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ON AN EXTENSION OF KUMMER-TYPE II TRANSFORMATION

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ABSTRACT. In the theory of hypergeometric and generalized hypergeometric series, Kummer's type I and II transformations play an important role.

In this short research paper, we aim to establish the explicit expression of

$$e^{-\frac{x}{2}}\,{}_2F_2\left[\begin{matrix} a, & d+n; & \\ & & x \\ 2a+n, & d; & \end{matrix}\right]$$

for $n = 3$.

For $n = 0$, we have the well known Kummer's second transformation. For $n = 1$, the result was established by Rathie and Pogany [12] and later on by Choi and Rathie [2]. For $n = 2$, the result was recently established by Rakha, et al. [10]. The result is derived with the help of Kummer's second transformation and its contiguous results recently obtained by Kim, et. al.[4]. The result established in this short research paper is simple, interesting, easily established and may be potentially useful.

Keywords: Generalized Hypergeometric Series, Kummer's type I and II transformations.

AMS Subject Classification: 33C20.

1. Introduction and Preliminaries

In the theory of hypergeometric and generalized hypergeometric series, summation and transformation formulas play an important role. For this, we start with the following Kummer-type I transformation [1, 6, 8], for the series ${}_1F_1$, viz.

$$e^{-x}\,{}_1F_1\left[\begin{matrix} a; & \\ & x \\ b; & \end{matrix}\right] = {}_1F_1\left[\begin{matrix} b-a; & \\ & -x \\ b; & \end{matrix}\right]. \tag{1}$$

Recently, Paris [7] generalized (1) in the form

$$e^{-x}\,{}_2F_2\left[\begin{matrix} a, & 1+d; & \\ & & x \\ 1+b, & d; & \end{matrix}\right] = {}_2F_2\left[\begin{matrix} b-a, & f+1; & \\ & & -x \\ b+1, & f; & \end{matrix}\right] \tag{2}$$

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where

$$f = \frac{d(a-b)}{a-d}. \tag{3}$$

The well known Kummer-type II transformation [6] is

$$e^{-\frac{x}{2}}\,{}_1F_1\left[\begin{matrix} a; \\ 2a; \end{matrix}\; x\right] = {}_0F_1\left[\begin{matrix} -; \\ a+\frac{1}{2}; \end{matrix}\; \frac{x^2}{16}\right]. \tag{4}$$

Bailey [1] established the result (4) by employing the Gauss second summation theorem and Choi and Rathie [2] established (4) by employing classical Gauss summation theorem.

Motivated by the extension of Kummer type I transformation (2) obtained by Paris [7], recently Rathie and Pogany [12] have given the following interesting extension of Kummer type II transformation in the form

$$\begin{aligned} & e^{-\frac{x}{2}}\,{}_2F_2\left[\begin{matrix} a, & 1+d; \\ 2a+1, & d; \end{matrix}\; x\right] \\ & = {}_0F_1\left[\begin{matrix} -; \\ a+\frac{1}{2}; \end{matrix}\; \frac{x^2}{16}\right] - \frac{x\left(1-\frac{2a}{d}\right)}{2(2a+1)}\,{}_0F_1\left[\begin{matrix} -; \\ a+\frac{3}{2}; \end{matrix}\; \frac{x^2}{16}\right]. \end{aligned} \tag{5}$$

Recently, Kim, et al. [4] have generalized the Kummer type II transformation (4) and obtained explicit expressions of

$$e^{-\frac{x}{2}}\,{}_1F_1\left[\begin{matrix} a; \\ 2a+j; \end{matrix}\; x\right] \tag{6}$$

for $j = 0, \pm1, \pm2, \ldots, \pm5$.

Very recently, Rakha et al. [10] have given another extension of Kummer type II transformation (4) in the following form

$$\begin{aligned} & e^{-\frac{x}{2}}\,{}_2F_2\left[\begin{matrix} a, & 2+d; \\ 2a+2, & d; \end{matrix}\; x\right] \\ & = {}_0F_1\left[\begin{matrix} -; \\ a+\frac{3}{2}; \end{matrix}\; \frac{x^2}{16}\right] + \frac{x\left(\frac{a}{d}-\frac{1}{2}\right)}{(a+1)}\,{}_0F_1\left[\begin{matrix} -; \\ a+\frac{3}{2}; \end{matrix}\; \frac{x^2}{16}\right] \\ & + \frac{cx^2}{2(2a+3)}\,{}_0F_1\left[\begin{matrix} -; \\ a+\frac{5}{2}; \end{matrix}\; \frac{x^2}{16}\right] \end{aligned} \tag{7}$$

where

$$c = \frac{\left(\frac{1}{2}-\frac{a}{d}\right)}{a+1} + \frac{a}{d(d+1)} \tag{8}$$

for $d \neq 0, -1, -2, \ldots$.

In this short research paper, we aim to establish another extension of Kummer type II transformation in the form

$$e^{-\frac{x}{2}}\,{}_2F_2\left[\begin{matrix} a, & 3+d; \\ 2a+3, & d; \end{matrix}\; x\right]. \tag{9}$$

The result is derived with the help of Kummer type II transformation (4) and its various contiguous results recently obtained by Kim, et al. [4]. For this the following results obtainable from (6) will be required in our present investigations.

$$e^{-\frac{x}{2}}\,{}_1F_1\left[\begin{array}{c} a; \\ \\ 2a+1; \end{array} x\right]$$
$$= {}_0F_1\left[\begin{array}{c} -; \\ \\ a+\frac{1}{2}; \end{array} \frac{x^2}{16}\right] - \frac{x}{2(2a+1)}\,{}_0F_1\left[\begin{array}{c} -; \\ \\ a+\frac{3}{2}; \end{array} \frac{x^2}{16}\right], \tag{10}$$

$$e^{-\frac{x}{2}}\,{}_1F_1\left[\begin{array}{c} a; \\ \\ 2a+2; \end{array} x\right]$$
$$= {}_0F_1\left[\begin{array}{c} -; \\ \\ a+\frac{3}{2}; \end{array} \frac{x^2}{16}\right] - \frac{x}{2(a+1)}\,{}_0F_1\left[\begin{array}{c} -; \\ \\ a+\frac{3}{2}; \end{array} \frac{x^2}{16}\right]$$
$$+ \frac{x^2}{4(a+1)(2a+3)}\,{}_0F_1\left[\begin{array}{c} -; \\ \\ a+\frac{5}{2}; \end{array} \frac{x^2}{16}\right] \tag{11}$$

and

$$e^{-\frac{x}{2}}\,{}_1F_1\left[\begin{array}{c} a; \\ \\ 2a+3; \end{array} x\right]$$
$$= {}_0F_1\left[\begin{array}{c} -; \\ \\ a+\frac{3}{2}; \end{array} \frac{x^2}{16}\right] - \frac{3x}{2(2a+3)}\,{}_0F_1\left[\begin{array}{c} -; \\ \\ a+\frac{5}{2}; \end{array} \frac{x^2}{16}\right]$$
$$+ \frac{x^2}{2(a+2)(2a+3)}\,{}_0F_1\left[\begin{array}{c} -; \\ \\ a+\frac{5}{2}; \end{array} \frac{x^2}{16}\right] - \frac{x^3}{4(a+2)(2a+3)(2a+5)}\,{}_0F_1\left[\begin{array}{c} -; \\ \\ a+\frac{7}{2}; \end{array} \frac{x^2}{16}\right]. \tag{12}$$

2. Main Result

The following extension of the Kummer type II transformation will be established in this short research paper

$$e^{-\frac{x}{2}}\,{}_2F_2\left[\begin{array}{cc} a, & 3+d; \\ & \\ 2a+3, & d; \end{array} x\right]$$
$$= {}_0F_1\left[\begin{array}{c} -; \\ \\ a+\frac{3}{2}; \end{array} \frac{x^2}{16}\right] + c_1 x\,{}_0F_1\left[\begin{array}{c} -; \\ \\ a+\frac{5}{2}; \end{array} \frac{x^2}{16}\right]$$
$$+ c_2 x^2\,{}_0F_1\left[\begin{array}{c} -; \\ \\ a+\frac{5}{2}; \end{array} \frac{x^2}{16}\right] + c_3 x^3\,{}_0F_1\left[\begin{array}{c} -; \\ \\ a+\frac{7}{2}; \end{array} \frac{x^2}{16}\right] \tag{13}$$

where

$$c_1 = \frac{3\left(\frac{1}{2} - \frac{a}{d}\right)}{(2a+3)} \tag{14}$$

$$c_2 = \frac{\left\{1 - \frac{3a}{d} + \frac{3a(a+1)}{d(d+1)}\right\}}{2(a+2)(2a+3)} \tag{15}$$

and

$$c_3 = \frac{\left\{\frac{3a}{2d} - \frac{1}{2} - \frac{3a(a+1)}{2d(d+1)} + \frac{a(a+1)(a+2)}{d(d+1)(d+2)}\right\}}{2(a+2)(2a+3)(2a+5)}. \tag{16}$$

Proof. Using the definition of the Pochhammer's symbol

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)},$$

it is not difficult to prove the following result

$$\frac{(d+3)_n}{(d)_n} = 1 + \frac{3n}{d} + \frac{3n(n-1)}{d(d+1)} + \frac{n(n-1)(n-2)}{d(d+1)(d+2)}. \tag{17}$$

Now, in order to establish our main result (13), we proceed as follows. Express ${}_2F_2$ as a series, we have

$${}_2F_2\left[\begin{matrix} a, & 3+d; & \\ & & x \\ 2a+3, & d; & \end{matrix}\right] = \sum_{n=0}^{\infty}\frac{(a)_n}{(2a+3)_n}\frac{x^n}{n!}\left\{\frac{(d+3)_n}{(d)_n}\right\}.$$

Using (17), we have

$$\begin{aligned}
&{}_2F_2\left[\begin{matrix} a, & 3+d; & \\ & & x \\ 2a+3, & d; & \end{matrix}\right] \\
&= \sum_{n=0}^{\infty}\frac{(a)_n}{(2a+3)_n}\frac{x^n}{n!}\left\{1 + \frac{3n}{d} + \frac{3n(n-1)}{d(d+1)} + \frac{n(n-1)(n-2)}{d(d+1)(d+2)}\right\} \\
&= \sum_{n=0}^{\infty}\frac{(a)_n}{(2a+3)_n}\frac{x^n}{n!} + \frac{3}{d}\sum_{n=1}^{\infty}\frac{(a)_n}{(2a+3)_n}\frac{x^n}{(n-1)!} \\
&+ \frac{3}{d(d+1)}\sum_{n=2}^{\infty}\frac{(a)_n}{(2a+3)_n}\frac{x^n}{(n-2)!} + \frac{3}{d(d+1)(d+2)}\sum_{n=3}^{\infty}\frac{(a)_n}{(2a+3)_n}\frac{x^n}{(n-3)!}.
\end{aligned}$$

Now replacing $n-1$ by N, $n-2$ by N and $n-3$ by N in 2^{nd}, 3^{rd} and 4^{th} series and using the results

$$(a)_{N+1} = a(a+1)_N$$
$$(a)_{N+2} = a(a+1)(a+2)_N$$

and

$$(a)_{N+3} = a(a+1)(a+2)(a+3)_N$$

and after some simplification, we have

$$
\begin{aligned}
&{}_2F_2\left[\begin{array}{cc} a, & 3+d; \\ & & x \\ 2a+3, & d; \end{array}\right] \\
&= \sum_{n=0}^{\infty} \frac{(a)_n}{(2a+3)_n} \frac{x^n}{n!} + \frac{ax}{(2a+3)} \sum_{N=0}^{\infty} \frac{(a+1)_N}{(2a+4)_N} \frac{x^N}{N!} \\
&+ \frac{a(a+1)x^2}{(2a+3)(2a+4)} \sum_{N=0}^{\infty} \frac{(a+2)_N}{(2a+5)_N} \frac{x^N}{N!} + \frac{a(a+1)(a+2)x^3}{(2a+3)(2a+4)(2a+5)} \sum_{N=0}^{\infty} \frac{(a+3)_N}{(2a+6)_N} \frac{x^N}{N!}.
\end{aligned}
$$

Finally, summing up the series, we have

$$
\begin{aligned}
&{}_2F_2\left[\begin{array}{cc} a, & 3+d; \\ & & x \\ 2a+3, & d; \end{array}\right] \\
&= {}_1F_1\left[\begin{array}{c} a; \\ & x \\ 2a+3; \end{array}\right] + \frac{3ax}{d(2a+3)}\, {}_1F_1\left[\begin{array}{c} a+1; \\ & x \\ 2a+4; \end{array}\right] \\
&+ \frac{3a(a+1)x^2}{d(d+1)(2a+3)(2a+4)}\, {}_1F_1\left[\begin{array}{c} a+2; \\ & x \\ 2a+5; \end{array}\right] \\
&+ \frac{a(a+1)(a+2)x^3}{d(d+1)(d+2)(2a+3)(2a+4)(2a+5)}\, {}_1F_1\left[\begin{array}{c} a+3; \\ & x \\ 2a+6; \end{array}\right]. \qquad (18)
\end{aligned}
$$

Now, multiply (18) both sides by $e^{-\frac{x}{2}}$, we have

$$
\begin{aligned}
&e^{-\frac{x}{2}}\, {}_2F_2\left[\begin{array}{cc} a, & 3+d; \\ & & x \\ 2a+3, & d; \end{array}\right] \\
&= e^{-\frac{x}{2}}\, {}_1F_1\left[\begin{array}{c} a; \\ & x \\ 2a+3; \end{array}\right] + \frac{3ax}{d(2a+3)} e^{-\frac{x}{2}}\, {}_1F_1\left[\begin{array}{c} a+1; \\ & x \\ 2a+4; \end{array}\right] \\
&+ \frac{3a(a+1)x^2}{d(d+1)(2a+3)(2a+4)} e^{-\frac{x}{2}}\, {}_1F_1\left[\begin{array}{c} a+2; \\ & x \\ 2a+5; \end{array}\right] \\
&+ \frac{a(a+1)(a+2)x^3}{d(d+1)(d+2)(2a+3)(2a+4)(2a+5)} e^{-\frac{x}{2}}\, {}_1F_1\left[\begin{array}{c} a+3; \\ & x \\ 2a+6; \end{array}\right]. \qquad (19)
\end{aligned}
$$

Now, it is easy to see that the first, second, third and fourth $e^{-\frac{x}{2}}\ {}_1F_1$ appearing on the right-hand side can be evaluated with the help of the known results (12), (11), (10) and (4) respectively and after some simplification, we arrive at the desired result (13). This completes the proof of (13). □

Remark 2.1. *Setting $d = 2a$ in (13), we see that $c_1 = c_3 = 0$ and $c_2 = \frac{1}{4(2a+1)(2a+3)}$, and we have*

$$e^{-\frac{x}{2}}\, {}_1F_1\left[\begin{matrix} a; \\ 2a; \end{matrix}\; x\right]$$

$$= {}_0F_1\left[\begin{matrix} -; \\ a+\frac{3}{2}; \end{matrix}\; \frac{x^2}{16}\right] + \frac{x^2}{4(2a+1)(2a+3)}\, {}_0F_1\left[\begin{matrix} -; \\ a+\frac{5}{2}; \end{matrix}\; \frac{x^2}{16}\right] \tag{20}$$

and it is not difficult to see that the right-hand side of (20) equals ${}_0F_1\left[\begin{matrix} -; \\ a+\frac{1}{2}; \end{matrix}\; \frac{x^2}{16}\right]$ *and thus we arive at the Kummer's second transformation (4). Thus our main result (13) may be regarded as an extension of (4).*

3. Concluding Remark

In this short research paper, we have obtained the extension of Kummer's second transformation viz

$$e^{-\frac{x}{2}}\, {}_2F_2\left[\begin{matrix} a, & d+n; \\ 2a+n, & d; \end{matrix}\; x\right]$$

for $n = 3$.

We conclude this short research paper by remarking that the extension of the Kummer's second transformation in the most general form any $n = 0, 1, 2, \dots$ are under investigation and together with some interesting applications it will be published soon.

References

[1] Bailey, W. N., Products of generalized hypergeometric series, Proc. London Math. Soc., 28, 242 - 250 (1928).

[2] Choi, J. and Rathie, A. K., Another proof of Kummer's second theorem, Commun. Korean Math. Soc., 13, 933 - 936 (1998).

[3] Kim, Y. S., Rakha M. A., and Rathei, A. K., Extensions of classical summation theorems for the series ${}_2F_1, {}_3F_2$ and ${}_4F_3$ with applications in Ramanujan's summations, Int. J. Math. & Math. Sci., ID 309503, 26 pages, (2010).

[4] Kim, Y. S., Rakha, M. A., and Rathei, A. K., Generalizations of Kummer's second theorem with applications, Comput. Math. & Math. Phys., 50 (3), 387 - 402 (2010).

[5] Kim, Y. S., Choi, J. and Rathie, A. K., Two results for the terminating ${}_3F_2(2)$ with applications, Bull. Korean Math. Soc., 49 (3),621 – 633 (2012).

[6] Kummer, E. E., Über die hypergeometridche Reihe ..., J. Reine Angew. Math., 15, 39 - 83 (1836).

[7] Paris, R. B., A Kummer type transformation for a ${}_2F_2$ hypergeometric function, J. Comput. Appl. Math., 173, 379 - 382 (2005).

[8] Rainville, E. D., Special Functions, The Macmillan Company, New York (1960).

[9] Rakha, M. A. and Rathie, A. K., Generalizations of classical summation theorems for the series ${}_2F_1$ and ${}_3F_2$ with applications, Integral Transform and Special Functions, 22 (11), 823 - 840 (2011).

[10] Rakha, M. A., Awad, M. M., and Rathie, A. K., On an extension of Kummer's second theorem, Abstract and Applied Analysis, Volume 2013, Article ID 128458, 6 pages.

[11] Rakha, M. A. A note on Kummer-type II transformation for the generalized hypergeometric function ${}_2F_2$, Mathematical Notes, 19 (1), 154 - 156 (2012).

[12] Rathie, A. K. and Pogany, K. New summation formula for ${}_3F_2\left(\frac{1}{2}\right)$ and Kummer-type II transformation, Math. Communic, 13, 63 - 66 (2008).