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A NUMERICAL SOLUTION OF THE MODIFIED REGULARIZED LONG WAVE (MRLW) EQUATION USING QUARTIC B-SPLINES

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Abstract. In this paper, a numerical solution of the modified regularized long wave (MRLW) equation is obtained by subdomain finite element method using quartic B-spline functions. Solitary wave motion, interaction of two and three solitary waves and the development of the Maxwellian initial condition into solitary waves are studied using the proposed method. Accuracy and efficiency of the proposed method are tested by calculating the numerical conserved laws and error norms L_2 and L_∞ . The obtained results show that the method is an effective numerical scheme to solve the MRLW equation. In addition, a linear stability analysis of the scheme is found to be unconditionally stable.

Keywords: MRLW equation, Finite element method, Subdomain, Quartic B-Splines, Solitary waves.

AMS Subject Classification: 97N40, 65N30, 65D07, 76B25, 74S05,74J35

1. Introduction

The one-dimensional nonlinear partial differential equation

$$U_t + U_x + \delta U U_x - \mu U_{xxt} = 0, \tag{1}$$

where δ and μ are positive parameters, is known as regularized long wave (RLW) equation. The equation was first introduced by Peregrine [1] to describe the development of an undular bore. This equation is one of the most improtant equations of the nonlinear dispersive waves having many applications in different areas, including ion-acoustic and magneto hydrodynamic waves in plasma, the transverse waves in shallow water, phonon packets in non-linear crystals, pressure waves in liquid-gas bubble mixtures and rotating flow down a tube. Benjamin et al. [2] also introduced a mathematical theory of the equation. Bona and Pryant [3] have discussed the existence and uniqueness of the equation. There are few analytical solutions available in the literature. Thus, the numerical solutions of the RLW equation have been subject of many papers. Various numerical studies including finite difference [4-7], finite element [8-21] and pseudo-spectral[22] method have been reported recently. A special property of the equation is the fact that the solutions

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may exhibit solitons whose magnitudes, shapes and velocities are not changed after the collision. RLW equation is a special case of the generalized long wave (GRLW) equation having the form

$$U_t + U_x + \delta U^p U_x - \mu U_{xxt} = 0, \tag{2}$$

where p is a positive integer. Zhang[23] solved the GRLW equation by finite difference method for a Cauchy problem. Kaya et.al [24] also studied the GRLW equation with Adomian decomposition method. Ramos[25] used quasilinearization method based on finite differences for solving the GRLW equation. Roshan[26] solved the GRLW equation numerically by the Petrov-Galerkin method using a linear hat function as the trial function and a quintic B-spline function as the test function. In this paper, we consider the modified regularized long wave (MRLW) equation which is a special form of the GRLW equation. Gardner et al.[27] have developed a collocation solution to the MRLW equation using quintic B-splines finite elements. A. K. Khalifa et al.[28, 29] obtained the numerical solutions of the MRLW equation using finite difference method and cubic B-spline collocation finite element method. Solutions based on collocation method using quadratic B-spline finite elements and the central finite difference method for time are investigated by K. R. Raslan[30]. K. R. Raslan and S. M. Hassan[31] have solved the MRLW equation by a collocation finite element method using quadratic, cubic, quartic and quintic B-splines to obtain the numerical solutions of the single solitary wave. S. B. Gazi Karakoc and T.Geyikli[32] has solved the equation by Petrov-Galerkin method in which the element shape functions are cubic and weight functions are quadratic B-splines. Fazal-i-Haq et al.[33] have designed a numerical scheme based on quartic B-spline collocation method for the numerical solution of MRLW equation.

In the present paper, we set up a subdomain finite element solution using quartic B-splines for the MRLW equation. The performance and accuracy of the method have been tested on four numerical experiments: the motion of single solitary waves, interaction of two and three solitary waves, and finally the Maxwellian initial condition.

2. The governing equation and quartic B-splines

The MRLW equation takes the form

$$U_t + U_x + 6U^2 U_x - \mu U_{xxt} = 0, \tag{3}$$

with the boundary conditions

$$\begin{array}{lll} U(a,t) = 0, & U(b,t) = 0, & \\ U_x(a,t) = 0, & U_x(b,t) = 0, & t > 0, \end{array} \tag{4}$$

and the initial condition

$$U(x,0) = f(x) \; a \le x \le b,$$

where μ is a positive parameter and the subscripts x and t denote the differentiation with the boundary conditions $U \to 0$ as $x \to \pm\infty$. The quartic B-splines $\phi_m(x)$, (m= -2(1)N+1), at the knots x_m which form a basis over the interval $[a, b]$ are defined by the relationships [35]

$$\phi_m(x) = \frac{1}{h^4} \begin{cases} (x - x_{m-2})^4, & x \in [x_{m-2}, x_{m-1}], \\ (x - x_{m-2})^4 - 5(x - x_{m-1})^4, & x \in [x_{m-1}, x_m], \\ (x - x_{m-2})^4 - 5(x - x_{m-1})^4 + 10(x - x_m)^4, & x \in [x_m, x_{m+1}], \\ (x_{m+3} - x)^4 - 5(x_{m+2} - x)^4, & x \in [x_{m+1}, x_{m+2}], \\ (x_{m+3} - x)^4, & x \in [x_{m+2}, x_{m+3}], \\ 0, & otherwise. \end{cases} \tag{5}$$

A global approximation $U_N(x,t)$ to the exact solution $U(x,t)$ can be expressed in terms of the quartic B-splines as:

$$U_N(x,t) = \sum_{j=-2}^{N+1} \delta_j(t)\phi_j(x) \tag{6}$$

where δ_j are time dependent quantities to be determined from both boundary and weighted residual conditions. Each quartic B-spline covers five elements so that each element $[x_m, x_{m+1}]$ is covered by five splines. The nodal values $U_m, U_m^{'}, U_m^{''}$ and $U_m^{'''}$ at the knots x_m are derived from Eq. (5) and Eq. (6) in the following form

$$\begin{aligned} U_m &= U(x_m) = \delta_{m-2} + 11\delta_{m-1} + 11\delta_m + \delta_{m+1}, \\ U_m' &= U'(x_m) = \tfrac{4}{h}(-\delta_{m-2} - 3\delta_{m-1} + 3\delta_m + \delta_{m+1}), \\ U_m'' &= U''(x_m) = \tfrac{12}{h^2}(\delta_{m-2} - \delta_{m-1} - \delta_m + \delta_{m+1}), \\ U_m''' &= U'''(x_m) = \tfrac{24}{h^3}(-\delta_{m-2} + 3\delta_{m-1} - 3\delta_m + \delta_{m+1}). \end{aligned} \tag{7}$$

A typical finite interval $[x_m, x_{m+1}]$ is mapped to the interval $[0, 1]$ by local coordinates ξ related to the global coordinates

$$h\xi = x - x_m, \ 0 \leq \xi \leq 1 \tag{8}$$

so the quartic B-spline shape functions over the element $[0, 1]$ can be defined as

$$\phi^e = \begin{cases} \phi_{m-2} = 1 - 4\xi + 6\xi^2 - 4\xi^3 + \xi^4, \\ \phi_{m-1} = 11 - 12\xi - 6\xi^2 + 12\xi^3 - \xi^4, \\ \phi_m = 11 + 12\xi - 6\xi^2 - 12\xi^3 + \xi^4, \\ \phi_{m+1} = 1 + 4\xi + 6\xi^2 + 4\xi^3 - \xi^4, \\ \phi_{m+2} = \xi^4. \end{cases} \tag{9}$$

Since all splines apart from $\phi_{m-2}(x), \phi_{m-1}(x), \phi_m(x), \phi_{m+1}(x)$ and $\phi_{m+2}(x)$ are zero over the element $[0, 1]$, approximation Eq.(6) over this element can be written in terms of basis functions given in Eq. (9) as

$$U_N(\xi,t) = \sum_{j=m-2}^{m+2} \delta_j(t)\phi_j(\xi)$$

where $\delta_{m-2}, \delta_{m-1}, \delta_m, \delta_{m+1}, \delta_{m+2}$ act as element parameters and B-splines $\phi_{m-2}(x)$, ϕ_{m-1}, ϕ_m, ϕ_{m+1}, ϕ_{m+2} as element shape functions.

3. The subdomain solution

The finite interval $[a, b]$ is partitioned into uniformly sized finite elements by the nodes x_m such that $a = x_0 < x_1 \cdots < x_N = b$ and $h = (x_{m+1} - x_m)$. Applying the subdomain approach to Eq.(3) with the weight function

$$W_m(x) = \begin{cases} 1, & x \in [x_m, x_{m+1}], \\ 0, & otherwise \end{cases} \tag{10}$$

we obtain the weak form of Eq. (3)

$$\int_{x_m}^{x_{m+1}} 1.(U_t + U_x + 6U^2U_x - \mu U_{xxt})dx = 0. \tag{11}$$

Substituting the transformation (8) into the weak form (11) and integrating Eq.(11) term by term with some manupulation by parts, results in

$$\begin{array}{l} \frac{h}{5}(\dot{\delta}_{m-2}+26\dot{\delta}_{m-1}+66\dot{\delta}_{m}+26\dot{\delta}_{m+1}+\dot{\delta}_{m+2})+Z_m(-\delta_{m-2}-10\delta_{m-1}+10\delta_{m+1}+\delta_{m+2}) \\ -\frac{4\mu}{h}(\dot{\delta}_{m-2}+2\dot{\delta}_{m-1}-6\dot{\delta}_{m}+2\dot{\delta}_{m+1}+\dot{\delta}_{m+2})=0, \end{array} \tag{12}$$

where the dot denotes differentiation with respect to t, and

$$Z_m = 6(\delta_{m-2}+11\delta_{m-1}+11\delta_m+\delta_{m+1})^2+1.$$

If time parameters δ_m and their time derivatives $\dot{\delta}_m$ in Eq. (12) are discretized by the Crank-Nicolson and forward difference approach respectively,

$$\delta_m = \frac{\delta_m^n+\delta_m^{n+1}}{2},\ \dot{\delta}_m = \frac{\delta_m^{n+1}-\delta_m^n}{\Delta t} \tag{13}$$

we obtain a recurrence relationship between the two time levels n and $n+1$ relating two unknown parameters δ_i^{n+1} and δ_i^n, for $i = m-2, m-1, ..., m+2$,

$$\begin{array}{l} \alpha_{m1}\delta_{m-2}^{n+1}+\alpha_{m2}\delta_{m-1}^{n+1}+\alpha_{m3}\delta_{m}^{n+1}+\alpha_{m4}\delta_{m+1}^{n+1}+\alpha_{m5}\delta_{m+2}^{n+1}= \\ \alpha_{m5}\delta_{m-2}^{n}+\alpha_{m4}\delta_{m-1}^{n}+\alpha_{m3}\delta_{m}^{n}+\alpha_{m2}\delta_{m+1}^{n}+\alpha_{m1}\delta_{m+2}^{n},\ m=0,1,...,N-1 \end{array} \tag{14}$$

where

$$\begin{array}{l} \alpha_{m1}=1-EZ_m-M,\ \alpha_{m2}=26-10EZ_m-2M,\ \alpha_{m3}=66+6M, \\ \alpha_{m4}=26+10EZ_m-2M,\ \alpha_{m5}=1+EZ_m-M, \end{array}$$

and

$$E=\frac{5\Delta t}{2h},\ M=\frac{20\mu}{h^2}.$$

The system (14) consists of N linear equations in $N+4$ unknowns $(\delta_{-2}, \delta_{-1}, ..., \delta_{N+1})$. To get a solution to this system, we need four additional constraints. These are obtained from the boundary conditions (4) and can be used to eliminate $\delta_{-2}, \delta_{-1}, \delta_N$ and δ_{N+1} from the system (14) which then becomes a matrix equation for the N unknowns $d = (\delta_0, \delta_1, ..., \delta_{N-1})$ of the form

$$Ad^{n+1} = Bd^n.$$

A lumped value for Z_m is obtained from $(U_m + U_{m+1})^2/4$ as

$$Z_m = \frac{6}{4}(\delta_{m-2}+12\delta_{m-1}+22\delta_m+12\delta_{m+1}+\delta_{m+2})^2+1.$$

The resulting system can be efficiently solved with a variant of the Thomas algorithm, and we need an inner iteration $(\delta^*)^{n+1} = \delta^n + \frac{1}{2}(\delta^{n+1}-\delta^n)$ at each time step to deal with the non-linear term Z_m. A typical member of the matrix system (14) can be written in terms of the nodal parameters δ_m^n as follows

$$\begin{array}{l} \gamma_1\delta_{m-2}^{n+1}+\gamma_2\delta_{m-1}^{n+1}+\gamma_3\delta_{m}^{n+1}+\gamma_4\delta_{m+1}^{n+1}+\gamma_5\delta_{m+2}^{n+1}= \\ \gamma_5\delta_{m-2}^{n}+\gamma_4\delta_{m-1}^{n+}+\gamma_3\delta_{m}^{n}+\gamma_2\delta_{m+1}^{n}+\gamma_1\delta_{m+2}^{n} \end{array} \tag{15}$$

where

$$\begin{array}{l} \gamma_1=\alpha-\beta-\lambda,\ \gamma_2=26\alpha-10\beta-2\lambda,\ \gamma_3=66\alpha+6\lambda, \\ \gamma_4=26\alpha+10\beta-2\lambda,\ \gamma_5=\alpha+\beta-\lambda. \end{array}$$

and

$$\alpha=1,\ \beta=EZ_m,\ \lambda=M.$$

Before the solution process begins iteratively, the initial vector $\delta^0 = (\delta_0, \delta_1, ..., \delta_{N-1})$ must be determined by using the initial condition and the following derivatives at the boundaries:

$$\begin{aligned}
U'(a,0) &= \frac{4}{h}(-\delta^0_{-2} - 3\delta^0_{-1} + 3\delta^0_0 + \delta^0_1) = 0,\\
U''(a,0) &= \frac{12}{h^2}(\delta^0_{-2} - \delta^0_{-1} - \delta^0_0 + \delta^0_1) = 0,\\
U(x_m,0) &= \delta^0_{m-2} + 11\delta^0_{m-1} + 11\delta^0_m + \delta^0_{m+1} = f(x), m = 0,1,...,N-1\\
U'(b,0) &= \frac{4}{h}(-\delta^0_{N-2} - 3\delta^0_{N-1} + 3\delta^0_N + \delta^0_{N+1}) = 0,\\
U''(b,0) &= \frac{12}{h^2}(\delta^0_{N-2} - \delta^0_{N-1} - \delta^0_N + \delta^0_{N+1}) = 0.
\end{aligned}$$

Eliminating $\delta^0_{-2}, \delta^0_{-1}, \delta^0_N, \delta^0_{N+1}$ from the system (14), we get $N \times N$ matrix system of the form:

$$W\delta^0 = B$$

where W is

$$W = \begin{bmatrix}
18 & 6 & & & & & & \\
11.5 & 11.5 & 1 & & & & & \\
1 & 11 & 11 & 1 & & & & \\
 & & & & & & & \\
 & & & & 1 & 11 & 11 & 1\\
 & & & & & 2 & 14 & 8
\end{bmatrix},$$

$\delta^0 = [\delta^0_0, \delta^0_1, ..., \delta^0_{N-1}]^T$ and $B = [U(x_0,0), U(x_1,0), ..., U(x_{N-1},0)]^T$. This matrix system can be solved efficiently by using a variant of Thomas algorithm.

4. Linear Stability analysis

The Von Neumann stability analysis will be applied. For this, the growth factor of a typical Fourier mode defined as

$$\delta^n_j = \xi^n e^{ijkh} \tag{16}$$

where k is mode number and h the element size, will be determined for a linearization of numerical scheme. In order to apply the stability analysis, the MRLW equation needs to be linearized by assuming that the quantity U in the non-linear term U^2U_x is locally constant. Substituting the Eq.(16) into the scheme (15) we have

$$g = \frac{a - ib}{a + ib}, \tag{17}$$

where

$$\begin{aligned}
a &= 33 + 3\lambda + (26 - 2\lambda)\cos kh + (1 - \lambda)\cos 2kh,\\
b &= 10\beta\sin(kh) + \beta\sin(2kh).
\end{aligned} \tag{18}$$

Taking the modulus of equation (17) gives $|g| = 1$, therefore we find that the scheme (15) is unconditionally stable.

5. Numerical examples and results

We obtain the numerical solutions of the MRLW equation for four problems: the motion of single solitary wave, interaction of two and three solitary waves and development of the Maxwellian initial condition into solitary waves. To measure the accuracy of the numerical solutions, difference between analytical and numerical solutions at some specified times is computed by both the error norm L_2

$$L_2 = \left\|U^{exact} - U_N\right\|_2 \simeq \sqrt{h\sum_{J=1}^{N}\left|U_j^{exact} - (U_N)_j\right|^2},$$

and the error norm L_∞

$$L_\infty = \left\|U^{exact} - U_N\right\|_\infty \simeq \max_j \left|U_j^{exact} - (U_N)_j\right|, j = 1, 2, ..., N-1.$$

The MRLW equation (3) possesses only three conservation constants given by

$$\begin{aligned} I_1 &= \textstyle\int_a^b U dx \simeq h\sum_{J=1}^{N} U_j^n, \\ I_2 &= \textstyle\int_a^b [U^2 + \mu(U_x)^2]dx \simeq h\sum_{J=1}^{N}[(U_j^n)^2 + \mu\,(U_x)_j^n], \\ I_3 &= \textstyle\int_a^b (U^4 - \mu U_x^2)dx \simeq h\sum_{J=1}^{N}\left[(U_j^n)^4 - \mu(U_x)_j^n\right], \end{aligned}$$

which correspond to conversation of mass, momentum and energy, respectively[34]. In the simulation of solitary wave motion, the invariants I_1, I_2 and I_3 are observed to check the conversation of the numerical algorithm.

5.1. **The motion of single solitary wave.**

For this problem, we consider Eq.(3) with the boundary conditions $U \to 0$ as $x \to \pm\infty$ and the initial condition

$$U(x, 0) = \sqrt{c}\sec h\left[p\left(x - x_0\right)\right].$$

The theoretical solitary wave solution of the MRLW has the following form

$$U(x, t) = \sqrt{c}\sec h\left[p\left(x - (c+1)t - x_0\right)\right]$$

where $p = \sqrt{\frac{c}{\mu(c+1)}}$, x_0 and c are arbitrary constants. The constants of motion, for a solitary wave of amplitude $\sqrt{c}$ and width depending on p may be evaluated analytically as [27]

$$I_1 = \frac{\pi\sqrt{c}}{p}, \quad I_2 = \frac{2c}{p} + \frac{2\mu pc}{3}, \quad I_3 = \frac{4c^2}{3p} - \frac{2\mu pc}{3}. \tag{19}$$

First, we have chosen the parameters $\mu = 1$, $c = 1$, $h = 0.2$, $k = 0.025$ and $x_0 = 40$ through the interval $[0, 100]$ to make a comparison with the results of Refs.[27, 28]. The computed values of the invariants with error norms L_2 and L_∞ are presented at some selected times up to $t = 10$ in Table1. As it is seen from the Table(1) the error norms L_2 and L_∞ are obtained sufficiently small and the the numerical values of invariants are in good agreement with their analytical values $I_1 = 4.4428829$, $I_2 = 3.2998316$, $I_3 = 1.4142135$. The percentage of the relative error of the conserved quantities I_1, I_2 and I_3 are calculated with respect to the conserved quantities at $t = 0$. Percentage of relative changes of I_1, I_2 and I_3 are found 0.041×10^{-3} %, 0.048×10^{-3} %, 0.097×10^{-3} %, respectively. Thus the quantities in the invariants remain almost constant during the computer run. Table(2) presents a comparison of the values of the invariants and error norms obtained by the present method with those obtained by other methods [27, 28]. It is clearly seen from the Table(2) that the error norm L_∞ obtained by the present method is smaller than those given in Ref.[28] whereas the error norm L_2 is almost the same those given in Ref.[28] but smaller than those obtained with the others. The motion of solitary wave using our method is plotted at different time levels in Fig.(1).

In a further simulation of the motion of a single solitary wave to allow the comparison with other existing schemes, parameters $\mu = 1$, $c = 0.3$, $h = 0.1$, $k = 0.01$ and

$x_0 = 40$ with range $[0, 100]$ are taken. Error norms L_2 and L_∞ and conserved quantities are illustrated in Table(3) for the time $t = 20$, together with results obtained with Ref.[28, 30]. It is seen that the predicted error norms L_2 and L_∞ are smaller than those obtained in Ref.[28, 30], and also invariants are reasonably in good agreement with their analytical values given by Eq.19. Percentage of relative changes of I_1, I_2 and I_3 are found 0.009×10^{-3} %, 0.009×10^{-3} %, 0.025×10^{-3} %, respectively. Moreover, the invariants I_1 and I_2 change from their initial values by less than 3×10^{-7} and 1×10^{-7} respectively, during the time of running whereas, the change of invariant I_3 approaches to zero throughout the run. Fig.(2) illustrates the motion of the solitary wave at different time leves. Error distributions at time $t = 10$ and $t = 20$ are depicted graphically for solitary waves amplitudes 1 and 0.3 in Fig.(3). It is seen that the maximum errors are about the tip of the solitary waves and between -6×10^{-3} and 6×10^{-3}, -2×10^{-4} and 2×10^{-4}, respectively.

Table 1. Invariants and error norms for single solitary wave with $c = 1$, $h = 0.2$, $k = 0.025$, $0 \le x \le 100$.

t	I_1	I_2	I_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
0	4.4428660	3.2998251	1.4142022	0.00000000	0.00000000
1	4.4428662	3.2998252	1.4142023	1.01730104	0.54356142
2	4.4428667	3.2998257	1.4142028	2.02079556	1.08637566
3	4.4428671	3.2998261	1.4142031	3.00493932	1.59302250
4	4.4428674	3.2998264	1.4142033	3.97153841	2.08447826
5	4.4428676	3.2998265	1.4142034	4.92641719	2.57019590
6	4.4428677	3.2998266	1.4142035	5.87417096	3.05402949
7	4.4428678	3.2998266	1.4142035	6.81766748	3.53682732
8	4.4428678	3.2998266	1.4142035	7.75861590	4.01910803
9	4.4428678	3.2998266	1.4142035	8.69803722	4.50111164
10	4.4428678	3.2998266	1.4142035	9.23663428	4.98295436

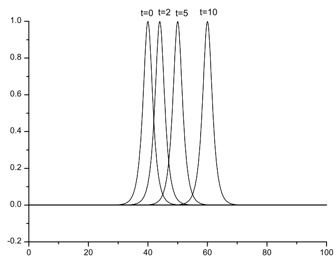

Figure 1. Single solitary wave with $c = 1, h = 0.2, \Delta t = 0.025, 0 \le x \le 100$ $t = 0, 2, 5$ and 10.

Table 2. Errors and invariants for single solitary wave with the order of convergence at $c = 1$, $h = 0.2$, $k = 0.025$, $0 \le x \le 100$, $t = 10$.

Method	I_1	I_2	I_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
Analytical	4.44288	3.29983	1.41421	0	0
Present	4.44287	3.29982	1.41420	9.23663	4.98295
Cubic B-splines coll-CN[27]	4.442	3.299	1.413	16.39	9.24
Cubic B-splines coll+PA-CN[27]	4.440	3.296	1.411	20.3	11.2
[28]	4.44288	3.29983	1.41420	9.30196	5.43718

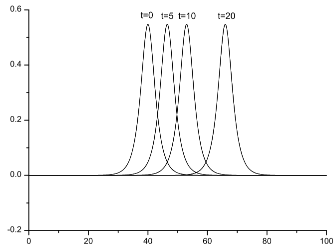

FIGURE 2. Single solitary wave with $c = 0.3$, $h = 0.1$, $\Delta t = 0.01$, $0 \leq x \leq 100$ at level times $t = 0, 5, 10$ and 20.

TABLE 3. Invariants and error norms for single solitary wave with $c = 0.3$, $h = 0.1$, $k = 0.01$, $0 \leq x \leq 100$.

t	I_1	I_2	I_3	$L_2 \times 10^4$	$L_\infty \times 10^4$
0	3.5820205	1.3450941	0.1537283	0.0000000	0.0000000
2	3.5820205	1.3450941	0.1537283	0.3959945	0.1793739
4	3.5820205	1.3450941	0.1537283	0.7903879	0.3500696
6	3.5820205	1.3450940	0.1537283	1.1833872	0.5158774
8	3.5820205	1.3450940	0.1537283	1.5741643	0.6800477
10	3.5820205	1.3450940	0.1537283	1.9627120	0.8456977
12	3.5820204	1.3450940	0.1537283	2.3494186	1.0096076
14	3.5820204	1.3450940	0.1537283	2.7346926	1.1724977
16	3.5820204	1.3450940	0.1537283	2.9188710	1.3347649
18	3.5820203	1.3450940	0.1537283	3.0022136	1.4966417
20	3.5820202	1.3450940	0.1537283	3.2849177	1.6582700
20[28]	3.58197	1.34508	0.153723	6.06885	2.96650
20[30]	3.582265	1.345182	0.1538901	3.379583	7.672911

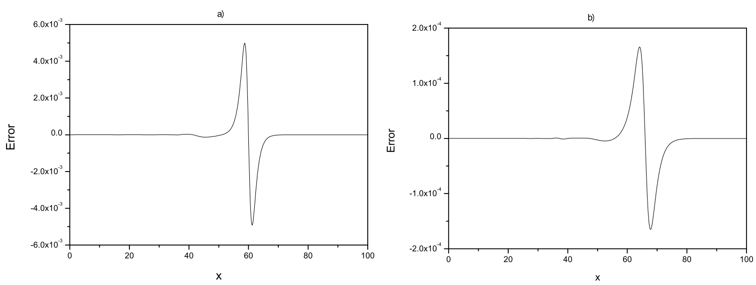

FIGURE 3. Error with a) $c = 1, h = 0.2, \Delta t = 0.025, t = 10, 0 \leq x \leq 100$ b) $c = 0.3, h = 0.1, \Delta t = 0.01, t = 20, 0 \leq x \leq 100$.

5.2. **Interaction of two solitary waves.** For this problem, we study the behavior of the interaction of two solitary waves having different amplitudes and travelling in the same direction. Initial condition of two well-seperated solitary waves of different amplitudes has the following form:

$$U(x,0) = \sum_{j=1}^{2} A_j \sec h(p_j[x - x_j]), \tag{20}$$

where $A_j = \sqrt{c_j}$, $p_j = \sqrt{\frac{c_j}{\mu(c_j+1)}}$, $j = 1, 2$, c_j and x_j are arbitrary constants. The analytical values of the conservation laws can be found from the Eq. (19) as

$$\begin{aligned} I_1 &= \sum_{j=1}^{2} \frac{\pi\sqrt{c_j}}{p_j} = 11.467698, \\ I_2 &= \sum_{j=1}^{2} \left(\frac{2c_j}{p_j} + \frac{2\mu p_j c_j}{3}\right) = 14.629243, \\ I_3 &= \sum_{j=1}^{2} \left(\frac{4c_j^2}{3p_j} - \frac{2\mu p_j c_j}{3}\right) = 22.880466. \end{aligned} \tag{21}$$

For the numerical simulation, we choose the parameters $\mu = 1$, $h = 0.2$, $k = 0.025$, $c_1 = 4$, $c_2 = 1$, $x_1 = 25$, $x_2 = 55$ over the interval $0 \leq x \leq 250$ to coincide with those used by Ref.[28]. The calculations are performed from $t = 0$ to $t = 20$ and the values of the invariant quantities I_1, I_2 and I_3 are recorded in Table(4). Table(4) displays a comparison of the values of the invariants obtained by the present method with those obtained in Ref. [28]. It is seen that the obtained values of the invariants remain almost constant during the computer run. Figure(4) illustrates the behaviour of the interaction of two solitary waves. It is observed from the Fig.(4) that at $t = 0$ the wave with larger amplitude is on the left of the second wave with smaller amplitude. Since the taller wave moves faster than the shorter one, it catches up and collides with the shorter one at $t = 8$ and then moves away from the shorter one as time increases. At $t = 20$, the amplitude of larger wave is 1.992788 at the point $x = 127.2$ whereas the amplitude of the smaller one is 0.994175 at the point $x = 92.2$. It is found that the absolute difference in amplitude is 5.82×10^{-3} for the smaller wave and 7.2×10^{-3} for the larger wave for this algorithm.

TABLE 4. Comparison of invariants for the interaction of two solitary waves with results from [28] with $h = 0.2$, $k = 0.025$ in the region $0 \leq x \leq 250$.

	Present method			[28]		
t	I_1	I_2	I_3	I_1	I_2	I_3
0	11.4677	14.6292	22.8803	11.4677	14.6291	22.8806
2	11.4675	14.6288	22.8785	11.4677	14.6292	22.8807
4	11.4673	14.6283	22.8766	11.4677	14.6292	22.8807
6	11.4672	14.6279	22.8747	11.4677	14.6295	22.8806
8	11.4684	14.6300	22.8793	11.4677	14.6451	22.8454
10	11.4682	14.6299	22.8784	11.4677	14.5963	22.8913
12	11.4663	14.6263	22.8704	11.4677	14.6287	22.8814
14	11.4664	14.6261	22.8687	11.4677	14.6295	22.8807
16	11.4664	14.6258	22.8669	11.4677	14.6294	22.8808
18	11.4663	14.6253	22.8650	11.4677	14.6293	22.8809
20	11.4661	14.6249	22.8631	11.4677	14.6292	22.8809

5.3. **Interaction of three solitary waves.** In this part, the behavior of the interaction of three solitary waves having different amplitudes and travelling in the same direction are studied. We consider the MRLW equation with initial condition given by the linear sum of three well-seperated solitary waves of different amplitudes

$$U(x,0) = \sum_{j=1}^{3} A_j \sec h(p_j[x - x_j]), \tag{22}$$

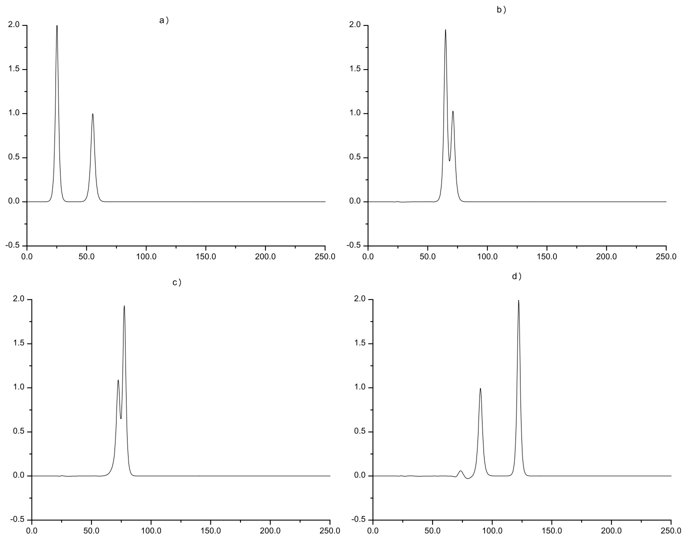

FIGURE 4. Interaction of two solitary waves at $a)t = 0$, $b)t = 8$, $c)t = 10$, $d)t = 19$.

where $A_j = \sqrt{c_j}$, $p_j = \sqrt{\frac{c_j}{\mu(c_j+1)}}$, $j = 1, 2, 3$, c_j and x_j are arbitrary constants. The analytical values of the conservation laws can be found from the Eq.(19) as

$$\begin{aligned} I_1 &= \sum_{j=1}^{3} \frac{\pi\sqrt{c_j}}{p_j} = 14.9801, \\ I_2 &= \sum_{j=1}^{3} \left(\frac{2c_j}{p_j} + \frac{2\mu p_j c_j}{3} \right) = 15.8218, \\ I_3 &= \sum_{j=1}^{3} \left(\frac{4c_j^2}{3p_j} - \frac{2\mu p_j c_j}{3} \right) = 22.9923. \end{aligned} \tag{23}$$

To ensure an interaction of three solitary waves take place, calculation is carried out with the parameters $\mu = 1, h = 0.2, k = 0.025, c_1 = 4, c_2 = 1$, $c_3 = 0.25$, $x_1 = 15, x_2 = 45, x_3 = 60$ over the region $0 \leq x \leq 250$. Simulations are run up to time $t = 45$. Table(5) compares the computed values of the invariants of the three solitary waves obtained by the Ref. [28]. It is observed that the obtained values of the invariants remain almost the same during the computer run and they are found to be very close to the values given in Ref. [28] which are all in good agreement with their analytical values given by Eq.(23). The absolute difference between the values of the conservative constants obtained by the present method at times $t = 0$ and $t = 45$ are $\Delta I_1 = 2.67 \times 10^{-2}$, $\Delta I_2 = 8.5 \times 10^{-3}, \Delta I_3 = 4.32 \times 10^{-2}$. Figure(5) shows the interaction of these solitary waves at different times. As it is seen from the Fig.(5),the interaction started about time $t = 10$, overlapping processes occured between time $t = 15$ and $t = 40$ and waves started to resume their original shapes after the time $t = 40$.

5.4. **The Maxwellian initial condition.** As our last problem, we have considered the evolution of an initial Maxwellian pulse into solitary waves using an initial condition of the form

$$U(x, 0) = \exp(-(x - 40)^2). \tag{24}$$

TABLE 5. Comparison of invariants for the interaction of three solitary waves with results from [28] with $h = 0.2$, $k = 0.025$ in the region $0 \leq x \leq 250$.

	Present method			[28]		
t	I_1	I_2	I_3	I_1	I_2	I_3
0	14.9801	15.8375	23.0081	13.6891	15.4549	22.8816
5	14.9799	15.8365	23.0036	13.6891	15.3109	22.6939
10	14.9850	15.8453	23.0207	13.6891	15.6514	22.8388
15	14.9809	15.8367	22.9986	13.6891	15.6548	22.9347
20	14.9790	15.8340	22.9927	13.6891	15.6557	22.9330
25	14.9780	15.8323	22.9876	13.6892	156559	22.9336
30	14.9777	15.8311	22.9827	13.6894	15.6559	22.9348
35	14.9778	15.8299	22.9779	13.6913	15.6564	22.9343
40	14.9795	15.8291	22.9728	13.7015	15.6566	22.9335
45	14.9534	15.8290	22.9649	13.7043	15.6563	22.9303

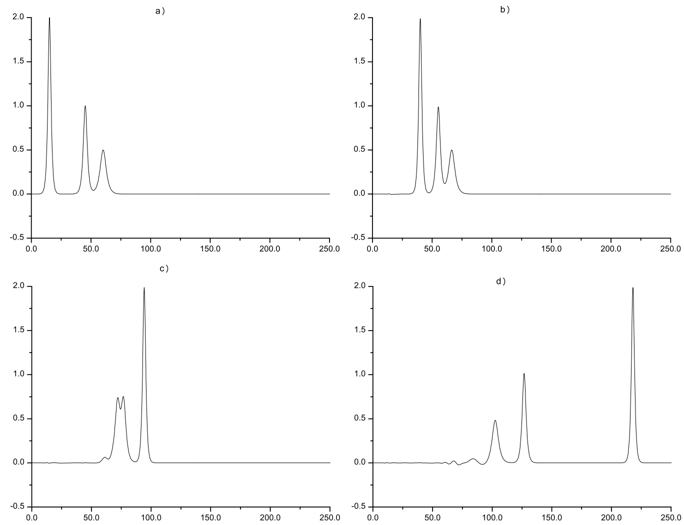

FIGURE 5. Interaction of three solitary waves at $a)t = 0$, $b)t = 5$, $c)t = 15$, $d)t = 40$.

It is known that with the Maxwellian condition (24), the behavior of the solution depends on the values of μ. We study each of the following cases: $\mu = 0.1$, $\mu = 0.015$ and $\mu = 0.01$. For $\mu = 0.1$, only a single soliton is formed as shown in Fig.(5a). When $\mu = 0.015$, three stable solitons are formed as shown in Fig.(5b). For $\mu = 0.01$, four solitary waves are formed as shown in Fig. (5c). The peaks of the well developed wave lie on a straight line so that their velocities are linearly dependent on their amplitudes and also we observe a small oscillating tail appearing behind the last wave in all Maxwellian figures. The recorded values of the invariants I_1, I_2 and I_3 are given in Table(6).

6. CONCLUSION

In this paper, Subdomain finite element method based on quartic B-splines was efficiently applied to the MRLW equation in order to examine the motion of a single solitary wave of which the analytic solution is known, the development of two and three solitary waves of which the analytical solution is unknown during the interaction. We have also studied the Maxwellian initial condition. To show how good and accurate the

Table 6. Invariants of MRLW equation using the Maxwellian initial condition.

t	μ	I_1	I_2	I_3	μ	I_1	I_2	I_3
0		1.7724809	1.3786633	0.7609104		1.7724809	1.2721328	0.8674409
3		1.7709450	1.3767872	0.7588507		1.7571086	1.2541293	0.8440123
6	0.1	1.7710230	1.3767936	0.7588503	0.015	1.7562844	1.2527946	0.8411456
9		1.7710380	1.3767952	0.7588493		1.7556066	1.2514706	0.8383714
12		1.7710450	1.3767955	0.7588481		1.7549308	1.2501178	0.8356652
15		1.7710491	1.3767954	0.7588469		1.7542550	1.2488122	0.8329835
0		1.7724809	1.2658663	0.8737074				
3		1.7491679	1.2385700	0.8368228				
6	0.01	1.7470079	1.2353249	0.8285796				
9		1.7451780	1.2312915	0.8212304				
12		1.7434037	1.2280212	0.8139112				
15		1.7416796	1.2253538	0.8066650				

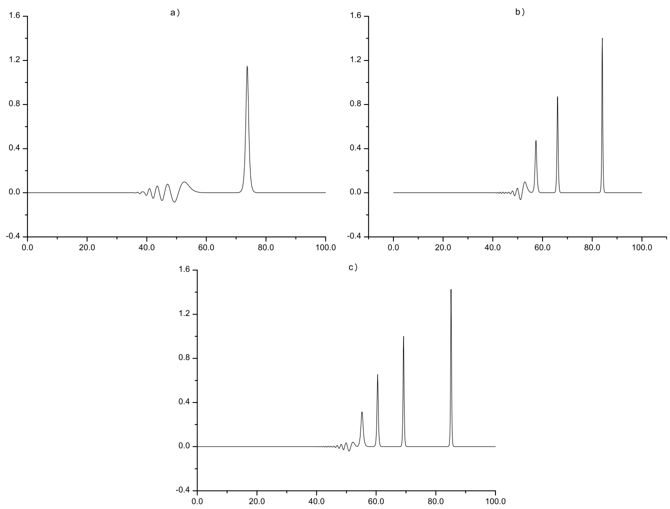

Figure 6. Maxwellian initial condition at $t = 14.5$ with $a)\mu = 0.1$, $b)\mu = 0.015$, $c)\mu = 0.01$.

numerical solutions of the test problems, we have calculated the error norms L_2 and L_∞. The method successfully models the motion, the interaction of solitary waves and Maxwellian initial condition. The obtained results show that the Subdomain method using quartic B-spline shape functions is a remarkably successful numerical technique for solving the MRLW equation and can also be efficiently applied to a broad class of physically important non-linear partial differential equations.

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