the nonlocal boundary value problem (14) the coercive inequalities*

$$\| v' \|_{C_0^{\beta,\gamma}(E)} + \| A(.)v \|_{C_0^{\beta,\gamma}(E)} + \| v' \|_{C(E_{\beta-\gamma})}$$

$$\leq M(\lambda)[\frac{1}{\beta-\gamma}||A(0)\varphi + f(\lambda) - f(0)||_{E_{\beta-\gamma}} + \beta^{-1}(1-\beta)^{-1} \| f \|_{C_0^{\beta,\gamma}(E)}]$$

hold, where $M(\lambda)$ *does not depend on* β, γ, φ *and* $f(t)$.

3. Applications

First, we consider the nonlocal boundary value problem for parabolic equation

$$\frac{\partial u}{\partial t} - a(t,x)\frac{\partial^2 u}{\partial x^2} + \delta u = f(t,x), 0 < t < T, 0 < x < 1, \tag{55}$$

$$u(0,x) = u(\lambda,x) + \varphi(x), \ 0 \leq x \leq 1,$$

$$u(t,0) = u(t,1), \ \ u_x(t,0) = u_x(t,1), \ \ 0 \leq t \leq T,$$

where $a(t,x)$, $\varphi(x)$ and $f(t,x)$ are given sufficiently smooth functions and $a(t,x) = a(t+\lambda,x) > 0$, $\delta > 0$ is a sufficiently large number.

We introduce the Banach spaces $C^{\beta}[0,1]$ $(0 < \beta < 1)$ of all continuous functions $\varphi(x)$ satisfying a Hölder condition for which the following norms are finite

$$\| \varphi \|_{C^{\beta}[0,1]} = \| \varphi \|_{C[0,1]} + \sup_{0 \leq x < x+\tau \leq 1} \frac{|\varphi(x+\tau) - \varphi(x)|}{\tau^{\beta}},$$

where $C[0,1]$ is the space of the all continuous functions $\varphi(x)$ defined on $[0,1]$ with the usual norm

$$\| \varphi \|_{C[0,1]} = \max_{0 \leq x \leq 1} |\varphi(x)|.$$

It is known that the differential expression

$$A^{t,x}v = -a(t,x)v''(t,x) + \delta v(t,x)$$

define a positive operator $A^{t,x}$ acting in $C^{\beta}[0,1]$ with domain $C^{\beta+2}[0,1]$ and satisfying the conditions $v(0) = v(1)$, $v_x(0) = v_x(1)$.

Therefore, we can replace the nonlocal boundary value problem (55) by the abstract nonlocal boundary value problem (14). We can obtain that

Theorem 3.1. *For the solution of nonlocal boundary value problem (55) the following coercive inequaly is valid:*

$$\| u \|_{C_0^{1+\beta,\gamma}(C^{\mu}[0,1])} + \| u \|_{C_0^{\beta,\gamma}(C^{2+\mu}[0,1])} + \| u \|_{C\left(C^{2(\beta-\gamma)+\mu}[0,1]\right)}$$

$$\leq \frac{M(\lambda,\mu)}{\beta(1-\beta)} \| f \|_{C_0^{\beta,\gamma}(C^{\mu}[0,1])} + \frac{M(\lambda,\mu)}{\beta-\gamma}||-a(0,\cdot)\frac{\partial^2 \varphi(\cdot)}{\partial x^2} + \delta\varphi(\cdot) + f(\lambda,\cdot) - f(0,\cdot)||_{C^{2(\beta-\gamma)+\mu}[0,1]},$$

$$0 < 2(\beta-\gamma) + \mu < 1, \ 0 \leq \gamma \leq \beta, 0 \leq \mu \leq 1.$$

Here $M(\lambda,\mu)$ *is independent of* $\gamma, \beta, \ f(t,x), \ \varphi(x)$.

The proof of Theorem 3.1 is based on the abstract Theorem 2.1 and on the following theorem on the structure of the fractional spaces $E_\alpha(C[0,1],A^{t,x})$.

Theorem 3.2. *$E_\alpha(C[0,1],A^{t,x})=C^{2\alpha}[0,1]$ for all $0<\alpha<\frac{1}{2},0\le t\le T$ [33].*

Second, let Ω be the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0<x_k<1,1\le k\le n)$ with boundary S, $\overline{\Omega}=\Omega\cup S$. In $[0,T]\times\overline{\Omega}$ we consider the nonlocal boundary value problem for the multidimensional parabolic equation

$$\frac{\partial u(t,x)}{\partial t}-\sum_{r=1}^{n}\alpha_r(t,x)\frac{\partial^2 u(t,x)}{\partial x_r^2}+\delta u(t,x)=f(t,x), \tag{56}$$

$$-\sum_{r=1}^{n}\alpha_r(t,x)\frac{\partial^2\varphi(x)}{\partial x_r^2}+\delta\varphi(x)+f(\lambda,x)-f(0,x)=0, x=(x_1,\dots,x_n)\in\overline{\Omega},0<t<T,$$

$$u(0,x)=u(\lambda,x)+\varphi(x),\ x\in\overline{\Omega},$$

$$u(t,x)=0,\ x\in S,$$

where $\alpha_r(t,x)$, $f(t,x)$ $(t\in[0,T],\ x\in\overline{\Omega})$, $\varphi(x)(x\in\overline{\Omega})$ are given smooth functions and $\alpha_r(t,x)=\alpha_r(t+\lambda,x)>0$, $\delta>0$ is a sufficiently large number.

We introduce the Banach spaces $C_{01}^{\beta}(\overline{\Omega})$ $(\beta=(\beta_1,\cdots,\beta_n),0<x_k<1,k=1,\dots,n)$ of all continuous functions satisfying a Hölder condition with the indicator $\beta=(\beta_1,\cdots,\beta_n),\beta_k\in(0,1),1\le k\le n$ and with weight $x_k^{\beta_k}(1-x_k-h_k)^{\beta_k},0\le x_k<x_k+h_k\le 1,1\le k\le n$ which equipped with the norm

$$\|f\|_{C_{01}^{\beta}(\overline{\Omega})}=\|f\|_{C(\overline{\Omega})}$$

$$+\sup_{0\le x_k<x_k+h_k\le 1,1\le k\le n}|f(x_1,\dots,x_n)-f(x_1+h_1,\dots,x_n+h_n)|\prod_{k=1}^{n}h_k^{-\beta_k}x_k^{\beta_k}(1-x_k-h_k)^{\beta_k},$$

where $C(\overline{\Omega})$-is the space of the all continuous functions defined on $\overline{\Omega}$, equipped with the norm

$$\|f\|_{C(\overline{\Omega})}=\max_{x\in\overline{\Omega}}|f(x)|.$$

It is known that the differential expression

$$A^{t,x}v=-\sum_{r=1}^{n}\alpha_r(t,x)\frac{\partial^2 v(t,x)}{\partial x^2}+\delta v(t,x)$$

defines a positive operator $A^{t,x}$ acting on $C_{01}^{\beta}(\overline{\Omega})$ with domain $D(A^{t,x})\subset C_{01}^{2+\beta}(\overline{\Omega})$ and satisfying the condition $v=0$ on S.

Therefore, we can replace the nonlocal boundary value problem (56) by the abstract nonlocal boundary value problem (14). We can obtain that

Theorem 3.3. *For the solution of the nonlocal boundary value problem (56) the following coercive inequality is valid:*

$$\|u\|_{C_0^{1+\beta,\gamma}(C_{01}^{\mu}(\overline{\Omega}))}+\sum_{r=1}^{n}\left\|\frac{\partial^2 u}{\partial x_r^2}\right\|_{C_0^{\beta,\gamma}(C_{01}^{\mu}(\overline{\Omega}))}$$

$$\le\frac{M(\mu)}{(\beta-\gamma)(1-\beta)}\|f\|_{C_0^{\beta,\gamma}(C_{01}^{\mu}(\overline{\Omega}))},$$

$$0<2(\beta-\gamma)+\mu<1,\ 0\le\gamma\le\beta,$$

$$\mu=\{\mu_1,\cdots,\mu_n\},0<\mu_k<1,1\le k\le n,$$

where $M(\mu)$ is independent of β, γ and $f(t,x)$, $\varphi(x)$.

The proof of Theorem 3.3 is based on the abstract Theorems 2.1, the coercivity inequality for an elliptic operator $A^{t,x}$ in $C_{01}^{\mu}(\overline{\Omega})$.

Third, we consider the nonlocal boundary value problem on the range $\{0 \leq t \leq T,\ x \in \mathbb{R}^n\}$ for the 2m-th order multidimensional parabolic equation

$$\frac{\partial u}{\partial t} + \sum_{|r|=2m} a_r(t,x) \frac{\partial^{|\tau|} u}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}} + \delta u(t,x) = f(t,x), \tag{57}$$

$$0 < t < T,\ x, r \in \mathbb{R}^n, |r| = r_1 + \cdots + r_n,$$

$$u(0,x) = u(\lambda, x) + \varphi(x), x \in \mathbb{R}^n,$$

where $a_r(t,x) = a_r(t+\lambda, x)$ and $f(t,x)$, $\varphi(x)$ are given sufficiently smooth functions and $\delta > 0$ is the sufficiently large number .

Let us consider a differential operator with constant coefficients of the form

$$B = \sum_{|r|=2m} b_r \frac{\partial^{r_1+\ldots+r_n}}{\partial_{x_1^{r_1}} \cdots \partial_{x_n^{r_n}}},$$

acting on functions defined on the entire space R^n. Here $r \in R^n$ is a vector with nonnegative integer components, $|r| = r_1 + \cdots + r_n$. If $\varphi(y)$ $(y = (y_1, \cdots, y_n) \in R^n)$ is an infinitely differentiable function that decays at infinity together with all its derivatives, then by means of the Fourier transformation one establishes the equality

$$F(B_\varphi)(\xi) = B(\xi) F(\varphi)(\xi).$$

Here the Fourier transform operator is defined by the rule

$$F(\varphi)(\xi) = (2\pi)^{-n/2} \int_{R^n} \exp\{-i(y,\xi)\} \varphi(y)\, dy,$$

$$(y,\xi) = y_1\xi_1 + \cdots + y_n\xi_n.$$

The function $B(\xi)$ is called the symbol of the operator B and is given by

$$B(\xi) = \sum_{|r|=2m} b_r (i\xi_1)^{r_1} \cdots (i\xi_n)^{r_n}.$$

We will assume that the symbol

$$B^{t,x}(\xi) = \sum_{|r|=2m} a_r(t,x) (i\xi_1)^{r_1} \cdots (i\xi_n)^{r_n}, \xi = (\xi_1, \cdots, \xi_n) \in R^n$$

of the differential operator of the form

$$B^{t,x} = \sum_{|r|=2m} a_r(t,x) \frac{\partial^{|r|}}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}} \tag{58}$$

acting on functions defined on the space $\mathbb{R}^n$, satisfies the inequalities

$$0 < M_1 |\xi|^{2m} \leq (-1)^m B^{t,x}(\xi) \leq M_2 |\xi|^{2m} < \infty$$

for $\xi \neq 0$. The problem (57) has a unique smooth solution. This allows us to reduce the nonlocal boundary value problem (57) by the abstract nonlocal boundary value problem (14) in a Banach space $E = C^{\mu}(R^n)$ of all continuous bounded functions defined on $\mathbb{R}^n$satisfying a Hölder condition with the indicator $\mu \in (0,1)$ with a strongly positive operator $A^{t,x} = B^{t,x} + \delta I$ defined by (58).

Theorem 3.4. *For the solution of the nonlocal boundary value problem (57) the following coercivity inequality is satisfied*

$$\| u \|_{C_0^{1+\beta,\gamma}(C^\mu(R^n))} + \sum_{|\tau|=2m} \| \frac{\partial^{|r|}u}{\partial x_1^{r_1}\cdots\partial x_n^{r_n}} \|_{C_0^{\beta,\gamma}(C^\mu(R^n))}$$

$$+ \| u \|_{C\left(C^{2(\beta-\gamma)+\mu}(R^n)\right)} \leq \frac{M(\lambda,\mu)}{\beta(1-\beta)} \| f \|_{C_0^{\beta,\gamma}(C^\mu[0,1])}$$

$$+\frac{M(\lambda,\mu)}{\beta-\gamma}|| \sum_{|r|=2m} a_r(0,\cdot)\frac{\partial^{|\tau|}\varphi(\cdot)}{\partial x_1^{r_1}\cdots\partial x_n^{r_n}} + \delta\varphi(\cdot) + f(\lambda,\cdot) - f(0,\cdot)||_{C^{2(\beta-\gamma)+\mu}(R^n)},$$

$$0 < 2(\beta-\gamma)+\mu < 1,\ 0 \leq \gamma \leq \beta, 0 \leq \mu \leq 1.$$

Here $M(\lambda,\mu)$ is independent of $\gamma, \beta,\ f(t,x),\ \varphi(x)$.

The proof of Theorem 3.4 is based on the abstract Theorems 2.1, the coercivity inequality for an elliptic operator $A^{t,x}$ in $C^\mu(R^n)$ and on the following theorem on the structure of the fractional spaces $E_\alpha(C^\mu(R^n), A^{t,x})$.

Theorem 3.5. $E_\alpha(C^\mu(R^n), A^{t,x}) = C^{2m\alpha+\mu}(R^n)$ *for all* $0 < \alpha < \frac{1}{2m}$ *and* $0 \leq t \leq T$ [10].

Acknowledgements: The authors would like to thank Prof. Pavel Sobolevskii (Jerusalem, Israel), for his helpful suggestions to the improvement of this paper.

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