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HARMONIC MAPPINGS RELATED TO THE CONVEX FUNCTIONS

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Abstract. The main purpose of this paper is to give the extent idea which was introduced by R. M. Robinson [5]. One of the interesting application of this extent idea is an investigation of the class of harmonic mappings related to the convex functions.

Keywords: Harmonic Mappings, Distortion theorem, Growth theorem, Coefficient inequality.

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1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} || z| < 1\}$ be the open unit disc in the complex plane $\mathbb{C}$. A complex-valued harmonic function $f : \mathbb{D} \to \mathbb{C}$ has the representation

$$f = h(z) + \overline{g(z)} \tag{1}$$

where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ and have the following power series expansions,

$$h(z) = \sum_{n=0}^{\infty} a_n z^n,$$

$$g(z) = \sum_{n=0}^{\infty} b_n z^n, z \in \mathbb{D},$$

where a_n,$b_n \in \mathbb{C}$, $n = 0, 1, 2, ...$ Choose (i.e, $b_0 = 0$) so the representation (1) is unique in $\mathbb{D}$ and is called the canonical representation of f.
For the univalent and sense-preserving harmonic mappings f in $\mathbb{D}$, it is convenient to make further normalization (without loss of generality), $h(0) = 0$ (i.e. , $a_0 = 0$) and $h'(0) = 1$ (i.e. , $a_1 = 1$). The family of such functions f is denoted by S_H [1] . The family of all functions $f \in S_H$ with the additional property that $g'(0) = 0$ (i.e. , $b_1 = 0$) is denoted by S_H^0 [1] . Observe that the classical family of univalent functions S consists of all functions $f \in S_H^0$ such that $g(z) \equiv 0$. Thus it is clear that $S \subset S_H^0 \subset S_H$ [1] .

Let Ω be the family of functions $\phi(z)$ regular in the open unit disc $\mathbb{D}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for every $z \in \mathbb{D}$.
Next, for arbitrary fixed real numbers A, B, $-1 \leq B < A \leq 1$, denoted by $P(A, B)$, the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + ...$ regular in $\mathbb{D}$ and such that $p(z)$ is in $P(A, B)$

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if and only if

$$p(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)}, \tag{2}$$

for some function $\phi(z) \in \Omega$ and every $z \in \mathbb{D}$. This class was introduced by Janowski W. [4].
Moreover, let $S^*(A, B)$ denote the family of functions $h(z) = z + a_2z^2 + ...$ regular in $\mathbb{D}$ and such that $h(z)$ is in $S^*(A, B)$ if and only if

$$z\frac{h'(z)}{h(z)} = p(z) \tag{3}$$

for some $p(z) \in P(A, B)$ and every $z \in \mathbb{D}$.
A set $\mathbb{D}$ in the plane is called convex if for every pair of points w_1 and w_2 in the interior of $\mathbb{D}$, the line segment joining w_1 and w_2 also in the interior of $\mathbb{D}$. If a function $h(z)$ maps $\mathbb{D}$ onto a convex domain, then $h(z)$ is called a convex function. The analytic statement of the convex function $h(z)$ is given by

$$Re(1 + z\frac{h''(z)}{h'(z)}) > 0 \tag{4}$$

and the class of such functions is denoted by C.
Finally, let $F_1(z) = z + \alpha_2z^2 + \alpha_3z^3 + ...$ and $F_2(z) = z + \beta_2z^2 + \beta_3z^3 + ...$ be analytic functions in $\mathbb{D}$. If there exists a function $\phi(z) \in \Omega$ such that $F_1(z) = F_2(\phi(z))$ for all $z \in \mathbb{D}$, then we say that $F_1(z)$ is subordinated to $F_2(z)$ and we write $F_1(z) \prec F_2(z)$. We also note that if $F_1(z) \prec F_2(z)$, then $F_1(\mathbb{D}) \subset F_2(\mathbb{D})$.
Now we consider the following class of harmonic mappings,

$$S_H^C(A, B) = \left\{ f = h(z) + \overline{g(z)} \mid w(z) = \frac{g'(z)}{h'(z)} \prec b_1\frac{1 + A\phi(z)}{1 + B\phi(z)}, h(z) \in C \right\} \tag{5}$$

In the present paper we will give some properties of the class of $S_H^C(A, B)$.
For the aim of this paper we need the following theorem and lemma.

Lemma 1.1. *([3]) Let $\phi(z)$ be regular in the open unit disc $\mathbb{D}$ with $\phi(0) = 0$ and $|\phi(z)| < 1$. If $|\phi(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_1, then we have $z_1.\phi'(z) = k\phi(z_1)$ for some $k \geq 1$.*

Theorem 1.1. *([2]) If $h(z) \in C$, then for $|z| = r$, $0 \leq r < 1$*

i. $Re(z\frac{h'(z)}{h(z)}) > \frac{1}{2}$

ii. $\frac{r}{1+r} \leq |h(z)| \leq \frac{r}{1-r}$,
$\frac{1}{(1+r)^2} \leq |h'(z)| \leq \frac{1}{(1-r)^2}$

for all $|z| = r < 1$.

2. Main Results

Theorem 2.1. *Let $f = (h(z) + \overline{g(z)})$ be an element of $S_H^C(A, B)$. Then*

$$\frac{g(z)}{h(z)} \prec b_1\frac{1 + Az}{1 + Bz}, \tag{6}$$

Proof. Since $f = h(z) + \overline{g(z)}$ be an element of $S_H^C(A,B)$, then

$$\frac{g'(z)}{h'(z)} \prec b_1 \frac{1+Az}{1+Bz} \Rightarrow \frac{1}{b_1}\frac{g'(z)}{h'(z)} = \frac{1+A\phi(z)}{1+B\phi(z)} \Rightarrow$$
$$\left|\frac{1}{b_1}\frac{g'(z)}{h'(z)} - \frac{1-ABr^2}{1-B^2r^2}\right| \leq \frac{(A-B)r}{1-B^2r^2} \Rightarrow$$
$$|b_1|\frac{1-Ar}{1-Br} \leq \left|\frac{g'(z)}{h'(z)}\right| \leq |b_1|\frac{1+Ar}{1+Br} \tag{7}$$

Therefore the relations (7) shows that the values of $(\frac{g'(z)}{h'(z)})$ are in the disc

$$D_r(b_1) = \begin{cases} \left\{\frac{g'(z)}{h'(z)} \mid \left|\frac{g'(z)}{h'(z)} - \frac{b_1(1-AB)r^2}{1-B^2r^2}\right| \leq \frac{|b_1|(A-B)r}{1-B^2r^2}\right\}, & B \neq 0; \\ \left\{\frac{g'(z)}{h'(z)} \mid \left|\frac{g'(z)}{h'(z)} - b_1\right| \leq |b_1|\, Ar\right\}, & B = 0. \end{cases} \tag{8}$$

Now we define a function $\phi(z)$ by

$$\frac{g(z)}{h(z)} = b_1 \frac{1+A\phi(z)}{1+B\phi(z)} \tag{9}$$

then $\phi(z)$ is analytic in $\mathbb{D}$ and $\phi(0) = 0$. Now we need to show that $|\phi(z)| < 1$. If we take derivative from (9), we obtain

$$\frac{g'(z)}{h'(z)} = b_1\left[\frac{1+A\phi(z)}{1+B\phi(z)} + \frac{(A-B)z\phi'(z)}{(1+B\phi(z))^2}\frac{h(z)}{zh'(z)}\right] \tag{10}$$

On the other hand since $h(z) \in C$, using Theorem 1.1

$$Re(z\frac{h'(z)}{h(z)}) > \frac{1}{2} \Rightarrow \left|z\frac{h'(z)}{h(z)} - \frac{1}{1-r^2}\right| \leq \frac{r}{1-r^2} \Rightarrow \frac{h(z)}{zh'(z)} = 1 + re^{i\theta}$$

this shows that the boundary value $\frac{h(z)}{zh'(z)}$. Taking derivative from (9) we get

$$w(z) = \frac{g'(z)}{h'(z)} = \begin{cases} b_1\left(\frac{1+A(\phi_z)}{1+B\phi(z)} + \frac{(A-B)z\phi'(z)}{(1+B\phi(z))^2}\frac{h(z)}{zh'(z)}\right), & B \neq 0; \\ b_1[(1+A\phi(z)) + Az\phi'(z)\frac{h(z)}{zh'(z)}], & B = 0. \end{cases} \tag{11}$$

In this step if we use the Jack's Lemma then we obtain

$$w(z_1) = \frac{g'(z_1)}{h'(z_1)} = \begin{cases} b_1\left(\frac{1+A(\phi(z_1)}{1+B\phi(z_1)} + \frac{k(A-B)\phi(z_1)}{(1+B\phi(z_1))^2}(1+re^{i\theta})\right] \notin w(\mathbb{D}_r), & B \neq 0; \\ b_1[(1+A\phi(z_1)) + kA\phi(z_1)(1+re^{i\theta})] \notin w(\mathbb{D}_r), & B = 0. \end{cases} \tag{12}$$

because $k \geq 1$ and $|\phi'(z_1)| = 1$ .This contradiction with

$$\frac{g'(z)}{h'(z)} \prec b_1 \frac{1+Az}{1+Bz}$$

therefore $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. □

Lemma 2.1. *Let* $f = (h(z) + \overline{g(z)})$ *be an element of* $S_H^C(A,B)$, *then*

$$\begin{cases} \frac{|b_1|(1-Ar)}{1-Br} \leq |w(z)| \leq \frac{|b_1|(1+Ar)}{1+Br}, & B \neq 0; \\ |b_1|\,(1-Ar) \leq |w(z)| \leq |b_1|\,(1+Ar), & B = 0. \end{cases} \tag{13}$$

Proof. Using Theorem 2.1, then we have

$$\begin{cases} \left|\frac{g'(z)}{h'(z)} - \frac{b_1(1-ABr^2)}{1-B^2r^2}\right| \leq \frac{|b_1|(A-B)r}{1-B^2r^2}, & B \neq 0; \\ \\ \left|\frac{g'(z)}{h'(z)} - b_1\right| \leq |b_1| \, Ar, & B = 0. \end{cases} \tag{14}$$

After the simple calculations from (14) we get (13). □

Corollary 2.1. *Let* $f = (h(z) + \overline{g(z)})$ *be an element of* $S_H^C(A, B)$*, then*

$$\frac{[(1-|b_1|)+(B-|b_1|\,A)r][(1+|b_1|)+(B+|b_1|\,A)r]}{(1+Br)^2} \leq (1-|w(z)|^2) \leq$$

$$\frac{[(1-|b_1|)-(B-|b_1|\,A)r][(1+|b_1|)-(B+|b_1|\,A)r]}{(1-Br)^2}$$

$$\frac{(1+|b_1|)-(B+|b_1|\,A)r}{1-Br} \leq (1+|w(z)|) \leq \frac{(1+|b_1|)+(B+|b_1|\,A)r}{1+Br}$$

$$\frac{(1-|b_1|)+(B-|b_1|\,A)r}{1+Br} \leq (1-|w(z)|) \leq \frac{(1-|b_1|)-(B-|b_1|\,A)r}{1-Br}$$

Proof. This corollary is a simple consequence of Lemma 2.1. □

Theorem 2.2. *Let* $f = (h(z) + \overline{g(z)})$ *be an element of* $S_H^C(A, B)$*, then*

$$\begin{cases} rF_1(A,B,-r) \leq |g(z)| \leq rF_1(A,B,r), & B \neq 0; \\ \\ rG_1(A,-r) \leq |g(z)| \leq rG_1(A,r), & B = 0. \end{cases} \tag{15}$$

where

$$F_1(A,B,r) = \frac{1}{1-r}\frac{|b_1|\,(1+Ar)}{1+Br}, B \neq 0$$

$$G_1(A,r) = \frac{1}{1-r}\,|b_1|\,(1+Ar), B = 0$$

and

$$\begin{cases} F_2(A,B,-r) \leq |g'(z)| \leq F_2(A,B,r), & B \neq 0; \\ \\ G_2(A,-r) \leq |g'(z)| \leq G_2(A,r), & B = 0. \end{cases} \tag{16}$$

where

$$F_2(A,B,r) = \frac{1}{(1-r)^2}\frac{|b_1|\,(1+Ar)}{1+Br}, B \neq 0$$

$$G_2(A,r) = \frac{1}{(1-r)^2}\,|b_1|\,(1+Ar), B = 0$$

Proof. Using the definition of the class $S_H^C(A, B)$ and Theorem (1.1) we obtain

$$\begin{cases} \frac{|b_1|(1-Ar)}{1-Br} \leq |w(z)| = \left|\frac{g'(z)}{h'(z)}\right| \leq \frac{|b_1|(1+Ar)}{1+Br}, & B \neq 0; \\ \\ |b_1|\,(1-Ar) \leq |w(z)| = \left|\frac{g'(z)}{h'(z)}\right| \leq |b_1|\,(1+Ar), & B = 0. \end{cases} \tag{17}$$

$$\begin{cases} |h'(z)| \frac{|b_1|(1-Ar)}{1-Br} \le |g'(z)| \le |h'(z)| \frac{|b_1|(1+Ar)}{1+Br}, & B \neq 0; \\ \\ |h'(z)| \, ||b_1|| \, (1-Ar) \le |g'(z)| \le |h'(z)| \, |b_1| \, (1+Ar), & B = 0. \end{cases} \tag{18}$$

similarly we obtain

$$\begin{cases} |h(z)| \frac{b_1(1-Ar)}{1-Br} \le |g(z)| \le |h(z)| \frac{b_1(1+Ar)}{1+Br}, & B \neq 0; \\ \\ |h(z)| \, |b_1| \, (1-Ar) \le |g(z)| \le |h(z)| \, |b_1| \, (1+Ar), & B = 0. \end{cases} \tag{19}$$

Using Theorem 1.1 in inequalities (18) and (19) we get (15) and (16). □

Corollary 2.2. *If $f = (h(z) + \overline{g(z)})$ be an element of $S_H^C(A,B)$, then*

$$\begin{cases} \frac{1}{(1+r)^4} F_3(A,B,|b_1|,-r) \le \left|J_{f(z)}\right| \le \frac{1}{(1-r)^4} F_3(A,B,|b_1|,r), & B \neq 0; \\ \\ \frac{1}{(1+r)^4} G_3(A,|b_1|,-r) \le \left|J_{f(z)}\right| \le \frac{1}{(1-r)^4} G_3(A,|b_1|,r), & B = 0. \end{cases} \tag{20}$$

where

$$F_3(A,B,|b_1|,r) = \frac{[(1-|b_1|) - (B - |b_1| A)r][(1+|b_1|) - (B + |b_1| A)r]}{(1-Br)^2}$$

$$G_3(A,|b_1|,r) = (1 - |b_1| + |b_1| Ar)(1 + |b_1| - |b_1| Ar)$$

Proof. Since

$$\begin{aligned} J_f = \left|h'(z)\right|^2 - \left|g'(z)\right|^2 &= \left|h'(z)\right|^2 - \left|h'(z)w(z)\right|^2 \\ &= \left|h'(z)\right|^2 (1 - |w(z)|^2), \end{aligned}$$

then using Theorem 2.2 and Corollary 2.1 we get (20).

□

Corollary 2.3. *Let $f = (h(z) + \overline{g(z)})$ be an element of $S_H^C(A,B)$, then*

$$\begin{cases} \ln e^{-\frac{B-1+|b_1|(1-A)}{(B-1)(1+r)}} (1+r)^{\frac{|b_1|(B-A)}{(B-1)^2}} (1+Br)^{-\frac{|b_1|(B-A)}{(B-1)^2}} \le |f| \\ \le \ln e^{\frac{(B+1)+|b_1|(1+A)}{(B+1)(1-r)}} (1-r)^{\frac{|b_1|(A-B)}{(B+1)^2}} (1+Br)^{\frac{|b_1|(B-A)}{(B+1)^2}}, & B \neq 0; \\ \\ \ln e^{\frac{-1+|b_1|(1-A)}{1+r}} (1+r)^{-A|b_1|} \le |f| \le \ln e^{\frac{1+|b_1|(1+A)}{1-r}} (1-r)^{A|b_1|}, & B = 0. \end{cases} \tag{21}$$

Proof. Since

$$\begin{aligned} &(\left|h'(z)\right| - \left|g'(z)\right|) |dz| \le |df| \le (\left|h'(z)\right| + \left|g'(z)\right|) |dz| \Rightarrow \\ &(\left|h'(z)\right| - \left|h'(z)w(z)\right|) |dz| \le |df| \le (\left|h'(z)\right| + \left|h'(z)w(z)\right|) |dz| \Rightarrow \\ &\left|h'(z)\right| (1 - |w(z)|) |dz| \le |df| \le \left|h'(z)\right| (1 + |w(z)|) |dz| \end{aligned} \tag{22}$$

Using Corollary 2.1 and Theorem 1.1 we obtain the following inequalities

$$\begin{cases} \int_0^r \frac{1}{(1+t)^2} \frac{(1-|b_1|)+(B-|b_1|A)t}{1+Bt} dt \le |f| \le \int_0^r \frac{1}{(1-t)^2} \frac{(1+|b_1|)+(B+|b_1|A)t}{1+Bt} dt, & B \neq 0; \\ \\ \int_0^r \frac{1}{(1+t)^2} [(1-|b_1|) - |b_1| At] dt \le |f| \le \int_0^r \frac{1}{(1-t)^2} [(1+|b_1|) + |b_1| At] dt, & B = 0. \end{cases} \tag{23}$$

and by calculating the integral we get (21). □

Theorem 2.3. *If* $f = (h(z) + \overline{g(z)})$ *be an element of* $S_H^C(A, B)$*, then*

$$\sum_{k=1}^{n} |b_k - b_1 a_k|^2 \leq |b_1| (A - B)^2 + \sum_{k=1}^{n-1} |Ab_1 a_k - Bb_k| \tag{24}$$

Proof. Using Theorem 2.1 then we can write

$$\frac{g(z)}{h(z)} \prec b_1 \frac{1 + Az}{1 + Bz} \Rightarrow \frac{g(z)}{h(z)} = b_1 \frac{1 + A\phi(z)}{1 + B\phi(z)} \Rightarrow$$

$$\frac{\frac{1}{b_1} g(z)}{h(z)} = \frac{1 + A\phi(z)}{1 + B\phi(z)} \Rightarrow$$

$$\frac{\frac{1}{b_1}(b_1 z + b_2 z^2 + ...)}{z + a_2 z^2 + a_3 z^3 + ...} = \frac{1 + A\phi(z)}{1 + B\phi(z)} \Rightarrow$$

$$\frac{z + \frac{b_2}{b_1} z^2 + \frac{b_3}{b_1} z^3 + ...}{z + a_2 z^2 + a_3 z^3 + ...} = \frac{1 + A\phi(z)}{1 + B\phi(z)} \Rightarrow$$

$$\frac{z + \alpha_2 z^2 + \alpha_3 z^3 + ...}{z + a_2 z^2 + a_3 z^3 + ...} = \frac{1 + A\phi(z)}{1 + B\phi(z)} \Rightarrow$$

$$\frac{1 + \alpha_2 z + \alpha_3 z^2 + ...}{1 + a_2 z + a_3 z^2 + ...} = \frac{1 + A\phi(z)}{1 + B\phi(z)} = \frac{G(z)}{H(z)} \Rightarrow$$

$$G(z) = 1 + \alpha_2 z + \alpha_3 z^2 + ...$$
$$H(z) = 1 + a_2 z + a_3 z^2 + ...$$
$$\phi(z) = c_1 z + c_2 z^2 + c_3 z^3 + ...$$

From

$$\frac{G(z)}{H(z)} = \frac{1 + A\phi(z)}{1 + B\phi(z)}$$

then we get

$$G(z) - H(z) = \phi(z)(-BG(z) + AH(z)) \tag{25}$$

From (25) we find

$$\sum_{k=1}^{\infty} (\alpha_k - a_k) z^k = (\sum_{k=1}^{\infty} c_k z^k)((A - B) + \sum_{k=1}^{\infty} (-B\alpha_k + Aa_k) z^k)$$

And from the calculations we get

$$\sum_{k=n+1}^{\infty} (\alpha_k - a_k) z^k - (\sum_{k=1}^{\infty} c_k z^k) + (\sum_{k=n}^{\infty} (-B\alpha_k + Aa_k) z^k) = \tag{26}$$
$$\sum_{k=n+1}^{\infty} (\alpha_k - a_k) z^k - (\sum_{k=n+1}^{\infty} d_k z^k)(\sum_{k=n+1}^{\infty} s_k z^k)$$

The inequality (26) can be written in the following form

$$F(z) = \phi(z) F_1(z), |\phi(z)| < 1.$$

Therefore we have

$$|F(z)|^2 = |\phi(z) F_1(z)|^2 = |\phi(z)|^2 |F_1(z)|^2 \Rightarrow |F(z)|^2 \leq |F_1(z)|^2 \Rightarrow$$
$$\frac{1}{2\pi} \int_0^{2\pi} \left| F(re^{i\theta}) \right|^2 d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \left| F_1(re^{i\theta}) \right|^2 d\theta$$

$$\Rightarrow \sum_{k=1}^{n} |\alpha_k - a_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |d_k|^2 r^{2k} \leq |A-B|^2 r^{2k} + \sum_{k=1}^{n-1} |Aa_k - B\alpha_k|^2 r^{2k} \tag{27}$$

Since

$$\left(\sum_{k=n+1}^{\infty} |d_k|^2 r^{2k}\right) > 0$$

then the equality (27) can be written in the following form

$$\sum_{k=1}^{n} |\alpha_k - a_k|^2 r^{2k} \leq |A-B|^2 r^{2k} + \sum_{k=1}^{n-1} |Aa_k - B\alpha_k|^2 r^{2k}$$

Taking $r \to 1$ we obtain

$$\sum_{k=1}^{n} |\alpha_k - a_k|^2 \leq |A-B|^2 + \sum_{k=1}^{n-1} |Aa_k - B\alpha_k|^2 \tag{28}$$

If we take $\alpha_k = \frac{b_k}{b_1}$ then (2.23) can be written

$$\sum_{k=1}^{n} \left|\frac{b_k}{b_1} - a_k\right|^2 \leq |A-B|^2 + \sum_{k=1}^{n-1} \left|Aa_k - B\frac{b_k}{b_1}\right|^2$$

$$\sum_{k=1}^{n} |b_k - b_1 a_k|^2 \leq |b_1| (A-B)^2 + \sum_{k=1}^{n-1} |Ab_1 a_k - Bb_k|^2$$

□

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