SUFFICIENT CONDITIONS FOR GENERALIZED SAKAGUCHI TYPE 
FUNCTIONS OF ORDER $\beta$

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ABSTRACT. In this paper, we obtain some sufficient conditions for generalized Sakaguchi type function of order $\beta$, defined on the open unit disk. Many interesting outcomes of our results are also calculated.

Keywords: Generalized Sakaguchi type function of order $\beta$, Univalent functions.

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1. INTRODUCTION

Let $A_n$ be the class of the form

$$f(z) = z + a_{n+1}z^{n+1} + \ldots$$

that are analytic in the unit disk $\Delta = \{z \in C : |z| < 1\}$ and let $A_1 = A$. An analytic function $f(z) \in A_n$ is said to be in the generalized Sakaguchi class $S_n(\beta, s, t)$ if it satisfies

$$\text{Re}\left\{ \frac{(s-t)zf'(z)}{f(sz) - f(tz)} \right\} > \beta, \quad z \in \Delta$$

for some $\beta(0 \leq \beta < 1)$, $s$ and $t$ are real parameters, $s > t$ and for all $z \in \Delta$.

For $n = 1$ the generalized Sakaguchi class $S_1(\beta, s, t)$ reduces to the subclass $S(\beta, s, t)$ studied by Frasin [2], see also [6], [7]). For $n = 1$, $s = 1$, this class is reduced to $S(\beta, t)$ studied by Owa et al. [9, 10], Goyal and Goswami [3] and Cho et al.[1]. The class $S(0, -1)$ was introduced by Sakaguchi [12]. Recently T. Mathur et al. [6], [7] have introduced and studied some properties of $S(\beta, s, t)$.

In this paper, we obtain some sufficient conditions for functions $f(z) \in S_n(\beta, s, t)$. To prove our results, we need the following:

Lemma 1.1 (8). Let $\Omega$ be a set in the complex plane $C$ and suppose that $\phi$ is a mapping from $C^2 \times \Delta$ to $C$ which satisfies $\phi(ix, y; z) \not\in \Omega$ for $z \in \Delta$, and for all real $x, y$ such that $y \leq -n(1 + x^2)/2$. If the function $p(z) = 1 + c_nz^n + \ldots$ is analytic in $\Delta$ and $\phi(p(z), zp'(z); z) \in \Omega$ for all $z \in \Delta$, then $\text{Re}(p(z)) > 0$.

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2. Main Results

**Theorem 2.1.** If \( f(z) \in A_n \), satisfies

\[
Re \left[ \frac{(s-t)^2zf'(sz)}{f(sz) - f(tz)} \left\{ \frac{\alpha szf''(sz)}{f'(sz)} + \frac{\alpha tzf'(tz)}{f(sz) - f(tz)} + 1 \right\} \right] > \alpha \beta \left\{ \beta + \frac{n}{2} (s-t) - (s-t) \right\} + \left\{ \beta - \frac{na}{2} \right\} (s-t)
\]

for \((z \in \Delta, 0 \leq \alpha \leq 1, 0 \leq \beta < 1 \) and \( t < s \), then \( f(z) \in S_n(\beta, s, t) \).

**Proof.** Define \( p(z) \) by

\[
\begin{align*}
\left\{ \frac{(s-t)zf'(sz)}{f(sz) - f(tz)} \right\} &= (1 - \beta)p(z) + \beta.
\end{align*}
\]

Then \( p(z) = 1 + cz^n + \ldots \) and is analytic in \( \Delta \).

A computation shows that

\[
\frac{szf''(sz)}{f'(sz)} + \frac{tzf'(tz)}{f(sz) - f(tz)} = \frac{(s-t)(1-\beta)zp'(z) + s[(1-\beta)p(z) + \beta]^2 - (s-t)[(1-\beta)p(z) + \beta]}{(s-t)[(1-\beta)p(z) + \beta]}
\]

and hence

\[
\begin{align*}
\frac{(s-t)^2zf'(sz)}{f(sz) - f(tz)} \left\{ \frac{\alpha szf''(sz)}{f'(sz)} + \frac{\alpha tzf'(tz)}{f(sz) - f(tz)} + 1 \right\} &= \alpha(s-t)(1-\beta)zp'(z) + \alpha s(1-\beta)^2p^2(z) + (1-\beta)[2s\alpha \beta + (s-t)(1-\alpha)]p(z) + \beta[s\alpha \beta + (s-t)(1-\alpha)]
\end{align*}
\]

or

\[
\begin{align*}
\phi(p(z), zp'(z); z) \quad \text{(say)}
\end{align*}
\]

where

\[
\phi(u, v; z) = \alpha(s-t)(1-\beta)v + \alpha s(1-\beta)^2u^2 + (1-\beta)[2s\alpha \beta + (s-t)(1-\alpha)]u + \beta[s\alpha \beta + (s-t)(1-\alpha)]
\]

For all real \( x \) and \( y \) satisfying \( y \leq -n(1 + x^2)/2 \), we have

\[
Re[\phi(ix, y; z)] \leq \alpha(s-t)(1-\beta)y - \alpha s(1-\beta)^2x^2 + \beta[s\alpha \beta + (s-t)(1-\alpha)]
\]

\[
\leq \alpha(s-t)(1-\beta) \left\{ \frac{-(1 + x^2)}{2} \right\} - \alpha s(1-\beta)^2x^2 + \beta[s\alpha \beta + (s-t)(1-\alpha)]
\]

\[
= \frac{-\alpha n}{2} (s-t)(1-\beta) - \left\{ \frac{\alpha n}{2} (s-t)(1-\beta) + \alpha \beta(1-\beta)^2 \right\} x^2 + \beta[s\alpha \beta + (1-\alpha)(s-t)]
\]

\[
\leq \frac{-\alpha n}{2} (s-t)(1-\beta) + \beta[s\alpha \beta + (1-\alpha)(s-t)]
\]

\[
= \alpha \beta \left\{ \beta + \frac{n}{2} (s-t) - (s-t) \right\} + \left\{ \beta - \frac{na}{2} \right\} (s-t)
\]

Let \( \Omega = \{ w; \text{Re}(w) > \alpha \beta \left\{ \beta + \frac{n}{2} (s-t) - (s-t) \right\} + \left\{ \beta - \frac{na}{2} \right\} (s-t) \} \)

Then \( \phi(p(z), zp'(z); z) \in \Omega \) and \( \phi(ix, y; z) \notin \Omega \) for all real \( x \) and \( y \leq -n(1 + x^2)/2, \ z \in \Delta \).

By an application of Lemma 1.1, the result follows.

**Remark 2.1.** On putting \( s = 1 \), in Theorem 2.1, we get the known results due to Goyal et al.[9]
Theorem 2.2. Let $0 \leq \beta < 1, t < s$ with $-1 \leq \frac{1}{2} + \beta < 1,$
\[ \lambda = (1 - \beta)^2 \left\{ \frac{n}{2}(s - t) + s(1 - \beta) \right\}^2, \quad \mu = \left\{ \frac{n}{2}(s - t)(1 - \beta + \beta(s - t - s\beta)) \right\}^2, \]
\[ \nu = \left\{ s(1 - \beta)^2 - \beta(s - t - s\beta) \right\}^2 \quad \text{and} \quad \sigma = \left\{ (1 - \beta)(2s\beta - t - s) \right\}^2 \] (5)
satisfy $(\lambda + \mu - \nu + \sigma)\beta^2 < (1 - 2\beta)\mu$.
Also suppose that $r_0$ be the positive real root of the equation
\[ 2\lambda(1 - \beta)^2 r^3 + \{ (1 - \beta)^2 (2\lambda + \mu - \nu + \sigma) + 3\lambda\beta^2 \} r^2 + 2\beta^2 (2\lambda + \mu - \nu + \sigma) r \]
\[ + (\lambda + 2\mu - \nu + \sigma)\beta^2 - (1 - \beta)^2 \mu = 0 \] (6)
and
\[ \rho^2 = \frac{(1 - \beta)^2 (1 + r_0)}{(s - t)^2 \{ (1 - \beta)^2 r_0 + \beta^2 \}} [\lambda r_0^2 + (\lambda + \mu - \nu + \sigma) r_0 + \mu] \] (7)
Now if $f(z) \in A_n$ satisfies
\[ \left| \left( \frac{(s - t)zf'(sz)}{f(sz) - f(tz)} - 1 \right) \left( \frac{szf''(sz)}{f'(sz)} + \frac{tzf'(tz)}{f(sz) - f(tz)} \right) \right| \leq \rho \quad (z \in \Delta) \]
then $f(z) \in S_n(\beta, s, t)$.

Proof. Define $p(z)$ by
\[ \left\{ \frac{(s - t)zf'(sz)}{f(sz) - f(tz)} \right\} = (1 - \beta)p(z) + \beta. \]
Then $p(z) = 1 + c_n z^n + \ldots$ and is analytic in $\Delta$.
A computation shows that
\[ \frac{szf''(sz)}{f'(sz)} + \frac{tzf'(tz)}{f(sz) - f(tz)} = (s - t)(1 - \beta)zp'(z) + s[(1 - \beta)p(z) + \beta]^2 - (s - t)[(1 - \beta)p(z) + \beta] \]
and hence
\[ = \frac{(1 - \beta)(p(z) - 1)}{(s - t)(1 - \beta)p(z) + \beta} \left\{ (s - t)(1 - \beta)zp'(z) + s[(1 - \beta)p(z) + \beta]^2 - (s - t)[(1 - \beta)p(z) + \beta] \right\} \]
\[ = \phi(p(z), zp'(z); z) \]
Then for all real $x$ and $y$ satisfying $y \leq -n(1 + x^2)/2$, we have
\[ |\phi(ix, y; z)|^2 = \frac{(1 - \beta)^2 (1 + x^2)}{(s - t)^2 [(1 - \beta)^2 x^2 + \beta^2]} \]
\[ \times \left[ (s - t)(1 - \beta)y - s(1 - \beta)^2 x^2 - \beta(s - t - s\beta) \right]^2 + (1 - \beta)^2 \left[ 2s\beta - (s - t) \right]^2 x^2 \]
\[ = \frac{(1 - \beta)^2 (1 + r)}{(s - t)^2 [(1 - \beta)^2 r + \beta^2]} \]
\[ \times \left[ (s - t)(1 - \beta)y - s(1 - \beta)^2 r - \beta(s - t - s\beta) \right]^2 + (1 - \beta)^2 \left[ 2s\beta - (s - t) \right]^2 r \]
\[ = g(r, y) \]
where $r = x^2 > 0$ and $y \leq -n(1 + x^2)/2$.
Since
\[ \frac{\partial g}{\partial y} = \frac{2(1 - \beta)^3 (1 + r)}{(s - t)^2 [(1 - \beta)^2 r + \beta^2]} \left\{ (s - t)(1 - \beta)y - \beta(s - t - s\beta) - s(1 - \beta)^2 r \right\} < 0 \]
therefore we have
\[ h(r) = g[r, -n(1 + r)/2] \leq g(r, y), \]
where
\[ h(r) = \frac{(1 - \beta)^2(1 + r)}{(s - t)^2[(1 - \beta)^2r + \beta^2]} \left[ \lambda r^2 + (\lambda + \mu - \nu + \sigma)r + \mu \right] \] (8)
where \( \lambda, \mu, \nu, \) and \( \sigma \) are given in (5).
Now differentiating (8) and using \( h'(r) = 0, \) we get
\[
2\lambda(1 - \beta)^2 r^3 + \left\{ (1 - \beta)^2(2\lambda + \mu - \nu + \sigma) + 3\lambda \beta^2 \right\} r^2
+ 2\beta^2(2\lambda + \mu - \nu + \sigma)r + (\lambda + 2\mu - \nu + \sigma)\beta^2 - (1 - \beta)^2 \mu = 0
\]
which is a cubic equation in \( r. \) Since \( r_0 \) is the positive real root of this equation we have
\[ h(r) \geq h(r_0) \]
and hence
\[ |\phi(ix, y; z)|^2 \geq h(r_0) = \rho^2. \]
Define \( \Omega = \{ w; |w| < \rho \}, \) then \( \phi(p(z), zp'(z); z) \in \Omega \) for all real \( x \) and \( y \leq -n(1+x^2)/2, \) \( z \in \Delta. \) Therefore by an application of Lemma 1.1 the result follows. \( \square \)

**Remark 2.2.** By taking \( s = 1 \) in Theorem 2.2 we get the known results of Goyal et al.[4]
For \( s = 1 \) and \( t = 0 \) in Theorem 2.2 gives the known results due to Ravichandran et al.[11]
and for \( n = 1, \beta = 0, \) \( t = 0 \), our Theorem 2.2 reduces to another known result of Li and Owa.[5]

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**References**