NEW SUFFICIENT CONDITIONS FOR
STARLIKE AND CONVEX FUNCTIONS

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Abstract. Let $A$ be the class of analytic functions $f(z)$ in the open unit disc. Applying the subordination, some sufficient conditions for starlikeness and convexity are discussed.

Keywords: Analytic function, Starlike of order $\alpha$, Convex of order $\alpha$, Subordination.

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1. Introduction

Let $A$ be the class of functions $f(z)$ of the form

$$f(z) = \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z) \in A$ is said to be the starlike function of order $\alpha$ if it satisfies

$$\text{Re} \left( \frac{z f'(z)}{f(z)} \right) > \alpha \quad (z \in U)$$

for some $\alpha (0 \leq \alpha < 1)$. Also a function $f(z) \in A$ is said to be the convex function of order $\alpha$ if it satisfies

$$\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in U)$$

for some $\alpha (0 \leq \alpha < 1)$. These classes are denoted by $S^*(\alpha)$ and $K(\alpha)$, respectively. We well-known that $S^*(0) \equiv S^*$ and $K(0) \equiv K$, and that the relation $f(z) \in K$ if and only if $zf'(z) \in S^*$.

By investigating expressions

$$\frac{z^2 f'(z)}{(f(z))^2} - (1 + \gamma) \frac{z}{f(z)} \frac{zf''(z)}{(f'(z))^2} - \gamma \frac{1}{f'(z)}$$

and

$$\frac{f(z) f''(z)}{(f'(z))^2} - (1 + \gamma) \frac{f(z)}{zf'(z)},$$

we would like to introduce some sufficient conditions for the classes $S^*(\alpha)$ and $K(\alpha)$.

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2. SUFFICIENT CONDITIONS FOR STARLIKENESS AND CONVEXITY

For analytic functions \( f(z) \) and \( g(z) \) in \( \mathbb{U} \), \( f(z) \) is said to be subordinante to \( g(z) \) in \( \mathbb{U} \) if there exists an analytic function \( w(z) \) in \( \mathbb{U} \) such that \( w(0) = 0, |w(z)| < 1 \) and \( f(z) = g(w(z)) \). We denote this subordination by

\[
 f(z) \prec g(z).
\]

If \( g(z) \) is univalent in \( \mathbb{U} \), \( f(z) \prec g(z) \) if and only if \( f(0) = g(0) \) and \( f(\mathbb{U}) \subset g(\mathbb{U}) \).

We make use of the case which \( \gamma \) is a non-negative real number of Theorem 2 of Miller, Mocanu and Reade [1] as following:

**Lemma 2.1.** Let \( F(z) \) and \( G(z) \) be analytic functions in \( \mathbb{U} \), \( \gamma \geq 0 \) and \( G'(0) \neq 0 \). Furthermore, in the case of \( \gamma = 0 \), \( F(0) = G(0) = 0 \). If

\[
 \text{Re} \left( 1 + \frac{zG''(z)}{G'(z)} \right) > k(\gamma) = \begin{cases} \frac{\gamma}{2} & (\gamma \leq 1) \\ -\frac{1}{2\gamma} & (\gamma \geq 1) \end{cases} \quad (z \in \mathbb{U})
\]

and

\[
 F(z) \prec G(z),
\]

then

\[
 z^{-\gamma} \int_0^z t^{\gamma-1} F(t) dt < z^{-\gamma} \int_0^z t^{\gamma-1} G(t) dt.
\]

For \( F(z) = 1 - \gamma p(z) - z p'(z) \), the following lemma was studied by Singh and Tuneski [3].

**Lemma 2.2.** Let \( p(z) \) and \( G(z) \) be analytic functions in \( \mathbb{U} \), \( \gamma \geq 0 \) and \( G'(0) \neq 0 \). If

\[
 \text{Re} \left( 1 + \frac{zG''(z)}{G'(z)} \right) > k(\gamma) \quad (z \in \mathbb{U})
\]

and

\[
 1 - \gamma p(z) - z p'(z) \prec G(z),
\]

then

\[
 p(z) - C z^{-\gamma} \prec z^{-\gamma} \int_0^z t^{\gamma-1}(1 - G(t)) dt,
\]

where \( C = p(0) \) for \( \gamma = 0 \) and \( C = 0 \) for \( \gamma > 0 \).

**Lemma 2.3.** (Tuneski [4]) Let us \( f(z) \in \mathcal{A} \). If it satisfies

\[
 \left| f'(z) - (1 - \gamma) \frac{f(z)}{z} - \gamma \right| < \lambda \quad (z \in \mathbb{U})
\]

for some \( \gamma \) (\( \gamma \geq 0 \)) and \( \lambda \) (\( \lambda > 0 \)), then

\[
 \left| \frac{f(z)}{z} - 1 \right| < \frac{\lambda}{1 + \gamma} \quad (z \in \mathbb{U})
\]

and

\[
 |f(z)| < 1 + \frac{\lambda}{1 + \gamma} \quad (z \in \mathbb{U}).
\]
Using similar manner of Lemma 2.3, Our first result is following

**Theorem 2.1.** If \( f(z) \in \mathcal{A} \) satisfies
\[
\left| \frac{z^2 f'(z)}{(f(z))^2} - (1 + \gamma) \frac{z}{f(z)} + \gamma \right| < \lambda \quad (z \in \mathbb{U})
\]
for some \( \gamma (\gamma \geq 0) \) and \( \lambda (\lambda > 0) \), then
\[
\left| \frac{z}{f(z)} - 1 \right| < \frac{\lambda}{1 + \gamma} \quad (z \in \mathbb{U}).
\]

**Proof.** Let us define the function \( G(z) \) by \( G(z) = 1 - \gamma + \lambda z \), then \( G'(0) = \lambda \) and
\[
\text{Re} \left( 1 + \frac{zG''(z)}{G'(z)} \right) = 1 \quad (z \in \mathbb{U}).
\]
Furthermore, let us suppose that \( p(z) = \frac{z}{f(z)} \), then \( p(0) = 1 \) and
\[
1 - \gamma p(z) - z p'(z) = 1 - (1 + \gamma) \frac{z}{f(z)} + \frac{z^2 f'(z)}{(f(z))^2}.
\]
On the other hand, we have
\[
1 - (1 + \gamma) \frac{z}{f(z)} + \frac{z^2 f'(z)}{(f(z))^2} < 1 - \gamma + \lambda z
\]
from inequality (1). Applying Lemma 2, we obtain
\[
\frac{z}{f(z)} - C z^{-\gamma} < z^{-\gamma} \int_0^z t^{\gamma-1}(1 - G(t))dt = 1 - \frac{\lambda}{1 + \gamma} z - C_1 z^{-\gamma},
\]
where \( C = C_1 = 1 \) for \( \gamma = 0 \) and \( C = C_1 = 0 \) for \( \gamma > 0 \). Thus, we arrive
\[
\left| \frac{z}{f(z)} - 1 \right| < \frac{\lambda}{1 + \gamma} \quad (z \in \mathbb{U}).
\]
The left hand side of the inequality (1) holds true for \( \lambda \) if we take the function
\[
f(z) = \frac{z}{1 + \gamma + \lambda e^{\theta} z}
\]
from inequality (2), implying that this result is sharp.

By virtue of Theorem 2.1, we obtain the sufficient condition of starlikeness below

**Theorem 2.2.** If \( f(z) \in \mathcal{A} \) satisfies
\[
\left| \frac{z^2 f'(z)}{(f(z))^2} - (1 + \gamma) \frac{z}{f(z)} + \gamma \right| < \lambda \quad (z \in \mathbb{U})
\]
for some \( \gamma (\gamma \geq 0) \) and \( \lambda \left( 0 < \lambda \leq \frac{1}{2} \right) \), then \( f(z) \in \mathcal{S}^\ast \left( \frac{(1 + \gamma)(1 - 2\lambda)}{1 + \gamma - \lambda} \right) \).
Proof. Supposing that a function $f(z)$ satisfies the inequality (3) and that there exists an analytic function $w(z)$ in $U$ such that $w(0) = 0$ and $|w(z)| < 1$, then we see
\[
\frac{zf'(z)}{f(z)} - (1 + \gamma) = \frac{f(z)}{z}(\lambda w(z) - \gamma).
\]
It follows that
\[
\left| \frac{zf'(z)}{f(z)} - (1 + \gamma) \right| = \left| \frac{f(z)}{z} \right| |\lambda w(z) - \gamma| < \frac{(1 + \gamma)(\gamma + \lambda)}{1 + \gamma - \lambda}.
\]
This shows that
\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 1 + \gamma - \frac{(1 + \gamma)(\gamma + \lambda)}{1 + \gamma - \lambda} = \frac{(1 + \gamma)(1 - 2\lambda)}{1 + \gamma - \lambda}.
\]
We complete the proof of the theorem.

Taking $\lambda = \frac{1}{2}$ in Theorem 2.2, we have

**Corollary 2.1.** If $f(z) \in A$ satisfies
\[
\left| \frac{zf'(z)}{(f(z))^2} - (1 + \gamma) \frac{z}{f(z)} + \gamma \right| < \frac{1}{2} \quad (z \in \mathbb{U})
\]
for some $\gamma$ ($\gamma \geq 0$), then $f(z) \in S^*$.  

Putting $zf'(z)$ instead of $f(z)$ in Theorem 2.1, we get

**Theorem 2.3.** If $f(z) \in A$ satisfies
\[
\left| \frac{zf''(z)}{(f'(z))^2} - \gamma \frac{1}{f'(z)} + \gamma \right| < \lambda \quad (z \in \mathbb{U})
\]
(4)
for some $\gamma$ ($\gamma \geq 0$) and $\lambda$ ($\lambda > 0$), then
\[
\left| \frac{1}{f'(z)} - 1 \right| < \frac{\lambda}{1 + \gamma} \quad (z \in \mathbb{U}).
\]
(5)

Proof. Letting $p(z) = \frac{1}{f'(z)}$ in the proof of Theorem 2.1, we arrive
\[
\left| \frac{1}{f'(z)} - 1 \right| < \frac{\lambda}{1 + \gamma} \quad (z \in \mathbb{U}).
\]
The left hand side of the inequality (4) holds true for $\lambda$ if we take the function
\[
f(z) = \frac{1 + \gamma}{\lambda \epsilon^{i\theta}} \log \left( 1 + \frac{\lambda}{1 + \gamma} e^{i\theta} z \right)
\]
from inequality (5), implying that this result is sharp.  \(\square\)
In view of Theorem 2.3, we obtain the sufficient condition of convexity below.

**Theorem 2.4.** If \( f(z) \in A \) satisfies
\[
\left| \frac{zf''(z)}{(f'(z))^2} - \frac{1}{f'(z)} + \frac{1}{2} f''(z) + \gamma \right| < \lambda \quad (z \in \mathbb{U})
\]
for some \( \gamma (\gamma \geq 0) \) and \( \lambda \left( 0 < \lambda \leq \frac{1}{2} \right) \), then \( f(z) \in \mathcal{K} \left( \frac{(1 + \gamma)(1 - 2\lambda)}{1 + \gamma - \lambda} \right) \).

**Proof.** As the same technique in the proof of Theorem 2.2, we see
\[
\left| \frac{zf''(z)}{f'(z)} - \gamma \right| < \frac{(1 + \gamma)(\gamma + \lambda)}{1 + \gamma + \lambda}.
\]
This shows that
\[
\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \frac{(1 + \gamma)(1 - 2\lambda)}{1 + \gamma - \lambda} \quad (z \in \mathbb{U})
\]
which proves the theorem. \( \square \)

Taking \( \lambda = \frac{1}{2} \) in Theorem 2.4, we have

**Corollary 2.2.** If \( f(z) \in A \) satisfies
\[
\left| \frac{zf''(z)}{(f'(z))^2} - \gamma \frac{1}{f'(z)} + \frac{1}{2} f''(z) + \gamma \right| < \frac{1}{2} \quad (z \in \mathbb{U})
\]
for some \( \gamma (\gamma \geq 0) \), then \( f(z) \in \mathcal{K} \).

Applying the same way as the proof of Theorem 2.1, we get

**Theorem 2.5.** If \( f(z) \in A \) satisfies
\[
\left| \frac{f(z)f''(z)}{(f'(z))^2} - (1 - \gamma) \frac{f(z)}{zf'(z)} + \gamma - 1 \right| < \lambda \quad (z \in \mathbb{U})
\]
for some \( \gamma (\gamma \geq 0) \) and \( \lambda (\lambda > 0) \), then
\[
\left| \frac{f(z)}{zf'(z)} - 1 \right| < \frac{\lambda}{1 + \gamma} \quad (z \in \mathbb{U}).
\]

**Proof.** Letting \( p(z) = \frac{f(z)}{zf'(z)} \) in the proof of Theorem 2.1, we arrive
\[
\left| \frac{f(z)}{zf'(z)} - 1 \right| < \frac{\lambda}{1 + \gamma} \quad (z \in \mathbb{U}).
\]
The left hand side of the inequality (6) holds true for \( \lambda \) if we take the function
\[
f(z) = \frac{z}{1 + \frac{\lambda}{1 + \gamma} e^{\theta z}}
\]
from the inequality (7), implying that this result is sharp.

In view of Theorem 2.5, we obtain the sufficient condition of convexity below

**Theorem 2.6.** If \( f(z) \in A \) satisfies
\[
  f(z) \in A, \quad \left| \frac{f(z) f''(z)}{(f'(z))^2} - (1 - \gamma) \frac{f(z)}{zf'(z)} + \gamma - 1 \right| < \lambda \quad (z \in \mathbb{U}),
\]
then \( f(z) \in K \left( 1 - \frac{2\gamma \lambda}{1 + \gamma - \lambda} \right) \) for some \( \gamma (\gamma \geq 1) \) and \( \lambda \left( 0 \leq \lambda \leq \frac{1 + \gamma}{2\gamma + 1} \right) \), or \( f(z) \in K \left( 1 + \frac{2\gamma^2 - 2\gamma \lambda - 2}{1 + \gamma - \lambda} \right) \) for some \( \gamma \left( \frac{1}{2} < \gamma \leq 1 \right) \) and \( \lambda \left( 0 < \lambda \leq \frac{2\gamma^2 + \gamma - 1}{2\gamma + 1} \right) \).

**Proof.** As the same technique in the proof of Theorem 2.2, we see
\[
  \left| \frac{zf''(z)}{f'(z)} + (1 - \gamma) \right| < \frac{(1 + \gamma)(\gamma - 1 + \lambda)}{1 + \gamma - \lambda} \quad (z \in \mathbb{U})
\]
when \( \gamma \geq 1 \) for the inequality (8). This shows that
\[
  \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 1 - \frac{2\gamma \lambda}{1 + \gamma - \lambda} \quad (z \in \mathbb{U}).
\]
Moreover, we see
\[
  \left| \frac{zf''(z)}{f'(z)} + (1 - \gamma) \right| < 1 - \frac{(1 + \gamma)(1 - \gamma + \lambda)}{1 + \gamma - \lambda} \quad (z \in \mathbb{U})
\]
when \( \left( \frac{1}{2} < \gamma \leq 1 \right) \) for the inequality (8). This shows that
\[
  \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 1 + \frac{2\gamma^2 - 2\gamma \lambda - 2}{1 + \gamma - \lambda} \quad (z \in \mathbb{U}).
\]
The proof of theorem is completed.

**References**


