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REMOTALITY OF CERTAIN SETS $L^p(I, X)$

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ABSTRACT. Let X be a Banach space and let (I, Ω, μ) be a measure space. For $1 \leq p < \infty$, let $L^p(I, X)$ denote the space of Bochner $p-$integrable functions defined on I with values in X. The object of this paper is to give sufficient conditions for remotality of $L^1(I, H) + L^1(I, G)$ in $L^1(I, X)$, where H and G are two bounded sets in X which include as a special case remotality of $L^1(I) \overset{\wedge}{\otimes} G + H \overset{\wedge}{\otimes} L^1(I)$ in $L^1(I \times I)$.

Keywords: Remotal set, Bochner $p-$integrable function, Banach space.

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1. Introduction

Let (I, Ω, μ) be a measure space and let $L^p(I, X)$ denotes the space of Bochner $p-$integrable functions (equivalent classes) defined on (I, Ω, μ) with values in a Banach space X. It is known [8] that $L^p(I, X)$ is a Banach space under the norm

$$\|f\|_p = \left(\int \|f(t)\|^p \, d\mu(t) \right)^{\frac{1}{p}}, \qquad 1 \leq p < \infty.$$

For $x \in X$ and a bounded subset G of X, set $\rho(x, G) = \sup\{\|x - y\| : y \in G\}$. A point $g_0 \in G$ is called a farthest point of G if there exists $x \in X$ such that $\|x - g_0\| = \rho(x, G)$. For $x \in X$, the farthest point map is defined by $F_G(x) = \{g \in G : \|x - g\| = \rho(x, G)\}$, i.e., the set of points of G farthest from x. Note that, this set may be empty. Let $R(G, X) = \{x \in X : F_G(x) \neq \phi\}$. Call a closed bounded set G remotal if $R(G, X) = X$ and densely remotal if $R(G, X)$ is norm dense in X. The study of existence of farthest points in an arbitrary bounded subset of a Banach space has generated some interest during the last two decades. Asplund, Lau, Baronti, Boszany and others have considered the problem in various general situations. Remotal sets in vector valued continuous functions was considered in [6]. Related results on the spaces of Bochner integrable functions $L^p(I, X)$ $1 \leq p \leq \infty$ are given in [2, 9, 11]. Remotal sets in Köthe Bochner function spaces are considered in [1].

If X and Y are Banach spaces, then $X \overset{\wedge}{\otimes} Y$ and $X \overset{\vee}{\otimes} Y$ denote the completions of the injective and projective tensor product of X with Y respectively; see [12]. Light and Cheney, (Theorem 2.26, [12]), proved that if G and H are finite dimensional subspaces of $L^1(S)$ and $L^1(I)$ respectively, then each element of $L^1(I \times S) = L^1(I) \overset{\vee}{\otimes} L^1(S)$ has a best approximation in the subspace $L^1(I) \overset{\wedge}{\otimes} G + H \overset{\wedge}{\otimes} L^1(S)$.

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The object of this paper is to discuss remotality of $L^1(I,H)+L^1(I,G)$ in $L^1(I,X)$, where H and G are two bounded subsets of X. Further, we conclude from our results remotality of $L^1(I)\overset{\wedge}{\otimes}G+H\overset{\wedge}{\otimes}L^1(I)$ in $L^1(I\times I)$.

Throughout this paper, let (I,μ) be a finite measure space and X be a real Banach space.

2. Distance Formula

The farthest distance formula is important in the study of farthest point. The farthest distance from an element $f\in L^P(I,X)$ to a bounded subset $L^P(I,H)+L^P(I,G)$ is computed by the following theorem.

Theorem 2.1. *Let (I,Ω,μ) be a measure space, X a Banach space and H,G be two bounded subsets of X. Then for each $f\in L^p(I,X)$*

$$\rho(f,L^P(I,H)+L^P(I,G))=\|\rho(f(.),H+G))\|_p.$$

Proof. Let $f\in L^p(I,X)$. Since f is strongly measurable, there exists a sequence of simple functions f_n such that

$$\|f_n(s)-f(s)\|\to 0,$$

for almost all $s\in I$. For $h\in H$ and $g\in G$, the inequality

$$\|f_n(s)-(h+g)\|\leq\|f_n(s)-f(s)\|+\|f(s)-(h+g)\|$$

implies that

$$\rho(f_n(s),H+G)\leq\|f_n(s)-f(s)\|+\rho(f(s),H+G),$$

for almost all $s\in I$. Consequently,

$$|\rho(f_n(s),H+G)-\rho(f(s),H+G)|\to 0$$

as $n\to\infty$ for almost all s in I. For $s\in I$, set $H_n(s)=\rho(f_n(s),H+G)$. Then for each n, H_n is a measurable function. Thus, $\rho(f(.),H+G)$ is measurable and

$$\rho(f(s),H+G)\geq\|f(s)-z\|,$$

for all z in $H+G$. Therefore,

$$\rho(f(s),H+G)\geq\|f(s)-g(s)\|$$

for all $g\in L^p(I,H)+L^p(I,G)$. Further,

$$\begin{aligned}\|\rho(f(s),H+G)\|_p &= \left(\int_I \rho^p(f(s),H+G)\,ds\right)^{\frac{1}{p}}\\ &\geq \left(\int_I \|f(s)-g(s)\|^p\,ds\right)^{\frac{1}{p}},\end{aligned}$$

for every $g\in L^p(I,H)+L^p(I,G)$. Since H and G are bounded, it follows that $\rho(f(.),H+G)\in L^p(I)$ and

$$\left(\int_I \rho^p(f(s),H+G)\,ds\right)^{\frac{1}{p}}\geq\rho(f,L^p(I,H)+L^p(I,G)). \tag{1}$$

Now, since simple functions are dense in $L^p(I,X)$, given $\epsilon > 0$, there exists a simple function φ in $L^p(I,X)$ such that $\|f-\varphi\|_p < \frac{\epsilon}{2}$. Write $\varphi = \sum\limits_{i=1}^{n} \chi_{A_i} y_i$, where χ_{A_i} is the characteristic function of the set A_i in Ω and $y_i \in X$. We may assume that $\sum\limits_{i=1}^{n} \chi_{A_i} = 1$, and $\mu(A_i) > 0$. Since $\varphi \in L^p(I,X)$, we have $\|y_i\| \, \mu(A_i) < \infty$ for $1 \le i \le n$. For each $i = 1, 2, \ldots, n$, if $\mu(A_i) < \infty$, select $h_i \in H, g_i \in G$ such that

$$\|y_i - (h_i + g_i)\| > \rho(y_i, H+G) - \frac{\epsilon}{(n\mu(A_i))^{\frac{1}{p}}}.$$

This could be done since $\rho(u, H+G) = \sup\limits_{g \in H+G} \|u-g\|$ for all $u \in X$. If $\mu(A_i) = \infty$, put $h_i = g_i = 0$. Let

$$w = \sum_{i=1}^{n} \chi_{A_i}(h_i + g_i) = \sum_{i=1}^{n} \chi_{A_i} h_i + \sum_{i=1}^{n} \chi_{A_i} g_i = w_1 + w_2.$$

Clearly, $w \in L^p(I,H) + L^p(I,G)$. Further,

$$\begin{aligned}
\left(\int_I \rho^p(\varphi(s), H+G)ds\right)^{\frac{1}{p}} &= \left(\sum_{i=1}^{n} \int_{A_i} \rho^p(\varphi(s), H+G\ )\, ds\right)^{\frac{1}{p}} \\
&= \left(\sum_{i=1}^{n} \int_{A_i} \rho^p(y_i, H+G\ )\, ds\right)^{\frac{1}{p}} \\
&= \left(\sum_{i=1}^{n} \rho^p(y_i, H+G\ )\, \mu(A_i)\right)^{\frac{1}{p}} \\
&\le \left(\sum_{i=1}^{n} \left(\|y_i - (h_i+g_i)\| + \frac{\epsilon}{(2n\mu(A_i))^{\frac{1}{p}}}\right)^p \mu(A_i)\right)^{\frac{1}{p}}.
\end{aligned}$$

Using triangle inequality for the l^p norm we get

$$\begin{aligned}
\left(\int_I \rho^p(\varphi(s), H+G)ds\right)^{\frac{1}{p}} &\leq \left(\sum_{i=1}^{n} (\|y_i - (h_i+g_i)\|)^p \mu(A_i)\right)^{\frac{1}{p}} \\
&\quad + \left(\sum_{i=1}^{n} \left(\frac{\epsilon}{(2n\mu(A_i))^{\frac{1}{p}}}\right)^p \mu(A_i)\right)^{\frac{1}{p}} \\
&= \left(\sum_{i=1}^{n} (\|y_i - (h_i+g_i)\|)^p \mu(A_i)\right)^{\frac{1}{p}} + \frac{\epsilon}{2} \\
&= \left(\sum_{i=1}^{n} \int_{A_i} (\|\varphi(s) - w(s)\|)^p ds\right)^{\frac{1}{p}} + \frac{\epsilon}{2} \\
&= \left(\int_I (\|\varphi(s) - w(s)\|)^p ds\right)^{\frac{1}{p}} + \frac{\epsilon}{2} \\
&\leq \|\varphi - f\|_p + \|f - w\|_p + \frac{\epsilon}{2} \\
&\leq \|f - w\|_p + \epsilon.
\end{aligned}$$

For any $a \in H+G$ and for $s \in I$, the inequality

$$\|f(s) - a\| \leq \|\varphi(s) - a\| + \|\varphi(s) - f(s)\|$$

implies that

$$\rho(f(s), H+G) \leq \rho(\varphi(s), H+G) + \|\varphi(s) - f(s)\|,$$

and this implies

$$\begin{aligned}
\left(\int_I \rho^p(f(s), H+G)\,ds\right)^{\frac{1}{p}} &\leq \left(\int_I \rho^p(\varphi(s), H+G)\,ds\right)^{\frac{1}{p}} + \frac{\epsilon}{2} \\
&\leq \|f - w\|_p + \epsilon + \frac{\epsilon}{2} \\
&\leq \frac{3\epsilon}{2} + \rho(f, L^p(I,H) + L^p(I,G)).
\end{aligned}$$

Since ϵ arbitrary, we get

$$\begin{aligned}
\left(\int_I \rho^p(f(s), H+G)\,ds\right)^{\frac{1}{p}} &= \|\rho(f(.), H+G)\|_p \\
&\leq \rho(f, L^p(I,H) + L^p(I,G)). \qquad (2)
\end{aligned}$$

From (1) and (2) the result holds. □

Corollary 2.1. *Let H and G be two bounded subsets of a Banach space X and (I, Ω, μ) be a measure space such that $\mu(I) < \infty$. Then, $g \in L^p(I,H) + L^p(I,G)$ is a farthest point for $f \in L^p(I,X)$ if and only if, for almost all $s \in I, g(s)$ is a farthest point in $H+G$ for $f(s)$.*

Proof. Let $g = g_1 + g_2$ be a farthest point in $L^p(I,H) + L^p(I,G)$ for $f \in L^p(I,X)$. Then,

$$\|f - g\|_p = \rho\left(f, L^p(I,H) + L^p(I,G)\right).$$

By Theorem 2.1, we have

$$\left(\int_I \|f(s) - (g_1(s) + g_2(s))\|^p\, ds\right)^{\frac{1}{p}} = \left(\int_I (\rho^p(f(s), H + G))^p\, ds\right)^{\frac{1}{p}}.$$

Since for any $y \in H$ and $z \in G$,

$$\rho(f(s), H + G) \geq \|f(s) - (y + z)\|,$$

and x^p is an increasing function for $p \geq 1$, it follows that

$$(\rho(f(s), H + G))^p \geq \|f(s) - (g_1(s) + g_2(s))\|^p.$$

Thus,

$$\|f(s) - (g_1(s) + g_2(s))\|^p = (\rho(f(s), H + G))^p,$$

for almost $s \in I$, and so

$$\|f(s) - (g_1(s) + g_2(s))\| = \rho(f(s), H + G),$$

for almost all $s \in I$. Hence, $g_1(s) + g_2(s)$ is a farthest point for $f(s) \in X$ for almost all $s \in I$.

Conversely, suppose that $g(s)$ is a farthest point for $f(s)$ in $H + G$ for almost all $s \in I$. Then,

$$\|f(s) - g(s)\| = \rho(f(s), H + G).$$

Hence, using Theorem 2.1,

$$\|f(.) - g(.)\|_p = \|\rho(f(.), H + G)\|_p = \rho(f, L^P(I,H) + L^P(I,G)).$$

□

Corollary 2.2. *Let H, G be two bounded subsets of a Banach space X, $f \in L^p(I,X)$ and g be a strongly measurable function such that $g(s) \in H + G$, is a farthest point to $f(s)$ from $H + G$ for almost all $s \in I$. Then, g is a farthest point to f from $L^p(I,H) + L^p(I,G)$ in $L^p(I,X)$.*

Proof. Since H and G are two bounded subsets, we can write $g(s)$ as $u_1(s) + u_2(s)$ where $u_1(s) \in H, u_2(s) \in G$ and $s \in I$. Then,

$$\|g(s)\| \leq \|u_1(s)\| + \|u_2(s)\| \leq M_1 + M_2 \leq M,$$

where $M = \max\{M_1, M_2\}$. The inequality,

$$|\ \|f(s)\| - \|g(s)\|\ | \leq \|f(s) - g(s)\| \leq \|f(s) - (u_1(s) + u_2(s))\| \leq \|f(s)\| + M$$

implies that

$$\|g(s)\| \leq 2\|f(s)\| + M.$$

Consequently, $g \in L^p(I,H) + L^p(I,G)$ and hence, by using the assumption and Corollary 2.1, we get the result. □

3. Remotality of $L^p(I,H)+L^p(I,G)$ in $L^p(I,X)$

A subspace Y of a Banach space X is called L^1-summand (M-summand) if there exists a bounded projection $P : X \to Y$ such that for any $x \in X$, $x = P(x) + (I - P)(x)$ and $\|x\| = \|P(x)\| + \|(I-P)(x)\|$, ($\|x\| = \max(\|P(x)\|, \|(I-P)(x)\|)$), where I is the identity function ; see [7].

Let G be a closed bounded subset of a Banach space X. In [9], it has been proved that if G is a remotal subset of X and $\overline{\text{span}G}$ is finite dimensional, then $L^1(I,G)$ is remotal in $L^1(I,X)$. In [2], this result was extended to the case of separable set. It has been proved if G is a separable remotal subset of X, then $L^P(I,G)$ is remotal in $L^P(I,X)$, $1 \le p < \infty$. In this section we give sufficient conditions for remotality of $L^1(I,H)+L^1(I,G)$ in $L^1(I,X)$, where H and G are two bounded sets in X which includes as a special case remotality of $L^1(I)\overset{\wedge}{\otimes} G + H \overset{\wedge}{\otimes} L^1(I)$ in $L^1(I\times I)$. We begin by the following Theorem.

Theorem 3.1. *Let H and G be two closed bounded subset of a Banach space X. If $L^p(I,H)+L^p(I,G)$ is remotal in $L^p(I,X)$, then $H+G$ is remotal in X.*

Proof. For $x \in X$, let $x \otimes 1 \in L^p(I,X)$, where 1 is the constant function. By assumption, there exists $g = f_1 + f_2 \in L^p(I,H)+L^p(I,G)$ farthest point from $x\otimes 1$. That means

$$\|x\otimes 1 - (f_1+f_2)\|_p \ge \|x\otimes 1 - w\|_p,$$

for every $w \in L^p(I,H)+L^p(I,G)$. By Corollary 2.1,

$$\|x\otimes 1(s) - (f_1+f_2)(s)\| \ge \|x\otimes 1(s) - w(s)\|,$$

for almost all $s \in I$ and for every $w \in L^p(I,H)+L^p(I,G)$. In particular,

$$\|x\otimes 1(s) - (f_1(s)+f_2(s))\| \ge \|x\otimes 1(s) - (h+g)\otimes 1(s)\|,$$

for almost all $s \in I$ and for every $h \in H$ and $g \in G$. Consequently,

$$\|x - (f_1(s)+f_2(s))\| \ge \|x-(h+g)\|,$$

which implies $f_1(s)+f_2(s)$ is a farthest point of x in $H+G$. □

Theorem 3.2. *Let Y be an L^1-Summand ($M-$ Summand) subspace of a Banach space X and H be a closed bounded subset of Y. If G is a closed bounded subset of X/Y such that $H+G$ is separable, then $H+G$ is remotal in X if and only if $L^p(I,H)+L^p(I,G)$ is remotal in $L^p(I,X)$.*

Proof. Necessity is in Theorem 3.1. Let us show sufficiency. Let $f \in L^p(I,X)$. Since $H+G$ is remotal separable subset of X and f is strongly measurable, then Theorem 3.7 in [2] guarantees that there exists a measurable function g defined on I with values in X such that $g(t)$ is a farthest point of $f(t)$ in $H+G$ for almost all $t \in I$. By the assumption that Y is L^1-Summand ($M-$ Summand), there exist continuous projections p_1, p_2 from X into $Y, X/Y$ respectively. Define $g_1 : I \to H$ and $g_2 : I \to G$, such that $g_1(t) = p_1 \circ g(t)$ and $g_2(t) = p_2 \circ g(t)$. Since g is measurable and p_1, p_2 are continuous, it follows that g_1, g_2 are measurable functions. Hence,

$$g = g_1 + g_2 \in L^p(I,H)+L^p(I,G)$$

and $g_1(t)+g_2(t) = g(t)$ is a farthest point to $f(t)$ in $H+G$. By Corollary 2.1 g_1+g_2 is a farthest point of f in $L^p(I,H)+L^p(I,G)$. □

Corollary 3.1. *Let Y be an L^1-Summand ($M-$ Summand) subspace of $L^1(I)$ and H be a closed bounded subset of Y. If G is a closed bounded subset of $L^1(I)/Y$ such that $H+G$ is a remotal separable subset of $L^1(I)$, then $L^1(I)\overset{\wedge}{\otimes} H + G\overset{\wedge}{\otimes} L^1(I)$ is remotal in $L^1(I\times I)$.*

Proof. Let Y be an L^1-Summand ($M-$Summand) subspace of $L^1(I)$, H be a closed bounded subset of Y, and G is a closed bounded subset of $L^1(I)/Y$. Since $H+G$ is remotal separable subset of $L^1(I)$, by Theorem 3.2, it follows that $L^p\left(I,H\right)+L^p\left(I,G\right)$ *is remotal in* $L^p\left(I,L^1(I)\right)$. Hence, using Theorem 1.15 in [12], we deduce that $L^1(I)\overset{\wedge}{\otimes}H+G\overset{\wedge}{\otimes}L^1(I)$ is remotal in $L^1(I\times I)$. □

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