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FIXED POINTS ON T_0-QPMS

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Abstract. In this paper, we proved some fixed point results for maps in quasi-pseudometric spaces. The theorems are a generalization of the results presented in [3].

Keywords: T_0-quasi-pseudometric, T-orbitally left-complete, d-sequentially continuous, ε-chainable.

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1. Introduction

Our purpose is to generalise some known fixed point results in the context of an "asymmetric metric space". Although the proofs follow closely the ones in the classical case, the results require a different type of assumptions. The terminology "asymmetric metric space" could be confusing in the sense that metrics are symmetric, but it is just to emphasize on the fact that we start from metric spaces and remove the "symmetry property". The distance from a point x to a point y may be different from the distance from y to x. Therefore, this could look like an orientation on the space. Theses "oriented distances" can be useful in practical application. For instance, in a hilly country, it makes a difference whether an auto-mobile climbs from a locality A to a locality B or goes down from B to A, considering the cost of transport.

2. Prelimaries

In this section, we recall some elementary definitions from the asymmetric topology which are necessary for a good understanding of the work below.

Definition 2.1. *Let X be a non empty set. A function $d : X \times X \to [0, \infty)$ is called a* ***quasi-pseudometric*** *on X if:*

i) $d(x, x) = 0 \quad \forall\, x \in X,$

ii) $d(x, z) \leq d(x, y) + d(y, z) \quad \forall\, x, y, z \in X.$

*Moreover, if $d(x, y) = 0 = d(y, x) \Longrightarrow x = y$, then d is said to be a T_0***-quasi-pseudometric***. The latter condition is referred to as the T_0-condition.*

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Remark 2.1.

- *Let d be a quasi-pseudometric on X, then the map d^{-1} defined by $d^{-1}(x,y) = d(y,x)$ whenever $x, y \in X$ is also a quasi-pseudometric on X, called the* ***conjugate*** *of d. In the literature, d^{-1} is also denoted d^t or $\bar{d}$.*
- *It is easy to verify that the function d^s defined by $d^s := d \vee d^{-1}$, i.e. $d^s(x,y) = \max\{d(x,y), d(y,x)\}$ defines a* ***metric*** *on X whenever d is a T_0-quasi-pseudometric.*

Let (X,d) be a quasi-pseudometric space. Then for each $x \in X$ and $\epsilon > 0$, the set

$$B_d(x,\epsilon) = \{y \in X : d(x,y) < \epsilon\}$$

denotes the open ϵ-ball at x with respect to d. It should be noted that the collection

$$\{B_d(x,\epsilon) : x \in X, \epsilon > 0\}$$

yields a base for the topology $\tau(d)$ induced by d on X. In a similar manner, for each $x \in X$ and $\epsilon \geq 0$, we define

$$C_d(x,\epsilon) = \{y \in X : d(x,y) \leq \epsilon\},$$

known as the closed ϵ-ball at x with respect to d.
Also the collection

$$\{B_{d^{-1}}(x,\epsilon) : x \in X, \epsilon > 0\}$$

yields a base for the topology $\tau(d^{-1})$ induced by d^{-1} on X. The set $C_d(x,\epsilon)$ is $\tau(d^{-1})$-closed, but not $\tau(d)$-closed in general.
The balls with respect to d are often called *forward balls* and the topology $\tau(d)$ is called *forward topology*, while the balls with respect to d^{-1} are often called *backward balls* and the topology $\tau(d^{-1})$ is called *backward topology.*

Definition 2.2. *Let (X,d) be a quasi-pseudometric space. The convergence of a sequence (x_n) to x with respect to $\tau(d)$, called* ***d-convergence*** *or* ***left-convergence*** *and denoted by $x_n \xrightarrow{d} x$, is defined in the following way*

$$x_n \xrightarrow{d} x \Longleftrightarrow d(x,x_n) \longrightarrow 0. \tag{1}$$

Similarly, the convergence of a sequence (x_n) to x with respect to $\tau(d^{-1})$, called ***d^{-1}-convergence*** *or* ***right-convergence*** *and denoted by $x_n \xrightarrow{d^{-1}} x$, is defined in the following way*

$$x_n \xrightarrow{d^{-1}} x \Longleftrightarrow d(x_n,x) \longrightarrow 0. \tag{2}$$

Finally, in a quasi-pseudometric space (X,d), we shall say that a sequence (x_n) ***d^s-converges*** *to x if it is both left and right convergent to x, and we denote it as $x_n \xrightarrow{d^s} x$ or $x_n \longrightarrow x$ when there is no confusion. Hence*

$$x_n \xrightarrow{d^s} x \iff x_n \xrightarrow{d} x \text{ and } x_n \xrightarrow{d^{-1}} x.$$

Definition 2.3. *A sequence (x_n) in a quasi-pseudometric (X,d) is called*

(a) ***left K-Cauchy*** *with respect to d if for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\forall\ n,k : n_0 \leq k \leq n \quad d(x_k,x_n) < \epsilon;$$

(b) ***right K-Cauchy*** *with respect to d if for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\forall\ n,k : n_0 \leq k \leq n \quad d(x_n,x_k) < \epsilon;$$

(c) ***d^s-Cauchy*** *if for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\forall n,k \geq n_0 \quad d(x_n,x_k) < \epsilon.$$

Remark 2.2.

- *A sequence is left K-Cauchy with respect to d if and only if it is right K-Cauchy with respect to d^{-1}.*
- *A sequence is d^s-Cauchy if and only if it is both left and right K-Cauchy.*

Definition 2.4. *A quasi-pseudometric space (X, d) is called*

i) ***left-complete*** *provided that any left K-Cauchy sequence is d-convergent.*
ii) ***right-complete*** *provided that any right K-Cauchy sequence is d-convergent.*

Finally, a T_0-quasi-pseudometric space (X, d) is called ***bicomplete*** *provided that the metric d^s on X is complete.*

Definition 2.5. *Let T be a self mapping on a quasi-pseudometric space (X, d). (X, d) is said to be* T**-orbitally left-complete with respect to quasi-pseudometric** *d if and only if every left K-Cauchy sequence which is contained in $\{a, Ta, T^2a, T^3a, \ldots\}$ for some $a \in X$ d-converges in X.*

Definition 2.6. *Let (X, d) be a quasi-pseudometric space, and $x, y \in X$. A* **path from** *x* **to** *y is a finite sequence of points of X starting at x and ending at y. Hence path from x to y can be written as $(x_0, \ldots, x_n)$ where $x_0 = x$ and $x_n = y$. In this case, n will be called the* **degree** *of the path.*
The quasi-pseudometric space (X, d) will be called ε**-chainable** *if for any two points $x, y \in X$, there is a path $(x_0, \cdots, x_n)$ such that $d(x_i, x_{i+1}) \leq \varepsilon$ for $i = 0, 1, \ldots, n-1$.*

Definition 2.7. *A self mapping T defined on a quasi-pseudometric space (X, d) is called* **locally contractive** *if for every $x \in X$, there exist ε_x and $\lambda_x \in [0, 1)$ such that $d(Tu, Tv) \leq \lambda_x d(u, v)$ whenever $u, v \in C_d(x, \varepsilon_x)$.*

Definition 2.8. *A self mapping T defined on a quasi-pseudometric space (X, d) is called* **uniformly locally contractive** *if it is locally contractive at all points $x \in X$, and ε_x and λ_x do not depend on x, i.e. $\lambda_x = \lambda$ and $\varepsilon_x = \varepsilon$ for all $x \in X$ and for some $\lambda, \varepsilon \geq 0$.*

3. Some first results

In this section, we give our first results. The conditions on the quasi-pseudometric space are motivated by the fact that these spaces are not canonically separated, unlike wise classical metric spaces.

Theorem 3.1. *Let (X, d) be a Hausdorff left-complete T_0-quasi-pseudometric space and $T : X \to X$ be a d-sequentially continuous function self mapping such that*

$$d(T^n x, T^n y) \leq a_n[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$ where $a_n (> 0)$ is independent of x, y and $a_1 + a_n < 1$ for $n \geq 1$. If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then T has a unique fixed point in X.

Proof.
Let $x_0 \in X$. We consider the sequence $\{x_n\}$ of iterates $x_n = T^n x_0, n = 1, 2, 3, \ldots$. Then for $n \in \mathbb{N}$ we have

$$d(T^n x_0, T^{n+1} x_0) \leq a_n[d(x_0, Tx_0) + d(Tx_0, T^2 x_0)].$$

Again $d(Tx_0, T^2x_0) \leq a_1[d(x_0, Tx_0) + d(Tx_0, T^2x_0)]$, which gives

$$d(Tx_0, T^2 x_0) \leq \frac{a_1}{1 - a_1} d(x_0, Tx_0) \tag{3}$$

Hence

$$d(T^n x_0, T^{n+1} x_0) \le a_n \left[1 + \frac{a_1}{1-a_1}\right] d(x_0, Tx_0) \tag{4}$$

By triangle inequality, we write

$$\begin{aligned} d(x_n, x_{n+m}) &= d(T^n x_0, T^{n+m} x_0) \\ &\le d(T^n x_0, T^{n+1} x_0) + d(T^{n+1} x_0, T^{n+2} x_0) + \cdots + d(T^{n+m-1} x_0, T^{n+m} x_0). \end{aligned}$$

So using (4), we get

$$d(T^n x, T^{n+m} x) \le [a_n + a_{n+1} + \cdots + a_{n+m-1}] \left[1 + \frac{a_1}{1-a_1}\right] d(x_0, Tx_0)$$

Now since $\sum_{n=1}^{\infty} a_n$ is convergent so each of $a_n, a_{n+1}, \ldots, a_{n+m-1}$ tends to 0 as $n \to \infty$. So

$$d(x_n, x_{n+m}) \to 0 \quad \text{as} \quad n \to \infty, \tag{5}$$

This implies that $\{x_n\}$ is a left K-Cauchy sequence in (X, d). Since (X, d) is left-complete and T d-sequentially continuous, there exists x^* such that $x_n \xrightarrow{d} x^*$ and $x_{n+1} \xrightarrow{d} Tx^*$. Since X is Hausdorff $x^* = Tx^*$.

For the uniqueness, assume by contradiction that there exists another fixed point z^*. So,

$$\begin{aligned} d(x^*, z^*) &= d(Tx^*, Tz^*) \le a_1[d(x^*, Tx^*) + d(z^*, Tz^*)] = 0, \\ &\text{and} \\ d(z^*, x^*) &= d(Tz^*, Tx^*) \le a_1[d(z^*, Tz^*) + d(x^*, Tx^*)] = 0. \end{aligned}$$

Hence $d(x^*, z^*) = 0 = d(z^*, x^*)$ and using the T_0-condition, we conclude that $y^* = z^*$. This completes the proof.

Another version of the above theorem, which seems stronger can be given as follow:

Theorem 3.2. *Let (X, d) be a Hausdorff T_0-quasi-pseudometric space and $T : X \to X$ be self mapping such that T is a d-sequentially continuous function and*

$$d(T^n x, T^n y) \le a_n[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$ where $a_n (> 0)$ is independent of x, y and $a_1 + a_n < 1$ for $n \ge 1$. If X is T-orbitally left-complete with respect to quasi-pseudometric d and the series $\sum_{n=1}^{\infty} a_n$ is convergent, then T has a unique fixed point in X.

Let Φ be the class of continuous functions $F : [0, \infty) \to [0, \infty)$ such that $F^{-1}(0) = \{0\}$. We have this very interesting result which generalizes Theorem (3.1).

Theorem 3.3. *Let S be a d-sequentially continuous self-mapping on a Hausdorff left-complete T_0-quasi-pseudometric space (X, d) satisfying*

$$F(d(S^n x, S^n y)) \le a_n[F(d(x, Sx)) + F(d(y, Sy))] \qquad (x, y \in X) \tag{6}$$

for all $x, y \in X$ where $a_n(>0)$ is independent of x, y, $a_1 + a_n < 1$ for $n \geq 1$ and for some $F \in \Phi$. If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then S has a unique fixed point in X. Moreover, for any arbitrary $x_0 \in X$ the orbit $\{S^n x_0, n \geq 1\}$ d-converges to the fixed point.

Proof.
Since $F^{-1}(0) = \{0\}$, $F(\eta) > 0$ for any $\eta > 0$. Let x_0 be an arbitrary point in X. We define the iterative sequence $(x_n)_n$ by $x_{n+1} = Sx_n$, $n = 1, 2, \cdots$. Using (6), we have

$$\begin{aligned} F(d(x_n, x_{n+1})) &= F(d(Sx_{n-1}, Sx_n)) = F(d(S^n x_0, S^{n+1} x_0)) \\ &\leq a_n[F(d(x_0, Sx_0)) + F(d(Sx_0, S^2 x_0))]. \end{aligned}$$

Again $F(d(Sx_0, S^2x_0)) \leq a_1[F(d(x_0, Sx_0)) + F(d(Sx_0, S^2x_0))]$, which gives

$$F(d(Sx_0, S^2x_0)) \leq \frac{a_1}{1-a_1} F(d(x_0, Sx_0)). \tag{7}$$

Hence

$$F(d(S^n x_0, S^{n+1} x_0)) \leq a_n \left[1 + \frac{a_1}{1-a_1}\right] F(d(x_0, Sx_0)) \tag{8}$$

Therefore, for every $m, n \in \mathbb{N}$ such that $m > n > 2$ we have,

$$\begin{aligned} F(d(x_n, x_m)) &= F(d(Sx_{n-1}, Sx_{m-1})) \\ &\leq [F(d(x_{n-1}, x_n)) + F(d(x_{m-1}, x_m))] \\ &\leq \left[a_{n-1}\left(1 + \frac{a_1}{1-a_1}\right) + a_{m-1}\left(1 + \frac{a_1}{1-a_1}\right)\right] F(d(x_0, Sx_0)). \end{aligned}$$

Letting $n \to \infty$, we obtain that $F(d(x_n, x_m)) \to 0$. Since $F \in \Phi$, we conclude that (x_n) is a left K-Cauchy and since (X, d) is left-complete, there exists $u \in X$ such that $x_n \xrightarrow{d} u$. S is d-sequentially continuous, so $x_{n+1} \xrightarrow{d} Su$. Since X Hausdorff $u = Su$.
For uniqueness, assume by contradiction that there exists another fixed point v. Then

$$F(d(Su, Sv)) = F(d(u, v)) \leq a_1[F(d(u, Su)) + F(d(v, Sv))] = 0,$$

and

$$F(d(Sv, Su)) = F(d(v, u)) \leq a_1[F(d(v, Sv)) + F(d(u, Su))] = 0,$$

which entails that $d(u, v) = 0 = d(v, u)$. Using the T_0-condition, we conclude that $u = v$.

Remark 3.1. *If we set $F = Id_{[0,\infty)}$ in the above theorem, we obtain the result of Theorem* (3.1).

Corollary 3.1. *Let (X, d) be a Hausdorff T_0-quasi-pseudometric space and $T : X \to X$ be self mapping such that T is a d-sequentially continuous function and*

$$F(d(T^n x, T^n y)) \leq a_n[F(d(x, Tx)) + F(d(y, Ty))]$$

for all $x, y \in X$ where $a_n(>0)$ is independent of x, y and $a_1 + a_n < 1$ for $n \geq 1$ and for some $F \in \Phi$. If X is T-orbitally left-complete with respect to quasi-pseudometric d and the series $\sum_{n=1}^{\infty} a_n$ is convergent, then T has a unique fixed point in X.

4. More resutls

Theorem 4.1. *If T is a (ε,λ) uniformly locally contractive mapping defined on a T-orbitally left-complete, $\frac{\varepsilon}{2}$-chainable and Hausdorff T_0-quasi-pseudometric space (X,d), then T has a unique fixed point.*

Proof.
Let $x \in X$. Since X is $\frac{\varepsilon}{2}$-chainable, there is a path $(x_0 = x, x_1, \ldots, x_n = Tx_0)$ from x to Tx such that $d(x_i, x_{i+1}) \leq \frac{\varepsilon}{2}$ for $i = 0, 1, \ldots, n-1$. It is very clear that

$$d(x, Tx) < \frac{n\varepsilon}{2}.$$

The result follows by the triangle inequality.
Since T is (ε,λ) uniformly locally contractive,

$$d(Tx_i, Tx_{i+1}) < \lambda d(x_i, x_{i+1}) < \frac{\lambda\varepsilon}{2} < \frac{\varepsilon}{2} \quad \forall i = 0, 1, \ldots, n-1.$$

Hence, by induction

$$d(T^m x_i, T^m x_{i+1}) < \lambda d(x_i, x_{i+1}) < \frac{\lambda^m \varepsilon}{2} \quad \forall m \in \mathbb{N}.$$

Moreover, by

$$d(T^m x, T^{m+1}x) \leq d(T^m x, T^m x_1) + d(T^m x_1, T^m x_2) + \cdots + d(T^m x_{n-1}, T^m(Tx)),$$

and the above induction, we conclude that

$$d(T^m x, T^{m+1}x) \leq \frac{\lambda^m n\varepsilon}{2} \quad \forall m \in \mathbb{N}. \tag{9}$$

Next, we show that $\{Tx_n\}$ is a left K-Cauchy sequence in X. For any $n, m \in \mathbb{N}$,

$$\begin{aligned} d(T^m x, T^{m+n}x) &\leq d(T^m x, T^{m+1}x) + d(T^{m+1}x, T^{m+2}x) + \ldots + d(T^{m+n-1}x, T^{m+n}x) \\ &< (\lambda^m + \lambda^{m+1} + \ldots + \lambda^{m+n})\frac{n\varepsilon}{2} \\ &< \frac{\lambda^m}{1-\lambda}\frac{n\varepsilon}{2}. \end{aligned}$$

This shows that $\{Tx_n\}$ is a left K-Cauchy sequence in X. Since X is T-orbitally left-complete, there exists $x^* \in X$ such that $T^m x \overset{d}{\longrightarrow} x^*$. Since (X,d) Hausdorff and T d-sequentially continuous, $T(x^*) = x^*$.
For uniqueness, let us assume that z^* is another fixed point. Since X is $\frac{\varepsilon}{2}$-chainable, we can find $\frac{\varepsilon}{2}$-chains

$$x^* = x_0, x_1, \ldots, x_n = z^* \text{ and } z^* = y_0, y_1, \ldots, y_k = x^*.$$

We can easily establish that

$$d^s(T^m x^*, T^m z^*) < \frac{\lambda^m l\varepsilon}{2}, \qquad \forall m \in \mathbb{N},$$

where $l = \max\{n, k\}$. This is easily by observing that

$$d(T^m x^*, T^m z^*) < \frac{\lambda^m n\varepsilon}{2} \leq \frac{\lambda^m l\varepsilon}{2}, \qquad \forall m \in \mathbb{N},$$

and

$$d(T^m z^*, T^m x^*) < \frac{\lambda^m k\varepsilon}{2} \leq \frac{\lambda^m l\varepsilon}{2}, \qquad \forall m \in \mathbb{N}.$$

Hence

$$d^s(x^*, z^*) = d^s(T^m x^*, T^m z^*) < \frac{\lambda^m n\varepsilon}{2}$$

which tends to 0 as $m \to \infty$, hence $x^* = z^*$.
This completes the proof.

We finish this section by an example.

Example 4.1. *Let $X = \{0,1,2\}$ and $d : X \times X \to [0,\infty)$ be defined by $d(0,1) = d(0,2) = 1 = d(1,0) = d(1,2)$, $d(2,1) = d(2,0) = 2$ and $d(x,x) = 0$ for all $x \in X$. Let $T : X \to X$ be the mapping*

$$T(x) = \begin{cases} 1 & , \text{ if } x \in \{0,1\}, \\ 0 & , \text{ if } x = 2. \end{cases}$$

It is easy to check that d is a T_0-quasi-pseudometric and that (X,d) is Hausdorff. It is also easy to see that (X,d) is $\frac{\varepsilon}{2}$-chainable where $\varepsilon = 4$. It can also be noticed that T is a (ε,λ) uniformly locally contractive mapping with $\lambda = \frac{1}{2}$ and T has a unique fixed point $x^* = 1$.

5. Conclusion

All the results we prove assumed a kind of continuity for the map T involved. This condition is meant to guarantee that we can pass the limit under the function and keep it. In [4], we will present more general fixed points results which do require any type of continuity.

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