ON THE THIRD BOUNDARY VALUE PROBLEM FOR PARABOLIC EQUATIONS IN A NON-REGULAR DOMAIN OF $\mathbb{R}^{N+1}$

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ABSTRACT. In this paper, we look for sufficient conditions on the lateral surface of the domain and on the coefficients of the boundary conditions of a $N-$space dimensional linear parabolic equation, in order to obtain existence, uniqueness and maximal regularity of the solution in a Hilbertian anisotropic Sobolev space when the right hand side of the equation is in a Lebesgue space. This work is an extension of solvability results obtained for a second order parabolic equation, set in a non-regular domain of $\mathbb{R}^3$ obtained in [1], to the case where the domain is cylindrical, not with respect to the time variable, but with respect to $N$ space variables, $N > 1$.

Keywords: Parabolic equations, Non-regular domains, Robin conditions, Anisotropic Sobolev spaces.

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1. INTRODUCTION

Let $\Omega$ be an open set of $\mathbb{R}^2$ defined by

$$\Omega = \{ (t, x_1) \in \mathbb{R}^2 : 0 < t < T; \varphi_1 (t) < x_1 < \varphi_2 (t) \}$$

where $T$ is a finite positive number, while $\varphi_1$ and $\varphi_2$ are Lipschitz continuous real-valued functions defined on $[0, T]$, and such that

$$\varphi (t) := \varphi_2 (t) - \varphi_1 (t) > 0$$

for $t \in [0, T]$. For fixed positive numbers $b_i, i = 1, ..., N - 1$, with $N > 1$, let $Q$ be the $(N+1)$-dimensional domain defined by

$$Q = \{ (t, x_1) \in \mathbb{R}^2 : 0 < t < T; \varphi_1 (t) < x_1 < \varphi_2 (t) \} \times \prod_{i=1}^{N-1} [0, b_i].$$

In $Q$, consider the boundary value problem

$$\begin{cases}
\partial_t u - \Delta u = f \in L^2 (Q), \\
\partial_{x_1} u + \beta_i u|_{\Sigma_i} = 0, i = 1, 2, \\
u|_{\partial Q \setminus (\Sigma_1 \cup \Sigma_T)} = 0, i = 1, 2,
\end{cases}$$

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where \( \Delta u = \sum_{k=1}^{N} \frac{\partial^2 u}{\partial x_k^2} \), \( \partial Q \) is the boundary of \( Q \), \( \Sigma_i \), \( i = 1, 2 \) is the part of \( \partial Q \) where \( x_1 = \varphi_i(t) \), \( i = 1, 2 \), \( \Sigma_T \) is the part of \( \partial Q \) where \( t = T \) and with the fundamental hypothesis \( \varphi(0) = 0 \).

The difficulty related to this kind of problems comes from this singular situation for evolution problems, i.e., \( \varphi_2 \) is allowed to coincide with \( \varphi_2 \) for \( t = 0 \), which prevent the domain \( Q \) to be transformed into a regular domain by means of a smooth transformation, see for example Sadallah [2]. On the other hand, the semi group generating the solution cannot be defined since the initial condition is defined on a set measure zero.

We are especially interested in the question of what sufficient conditions, as weak as possible, the functions \( \varphi_1 \), \( \varphi_2 \) and the coefficients \( \beta_i \), \( i = 1, 2 \), must verify in order that Problem (1) has a solution with optimal regularity, that is a solution \( u \) belonging to the anisotropic Sobolev space

\[
H^{1,2}_\gamma(Q) = \left\{ u \in H^{1,2}(Q) : u|_{\partial Q \setminus (\Sigma_1 \cup \Sigma_T)} = \partial_{x_1} u + \beta_i u|_{\Sigma_i} = 0, \quad i = 1, 2 \right\}
\]

with

\[
H^{1,2}(Q) = \left\{ u \in L^2(Q) : \partial_t u, \partial_{x_1}^i \partial_{x_2}^j u \in L^2(Q), \quad 1 \leq i_1 + i_2 + \ldots + i_N \leq 2 \right\}.
\]

Note that the Robin type condition \( \partial_{x_1} u + \beta_i u|_{\Sigma_i} = 0, \quad i = 1, 2 \) is a perturbation by \( \beta_i \), \( i = 1, 2 \) of the Neumann type one and it is well known that Dirichlet and Neumann type boundary conditions correspond to two extreme cases, namely \( \beta_i = \infty \) and \( \beta_i = 0 \), \( i = 1, 2 \), respectively. We can find in [3], [4], [5], [6], [7], [8] and [9] solvability results of this kind of problems with Dirichlet boundary conditions. In Nazarov [10], results for the Neumann problem in a conical domain were proved. We can find in Savaré [11] an abstract study for parabolic problems with mixed (Dirichlet-Neumann) lateral boundary conditions. The case of Robin type conditions in a non-rectangular domain is studied in [12].

The organization of this paper is as follows. In Section 2, we prove that Problem (1) admits a (unique) solution in the case of a truncated domain. In Section 3 we approximate \( Q \) by a sequence \( (Q_n) \) of such domains and we establish (for \( T \) small enough) a uniform estimate of the type

\[
\|u_n\|_{H^{1,2}(Q_n)} \leq K \|f\|_{L^2(Q_n)},
\]

where \( u_n \) is the solution of Problem (1) in \( Q_n \) and \( K \) is a constant independent of \( n \). Finally, in Section 4 we prove the two main results of this paper.

The main assumptions on the functions \( \varphi_1 \), \( \varphi_2 \) and on the coefficients \( \beta_i \), \( i = 1, 2 \), are

\[
\varphi_i(t) \varphi(t) \rightarrow 0 \quad \text{as } t \rightarrow 0, \quad i = 1, 2.
\]

The coefficients \( \beta_i \), \( i = 1, 2 \) are real numbers such that

\[
\beta_1 < 0 \quad \text{and} \quad \beta_2 > 0,
\]

\[
(-1)^i \left( \beta_i - \frac{\varphi_i(t)}{2} \right) \geq 0 \quad \text{a.e. } t \in [0, T[, \quad i = 1, 2.
\]

2. Resolution of the problem (1) in truncated domains \( Q_n \)

In this section, we replace \( Q \) by \( Q_n, n \in \mathbb{N}^* \) and \( \frac{1}{n} < T \):

\[
Q_n = \left\{ (t, x) \in Q : \frac{1}{n} < t < T \right\},
\]

where \( x = (x_1, x_2, \ldots, x_N) \).
Theorem 2.1. Under the assumptions (3) and (4) on the functions of parametrization \( \varphi_i \) and on the coefficients \( \beta_i, i = 1, 2 \), and for each \( n \in \mathbb{N}^* \) such that \( \frac{1}{n} < T \), the following problem admits a (unique) solution \( u_n \in H^{1,2}(Q_n) \)

\[
\begin{align*}
\partial_t u_n - \Delta u_n &= f_n \in L^2(Q_n), \\
\partial_{x_i} u_n + \beta_i u_n \big|_{\Sigma_i,n} &= 0, \ i = 1, 2, \\
u_n \big|_{\partial Q_n \setminus (\Sigma_i,n \cup \Sigma_{T,n})} &= 0, \ i = 1, 2.
\end{align*}
\]

Here

\[
\Sigma_{i,n} = \left\{ (t, \varphi_i(t)) : \frac{1}{n} < t < T \right\} \times \prod_{k=1}^{N-1} [0, b_k[ , \ i = 1, 2
\]

and \( \Sigma_{T,n} \) is the part of the boundary of \( Q_n \) where \( t = T \).

Proof. The uniqueness of the solution is easy to check, thanks to (4). Let us prove its existence. The change of variables

\[
\Phi : (t, x) \longmapsto (t, y) = \left( t, \frac{x_1 - \varphi_1(t)}{\varphi(t)}, x' \right)
\]

transforms \( Q_n \) into the cylinder \( P_n = ]\frac{1}{n}, T[ \times [0,1[ \times \prod_{k=1}^{N-1} [0, b_k[. \) Here and in the sequel \( x = (x_1, x_2, ..., x_N), \ x' = (x_2, ..., x_N) \) and \( y = (y_1, y_2, ..., y_N). \) Putting

\[
w_n(t, y) = u_n(t, x) \quad \text{and} \quad g_n(t, y) = f_n(t, x),
\]

then Problem (5) is transformed, in \( P_n \) into the variable-coefficient parabolic problem

\[
\begin{align*}
\partial_t w_n + a(t, y_1) \partial_{y_1} w_n - \frac{1}{b^2(t)} \partial_{y_1}^2 w_n - \sum_{k=2}^{N} \partial_{y_k}^2 w_n &= g_n, \\
\partial_{y_2} w_n + \beta_i \varphi(t) w_n \big|_{\Sigma_i,P_n} &= 0, \ i = 1, 2, \\
w_n \big|_{\partial P_n \setminus (\Sigma_i,n \cup \Sigma_{T,P_n})} &= 0, \ i = 1, 2,
\end{align*}
\]

where \( \Sigma_{1,P_n} = ]0, T[ \times \{0\} \times \prod_{k=1}^{N-1} [0, b_k[ , \Sigma_{2,P_n} = ]0, T[ \times \{1\} \times \prod_{k=1}^{N-1} [0, b_k[ , \Sigma_{T,P_n} = \{T\} \times \{0\} \times \prod_{k=1}^{N-1} [0, b_k[ , b(t) = \varphi(t) \) and \( a(t, y_1) = -\frac{y_1 \varphi'(t) + \varphi'(t)}{\varphi(t)}. \)

Since the functions \( a \) and \( \varphi \) are bounded when \( t \in ]\frac{1}{n}, T[ \), then the above change of variables which is \( (N + 1) \)-Lipschitz preserves the spaces \( H^{1,2} \) and \( L^2 \). In other words

\[
f_n \in L^2(Q_n) \Leftrightarrow g_n \in L^2(P_n) , \quad u_n \in H^{1,2}(Q_n) \Leftrightarrow w_n \in H^{1,2}(P_n).
\]

In the sequel, the variables \( (t, y) \) will be denoted again by \( (t, x) \). Consider the simplified problem

\[
\begin{align*}
\partial_t w_n - \frac{1}{b^2(t)} \partial_{y_1}^2 w_n - \sum_{k=2}^{N} \partial_{y_k}^2 w_n &= g_n, \\
\partial_{y_2} w_n + \beta_i \varphi(t) w_n \big|_{\Sigma_i,P_n} &= 0, \ i = 1, 2, \\
w_n \big|_{\partial P_n \setminus (\Sigma_i,n \cup \Sigma_{T,P_n})} &= 0, \ i = 1, 2.
\end{align*}
\]

Lemma 2.1. For each \( n \in \mathbb{N}^* \) such that \( \frac{1}{n} < T \) and for every \( g_n \in L^2(P_n) \), there exists a unique \( w_n \in H^{1,2}(P_n) \) solution of (7).

Proof. Since the coefficient \( b(t) \) is continuous in \( P_n \), the optimal regularity result is given by Ladyzhenskaya-Solonnikov-Ural’tseva [13].
Lemma 2.2. For each \( n \in \mathbb{N}^* \) such that \( \frac{1}{n} < T \), the following operator is compact
\[
a(t, x_1) \partial_{x_1} : H^{1,2}_\gamma(P_n) \rightarrow L^2_\omega(P_n).
\]

Here, for \( i = 1, 2 \)
\[
H^{1,2}_\gamma(P_n) = \{ w_n \in H^{1,2}_\gamma(P_n) : w_n|_{\partial P_n \setminus \Sigma \cup \Sigma T} = \partial_{x_1} w_n + \beta_i \varphi(t) w_n|_{\Sigma i, P_n} = 0 \}.
\]

Proof. \( P_n \) has the "horn property" of Besov [14], so
\[
\partial_{x_1} : H^{1,2}_\gamma(P_n) \rightarrow H^{\frac{1}{2},1}(P_n), \ w_n \mapsto \partial_{x_1} w_n,
\]
is continuous. Since \( P_n \) is bounded, the canonical injection is compact from \( H^{\frac{1}{2},1}(P_n) \) into \( L^2(P_n) \), where
\[
H^{\frac{1}{2},1}(P_n) = L^2 \left( \frac{1}{n}, T; H^1 \left( \left[ 0, 1 \right] \times \prod_{i=1}^{N-1} \left| 0, b_i \right| \right) \right) \cap H^{\frac{1}{2}} \left( \frac{1}{n}, T; L^2 \left( \left| 0, 1 \right| \times \prod_{i=1}^{N-1} \left| 0, b_i \right| \right) \right).
\]

For the complete definitions of the \( H^{r,s} \) Hilbertian Sobolev spaces see for instance [15]. Consider the composition
\[
\partial_{x_1} : H^{1,2}_\gamma(P_n) \rightarrow H^{\frac{1}{2},1}(P_n) \rightarrow L^2(P_n), \ w_n \mapsto \partial_{x_1} w_n \mapsto \partial_{x_1} w_n,
\]
then, \( \partial_{x_1} \) is a compact operator from \( H^{1,2}_\gamma(P_n) \) into \( L^2(P_n) \). Since \( a(\cdot, \cdot) \) is a bounded function for \( \frac{1}{n} < t < T \), the operator \( a \partial_{x_1} \) is also compact from \( H^{1,2}_\gamma(P_n) \) into \( L^2(P_n) \). \( \square \)

Lemma 2.1 shows that the operator \( \partial_t - \frac{1}{b^2(t)} \partial^2_{x_1} - \sum_{k=2}^N \partial^2_{x_k} \) is an isomorphism from \( H^{1,2}_\gamma(P_n) \) into \( L^2(P_n) \). On the other hand, the operator \( a \partial_{x_1} \) is compact (see Lemma 2.2). Consequently, the operator \( \partial_t + a(\cdot, \cdot) \partial_{x_1} - \frac{1}{b^2(t)} \partial^2_{x_1} - \sum_{k=2}^N \partial^2_{x_k} \) is a Fredholm operator from \( H^{1,2}_\gamma(P_n) \) into \( L^2(P_n) \). Thus the invertibility of \( \partial_t + a(\cdot, \cdot) \partial_{x_1} - \frac{1}{b^2(t)} \partial^2_{x_1} - \sum_{k=2}^N \partial^2_{x_k} \) follows from its injectivity.

Let \( w_n \in H^{1,2}_\gamma(P_n) \) be a solution of
\[
\partial_t w_n + a(t, x_1) \partial_{x_1} w_n - \frac{1}{b^2(t)} \partial^2_{x_1} w_n - \sum_{k=2}^N \partial^2_{x_k} w_n = 0
\]
in \( P_n \). We perform the inverse change of variable of \( \Phi \). Thus we set
\[
u_n = w_n \circ \Phi.
\]

It turns out that \( \nu_n \in H^{1,2}_\gamma(Q_n) \), and
\[
\partial_t \nu_n - \Delta \nu_n = 0, \text{ in } Q_n.
\]

In addition \( \nu_n \) fulfills the boundary conditions
\[
\partial_{x_1} \nu_n + \beta_i \nu_n|_{\Sigma i, n} = \nu_n|_{\partial Q_n \setminus \Sigma \cup \Sigma T, n} = 0, \ i = 1, 2,
\]
which imply that \( \nu_n \) vanishes (see Theorem 4.1); this is the desired injectivity and ends the proof of Theorem 2.1.

Lemma 2.3. For each \( n \in \mathbb{N}^* \) such that \( \frac{1}{n} < T \), the space
\[
W = \left\{ u_n \in D \left( \left[ \frac{1}{n}, T \right]; H^4 \left( \left| 0, 1 \right| \times \prod_{i=1}^{N-1} \left| 0, b_i \right| \right) \right) : \partial_{x_1} u_n + \beta_i u_n|_{\Sigma i, P_n} = 0, \ i = 1, 2 \right\},
\]
Remark 2.1. is dense in
\[ H^{1,2}_\gamma(P_n) = \left\{ u_n \in H^{1,2}(P_n) : \partial_{x_i} u_n + \beta_i u_n |_{\Sigma_i,P_n} = 0, \ i = 1,2 \right\}. \]

The above lemma is a particular case of [15, Theorem 2.1], from which we can derive the following result in order to justify the calculus of the section 3.

Lemma 2.4. For each \( n \in \mathbb{N}^* \) such that \( \frac{1}{n} < T \), the space
\[ \left\{ u_n \in H^1(P_n) : u_n|_{\partial P_n \setminus (\Sigma_i,P_n \cup \Sigma_T,P_n)} = \partial_{x_1} u_n + \beta_i u_n |_{\Sigma_i,P_n} = 0, \ i = 1,2 \right\} \]
is dense in the space
\[ \left\{ u_n \in H^{1,2}(P_n) : u_n|_{\partial P_n \setminus (\Sigma_i,P_n \cup \Sigma_T,P_n)} = \partial_{x_1} u_n + \beta_i u_n |_{\Sigma_i,P_n} = 0, \ i = 1,2 \right\}. \]

Remark 2.1. In Lemma 2.4, we can replace \( P_n \) by \( Q_n \) with the help of the change of variables defined above.

3. A UNIFORM ESTIMATE

For each \( n \in \mathbb{N}^* \) such that \( \frac{1}{n} < T \), we denote by \( u_n \in H^{1,2}(Q_n) \) the solution of Problem (5) in \( Q_n \). Such a solution \( u_n \) exists by Theorem 2.1.

Theorem 3.1. For each \( n \in \mathbb{N}^* \) such that \( \frac{1}{n} < T \) with \( T \) small enough, there exists a constant \( K > 0 \) independent of \( n \) such that
\[ \| u_n \|^2_{H^{1,2}(Q_n)} \leq K \| f \|^2_{L^2(Q_n)} \leq K \| f \|^2_{L^2(Q)}, \]
where
\[ \| u_n \|^2_{H^{1,2}(Q_n)} = \left( \| \partial_t u_n \|^2_{L^2(Q_n)} + \| u_n \|^2_{L^2(Q_n)} + \sum_{1 \leq i_1 + \ldots + i_N \leq 2} \| \partial_{x_1}^{i_1} \ldots \partial_{x_N}^{i_N} u_n \|^2_{L^2(Q_n)} \right). \]

In order to prove Theorem 3.1, we need some preliminary results. The proof of the following Lemma can be found in [1].

Lemma 3.1. Under the assumption (3) on \( (\beta_i)_{i=1,2} \), there exists a positive constant \( C_1 \) (independent of \( a \) and \( b \)) such that
\[ \| v(k) \|^2_{L^2(a,b)} \leq C_1 (b-a)^{2(2-k)} \| v(2) \|^2_{L^2(a,b)}, \]
for each \( v \in H^2(a,b) \), with
\[ H^2(a,b) = \left\{ v \in H^2(a,b) : v(a) = 0, v(b) = 0 \right\}. \]

Lemma 3.2. For every \( \epsilon > 0 \) chosen such that \( \varphi(t) \leq \epsilon, \) there exists a constant \( C > 0 \) independent of \( n \), such that
\[ \| \partial_{x_1} u_n \|^2_{L^2(Q_n)} \leq C \epsilon^{2(2-j)} \| \partial_{x_1} u_n \|^2_{L^2(Q_n)} , \ j = 0,1. \]

Proof. Replacing in Lemma 3.1 \( v \) by \( u_n \) and \( [a,b] \) by \( |\varphi_1(t)\varphi_2(t)| \), for a fixed \( t \), we obtain
\[ \int_{\varphi_1(t)}^{\varphi_2(t)} \left( \partial_{x_1} u_n \right)^2 dx_1 \leq C \varphi(t)^{2(2-j)} \int_{\varphi_1(t)}^{\varphi_2(t)} \left( \partial_{x_1} u_n \right)^2 dx_1 \]
where \( C \) is the constant of Lemma 3.1. Integrating with respect to \( t \), then with respect to \( x_2, x_3, \ldots, x_N \), we obtain the desired estimates. \( \square \)
Proposition 3.1. For each \( n \in \mathbb{N}^* \) such that \( \frac{1}{n} < T \) with \( T \) small enough, there exists a constant \( C > 0 \) independent of \( n \) such that
\[
\| \partial_t u_n \|_{L^2(Q_n)}^2 + \sum_{i_1, i_2, \ldots, i_N = 0}^2 \| \partial_{x_1}^2 \partial_{x_2}^2 \cdots \partial_{x_N}^2 u_n \|_{L^2(Q_n)}^2 \leq C \| f \|_{L^2(Q)}^2.
\]

Then, Theorem 3.1 is a direct consequence of Lemma 3.2 and Proposition 3.1, since \( \epsilon \) is independent of \( n \).

Proof. Step 1. First, we estimate the inner products
\[
\sum_{k=1}^N (\partial_t u_n, \partial_{x_k}^2 u_n) \text{ and } \sum_{k=1}^N (\sum_{j=1}^N \partial_{x_j}^2 u_n, \sum_{j=1}^N \partial_{x_j}^2 u_n), k \neq j
\]
in \( L^2(Q_n) \) making use of the boundary conditions (particulary, of the relation \( \partial_{x_1} u_n + \beta_i u_n = 0 \) on the parts of the boundary of \( Q_n \) where \( x = \varphi_i(t), i = 1, 2 \)). We use these estimates (step 2) when we develop the expression of \( \| f_n \|_{L^2(Q_n)}^2 \).

1) Estimation of \(-2(\partial_t u_n, \partial_{x_1}^2 u_n)\): We have
\[
-2(\partial_t u_n, \partial_{x_1}^2 u_n) = \partial_{x_1} (\partial_t u_n \partial_{x_1} u_n) - \frac{1}{2} \partial_t (\partial_{x_1} u_n)^2.
\]
Then
\[
-2(\partial_t u_n, \partial_{x_1}^2 u_n) = -2 \int_{Q_n} \partial_{x_1} (\partial_t u_n \partial_{x_1} u_n) dt \, dx + \int_{Q_n} \partial_t (\partial_{x_1} u_n)^2 dt \, dx
\]
\[
= \int_{Q_n} (\partial_{x_1} u_n)^2 \nu_t - 2 \partial_t u_n \partial_{x_1} u_n \nu_{x_1} \, d\sigma,
\]
where \( \nu_t, \nu_{x_1}, \ldots, \nu_{x_N} \) are the components of the unit outward normal vector at \( \partial Q_n \) and \( dx = dx_1 dx_2 \cdots dx_N \). We shall rewrite the boundary integral making use of the boundary conditions. On the parts of the boundary of \( Q_n \) where \( t = \frac{1}{n}, x_k = 0, k = 2, \ldots, N \) and \( x_k = b_{k-1}, k = 2, \ldots, N \) we have \( u_n = 0 \) and consequently \( \partial_{x_1} u_n = 0 \). The corresponding boundary integral vanishes. On the boundary of the boundary where \( t = T \), we have \( \nu_{x_1} = 0 \) and \( \nu_t = 1 \). Accordingly the corresponding boundary integral
\[
\int_0^{b_{N-1}} \cdots \int_0^{b_1} \int_{\varphi_1(T)}^{\varphi_2(T)} (\partial_{x_1} u_n)^2 \, dx
\]
is nonnegative. On the parts of the boundary where \( x_1 = \varphi_i(t), i = 1, 2 \), we have
\[
\nu_{x_1} = \frac{(-1)^i}{\sqrt{1 + (\varphi_i')^2(t)}}, \quad \nu_t = \frac{(-1)^{i+1} \varphi_i'(t)}{\sqrt{1 + (\varphi_i')^2(t)}}
\]
and
\[
\partial_{x_1} u_n (t, \varphi_i(t), x') + \beta_i u_n (t, \varphi_i(t), x') = 0, i = 1, 2.
\]
Consequently the corresponding boundary integral is
\[
I_{n,k} = (-1)^{k+1} \int_0^{b_{N-1}} \cdots \int_0^{b_1} \int_{\varphi_1(T)}^{\varphi_2(T)} \partial_{x_1} u_n (t, \varphi_k(t), x')^2 dt \, dx', k = 1, 2,
\]
\[
J_{n,k} = (-1)^k 2 \int_0^{b_{N-1}} \cdots \int_0^{b_1} \int_{\varphi_1(T)}^{\varphi_2(T)} \beta_k (\partial_t u_n, u_n) (t, \varphi_k(t), x') dt \, dx', k = 1, 2,
\]
where \( dx' = dx_2 \cdots dx_N \). Then, we have
\[
-2(\partial_t u_n, \partial_{x_1}^2 u_n) \geq -|I_{n,1}| - |I_{n,2}| - |J_{n,1}| - |J_{n,2}|. \tag{8}
\]

2) Estimation of \(-2 \sum_{k=2}^N (\partial_t u_n, \partial_{x_k}^2 u_n)\): We have
\[
\partial_t u_n \partial_{x_k}^2 u_n = \partial_{x_k} (\partial_t u_n \partial_{x_k} u_n) - \frac{1}{2} \partial_t (\partial_{x_k} u_n)^2.
\]
Then
\[-2\langle \partial_t u_n, \partial_x^2 u_n \rangle = -2 \int_{Q_n} \partial_x^2 (\partial_t u_n \partial_x u_n) \, dt \, dx + \int_{Q_n} \partial_t (\partial_x u_n)^2 \, dt \, dx \]
\[= \int_{Q_n} (\partial_x^2 u_n)^2 \, \nu_t - 2 \partial_t u_n \partial_x u_n \nu_x \, d\sigma.\]

On the part of the boundary where \( t = \frac{1}{n}, x_k = 0, k = 2, \ldots, N \) and \( x_k = b_{k-1}, k = 2, \ldots, N \) we have \( u_n = 0 \) and consequently \( \partial_x u_n = 0 \). The corresponding boundary integral vanishes. On the part of the boundary where \( t = T \), we have \( \nu_x = 0, \nu_{x_k} = 0, k = 2, \ldots, N \) and \( \nu_t = 1 \). The corresponding boundary integral
\[\int_0^{b_{N-1}} \cdots \int_0^{b_1} \int_{\varphi_x(T)} (\partial_x^2 u_n)^2 \, dx \]
is nonnegative. On the parts of the boundary of \( Q_n \) where \( x_1 = \varphi_i(t), i = 1, 2 \), we have \( \nu_x = \frac{(-1)^i}{\sqrt{1 + (\varphi_i')^2(t)}} \), \( \nu_t = \frac{(-1)^{i+1} \varphi_i'(t)}{\sqrt{1 + (\varphi_i')^2(t)}} \) and \( \nu_{x_k} = 0, k = 2, \ldots, N \). Consequently the corresponding boundary integral is
\[M_{n,j} = (-1)^{j+1} \int_0^{b_{N-1}} \cdots \int_0^{b_1} \int_0^T \varphi_x(t) \left[ \partial_x u_n \left(t, \varphi_j(t), x'\right)\right]^2 \, dtdx', j = 1, 2. \]
Then, we have
\[-2\langle \partial_t u_n, \partial_x^2 u_n \rangle \geq M_{n,1} + M_{n,2}, k = 2, \ldots, N. (9)\]

3) Estimation of \(2 \sum_{k=2}^N (\partial_x^2 u_n, \partial_x^2 u_n)\): We have
\[\partial_x^2 u_n, \partial_x^2 u_n = \partial_x \left( \partial_x u_n, \partial_x^2 u_n \right) - \partial_x^3 \left( \partial_x u_n, \partial_x u_n \partial_x u_n \right) + (\partial_x u_n)^2.\]

Then
\[2\langle \partial_x^2 u_n, \partial_x^2 u_n \rangle = 2 \int_{Q_n} \partial_x \left( \partial_x u_n, \partial_x^2 u_n \right) \, dt \, dx - 2 \int_{Q_n} \partial_x^3 \left( \partial_x u_n, \partial_x u_n \partial_x u_n \right) \, dt \, dx + 2 \int_{Q_n} (\partial_x u_n)^2 \, dt \, dx \]
\[= 2 \int_{Q_n} (\partial_x^2 u_n)^2 \, dt \, dx + 2 \int_{Q_n} [\partial_x u_n \partial_x^2 u_n \nu_x - \partial_x u_n \partial_x u_n \partial_x u_n \nu_{x_k}] \, d\sigma.\]

On the part of the boundary where \( t = \frac{1}{n}, x_k = 0, k = 2, \ldots, N \) and \( x_k = b_{k-1}, k = 2, \ldots, N \) we have \( u_n = 0 \) and consequently \( \partial_x u_n = 0 \). On the part of the boundary where \( t = T \), we have \( \nu_x = 0, \nu_{x_k} = 0, k = 2, \ldots, N \) and \( \nu_t = 1 \). The corresponding boundary integral vanishes. On the parts of the boundary of \( Q_n \) where \( x_1 = \varphi_i(t), i = 1, 2 \), we have
\[\nu_{x_1} = \frac{(-1)^i}{\sqrt{1 + (\varphi_i')^2(t)}}, \nu_t = \frac{(-1)^{i+1} \varphi_i'(t)}{\sqrt{1 + (\varphi_i')^2(t)}} \] and \( \nu_{x_k} = 0, k = 2, \ldots, N \)
and
\[\partial_x u_n \left(t, \varphi_i(t), x'\right) + \beta_1 u_n \left(t, \varphi_i(t), x'\right) = 0, i = 1, 2.\]
Consequently, the corresponding boundary integral is
\[H_{n,j} = (-1)^j 2 \int_0^{b_{N-1}} \cdots \int_0^{b_1} \int_0^T \beta_i \left[ \partial_x u_n \left(t, \varphi_j(t), x'\right)\right]^2 \, dtdx', j = 1, 2. \]
Then, we have
\[2\langle \partial_x^2 u_n, \partial_x^2 u_n \rangle = 2 \| \partial_x u_n \partial_x u_n \|_{L^2(Q_n)}^2 + H_{n,1} + H_{n,2}. (10)\]

Summing up the estimates (9) and (10) and using the hypothesis (4), we obtain
\[-2\langle \partial_t u_n, \partial_x^2 u_n \rangle + 2\langle \partial_x^2 u_n, \partial_x^2 u_n \rangle \geq 2 \| \partial_x u_n \partial_x u_n \|_{L^2(Q_n)}^2, k = 2, \ldots, N. (11)\]
Indeed, for \( k = 2, \ldots, N \) we have
\[
\sum_{j=1}^{2} M_{n,j} + H_{n,j} = \sum_{j=1}^{2} \int_0^{b_{N-1}} \cdots \int_0^{b_1} \int_0^{T} (-1)^k \left( 2\beta_j^2(t) \right) \left[ \partial_{x_k} u_n (t, \varphi_j (t), x') \right]^2 \, dt \, dx',
\]
which is nonnegative, thanks to the hypothesis (4). By a similar argument, we obtain
\[
2\langle \partial_{x_k}^2 u_n, \partial_{x_k}^2 u_n \rangle \geq 2 \left\| \partial_{x_k} \partial_{x_k} u_n \right\|_{L^2(Q_n)}^2, \quad k = 3, \ldots, N,
\]
\[
2\langle \partial_{x_k}^2 u_n, \partial_{x_k}^2 u_n \rangle \geq 2 \left\| \partial_{x_k} \partial_{x_k} u_n \right\|_{L^2(Q_n)}^2, \quad k = 4, \ldots, N,
\]
\[
\vdots
\]
\[
2\langle \partial_{x_{N-1}}^2 u_n, \partial_{x_{N-1}}^2 u_n \rangle \geq 2 \left\| \partial_{x_{N-1}} \partial_{x_{N-1}} u_n \right\|_{L^2(Q_n)}^2.
\]
\[\tag{12}\]

**Step 2. Estimation of \( I_{n,k} \), \( J_{n,k} \):** We have
\[
\left\| f_n \right\|^2_{L^2(Q_n)} = \langle \partial_{u_n} - \sum_{k=1}^N \partial_{x_k}^2 u_n \partial_{x_k} u_n - \sum_{k=1}^N \partial_{x_k}^2 u_n \rangle = \left\| \partial_{u_n} \right\|^2_{L^2(Q_n)} + \sum_{k=1}^N \left\| \partial_{x_k}^2 u_n \right\|^2_{L^2(Q_n)}.
\]

It is the reason for which we look for an estimate of the type
\[
|I_{n,1}| + |I_{n,2}| + |I_{n,1}| + |J_{n,2}| \leq K \epsilon \left\| \partial_{x_1} u_n \right\|^2_{L^2(Q_n)}.
\]

**A. Estimation of \( I_{n,k} \), \( k = 1, 2 \)**

**Lemma 3.3.** There exists a constant \( K > 0 \) independent of \( n \) such that
\[
|I_{n,k}| \leq K \epsilon \left\| \partial_{x_1}^2 u_n \right\|^2_{L^2(Q_n)}, \quad k = 1, 2.
\]

**Proof.** We convert the boundary integral \( I_{n,1} \) into a surface integral by setting
\[
\left[ \partial_{x_1} u_n (t, \varphi_1 (t), x') \right]^2 = -\frac{\varphi_2(t)-x_1}{\varphi_2(t)-\varphi_1(t)} \left[ \partial_{x_1} u_n (t, x) \right]_{x_1=\varphi_1(t)}^{x_1=\varphi_2(t)} + \frac{\varphi_2(t)-x_1}{\varphi_2(t)-\varphi_1(t)} \left[ \partial_{x_1} u_n (t, x) \right]_{x_1=\varphi_1(t)}^{x_1=\varphi_2(t)} \, dx_1.
\]

Then, we have
\[
I_{n,1} = \int_0^{b_{N-1}} \cdots \int_0^{b_1} \int_0^{T} \varphi'_1 (t) \left[ \partial_{x_1} u_n (t, \varphi_1 (t), x') \right]^2 \, dt \, dx' = \int_{Q_n} \frac{\varphi'_1(t)}{\varphi(t)} \left[ \partial_{x_1} u_n \right]_{x_1=\varphi_1(t)} \left[ \partial_{x_1} u_n \right]_{x_1=\varphi_2(t)} \, dt \, dx.
\]

Thanks to Lemma 3.2, we can write
\[
\int_{Q_n} \frac{\varphi'_1(t)}{\varphi(t)} \left[ \partial_{x_1} u_n \right]_{x_1=\varphi_1(t)} \left[ \partial_{x_1} u_n \right]_{x_1=\varphi_2(t)} \, dt \, dx \leq C \left\| \varphi \right\|^2 \int_{Q_n} \left[ \partial_{x_1}^2 u_n \right]_{x_1=\varphi_1(t)} \left[ \partial_{x_1}^2 u_n \right]_{x_1=\varphi_2(t)} \, dt \, dx.
\]

Therefore
\[
\int_{Q_n} \frac{\varphi'_1(t)}{\varphi(t)} \left[ \partial_{x_1} u_n \right]_{x_1=\varphi_1(t)} \left[ \partial_{x_1} u_n \right]_{x_1=\varphi_2(t)} \, dt \, dx \leq C \left\| \varphi \right\|^2 \int_{Q_n} \left[ \partial_{x_1}^2 u_n \right]_{x_1=\varphi_1(t)} \left[ \partial_{x_1}^2 u_n \right]_{x_1=\varphi_2(t)} \, dt \, dx.
\]
consequently,
\[
|I_{n,1}| \leq C \int_{Q_n} \left\| \varphi \right\|^2 \left[ \partial_{x_1}^2 u_n \right]^2 \, dt \, dx + 2 \int_{Q_n} \left\| \varphi \right\|^2 \left[ \partial_{x_3} u_n \right] \left[ \partial_{x_1}^2 u_n \right] \, dt \, dx.
\]
since \( \left| \frac{\varphi_2(t) - x_1}{\varphi(t)} \right| \leq 1 \). Using the inequality

\[
2 |\varphi'_1 \partial x_1 u_n| |\partial^2 x_1 u_n| \leq E \left( \partial^2 x_1 u_n \right)^2 + \frac{1}{E} (\varphi'_1)^2 (\partial x_1 u_n)^2
\]

for all \( E > 0 \), we obtain

\[
|I_{n,1}| \leq C \int_{Q_n} |\varphi'_1| \varphi(t) (\partial^2 x_1 u_n)^2 \, dt \, dx + \int_{Q_n} |E (\partial^2 x_1 u_n)^2 + \frac{1}{E} (\varphi'_1)^2 (\partial x_1 u_n)^2| \, dt \, dx.
\]

Lemma 3.2 yields

\[
\frac{1}{E} \int_{Q_n} (\varphi'_1)^2 (\partial x_1 u_n)^2 \, dt \, dx \leq C \frac{1}{E} \int_{Q_n} (\varphi'_1)^2 \varphi(t) \partial^2 x_1 u_n^2 \, dt \, dx.
\]

Thus, there exists a constant \( K > 0 \) independent of \( n \) such that

\[
|I_{n,1}| \leq C \int_{Q_n} \left[ |\varphi'_1| \varphi(t) + \frac{1}{E} (\varphi'_1)^2 \varphi(t)^2 \right] \partial^2 x_1 u_n^2 \, dt \, dx + \int_{Q_n} E (\partial^2 x_1 u_n)^2 \, dt \, dx,
\]

because \( |\varphi'_1 \varphi(t)| \leq E \). The inequality

\[
|I_{n,2}| \leq K E \left\| \partial^2 x_1 u_n \right\|_{L^2(Q_n)}^2,
\]

can be proved by a similar argument.

**B. Estimation of** \( J_{n,k}, k = 1,2 \): We have

\[
J_{n,1} = -2 \int_0^{b_n-1} \cdots \int_0^{b_1} \int_{1/n}^T \beta_1 \partial t u_n(t, \varphi_1(t), x') u_n(t, \varphi_1(t), x') \, dt \, dx \, dx'.
\]

By setting, for each fixed \( x' \) in \( \prod_{i=1}^{N-1} [0, b_i[, h(t) = u_n^2(t, \varphi_1(t), x') \), we obtain

\[
J_{n,1} = -\int_0^{b_n-1} \cdots \int_0^{b_1} \int_{1/n}^T \beta_1 (h(t) - \varphi_1(t) \partial x_1 u_n^2(t, \varphi_1(t), x')) \, dt \, dx \, dx'.
\]

Since \( \beta_1 \) is negative and \( u_n^2 \left( \frac{1}{n}, \varphi_1 \left( \frac{1}{n} \right), x' \right) = 0 \), we have \( \int_0^{b_n-1} \cdots \int_0^{b_1} -\beta_1 h(t) \frac{T}{n} \, dx \geq 0 \).

The last boundary integral in the expression of \( J_{n,1} \) can be treated by a similar argument used in Lemma 3.3. So, we obtain the existence of a positive constant \( K \) independent of \( n \), such that

\[
\left| \int_0^{b_n-1} \cdots \int_0^{b_1} \beta_1 \varphi'_1(t) \partial x_1 u_n^2(t, \varphi_1(t), x') \, dt \, dx \, dx' \right| \leq K E \left\| \partial^2 x_1 u_n \right\|_{L^2(Q_n)}^2,
\]

and consequently,

\[
|J_{n,1}| \geq -K E \left\| \partial^2 x_1 u_n \right\|_{L^2(Q_n)}^2. \tag{13}
\]

By a similar method and using the fact that \( \beta_2 \) is positive and \( u_n^2 \left( \frac{1}{n}, \varphi_2 \left( \frac{1}{n} \right), x' \right) = 0 \), we obtain the existence of a positive constant \( K \) independent of \( n \), such that

\[
|J_{n,2}| \geq -K E \left\| \partial^2 x_1 u_n \right\|_{L^2(Q_n)}^2. \tag{14}
\]
Summing up the estimates (8), (11), (12), (13), (14) and making use of Lemma 3.2, we then obtain

$$\|f_n\|_{L^2(Q_n)}^2 \geq \|\partial_t u_n\|_{L^2(Q_n)}^2 + \sum_{k=1}^N \|\partial_{x_k}^2 u_n\|_{L^2(Q_n)}^2 - 4K\|\epsilon\|_{L^2(Q_n)}^2$$

$$+ 2\sum_{k=2}^N \|\partial_{x_1}\partial_{x_k} u_n\|_{L^2(Q_n)}^2 + 2\sum_{k=3}^N \|\partial_{x_2}\partial_{x_k} u_n\|_{L^2(Q_n)}^2$$

$$+ 2\sum_{k=4}^N \|\partial_{x_3}\partial_{x_k} u_n\|_{L^2(Q_n)}^2 + \ldots + 2\|\partial_{x_{N-1}}\partial_{x_N} u_n\|_{L^2(Q_n)}^2$$

Then, it is sufficient to choose $\epsilon$ such that $(1 - 4K\epsilon) > 0$, to get a constant $K_0 > 0$ independent of $n$ such that

$$\|f_n\|_{L^2(Q_n)}^2 \geq K_0 \left( \|\partial_t u_n\|_{L^2(Q_n)}^2 + \sum_{i_1,i_2,\ldots,i_N=0}^2 \|\partial_{x_1}^{i_1}\partial_{x_2}^{i_2}\ldots\partial_{x_N}^{i_N} u_n\|_{L^2(Q_n)}^2 \right).$$

But $\|f_n\|_{L^2(Q_n)} \leq \|f\|_{L^2(Q)}$, then, there exists a constant $C > 0$, independent of $n$ satisfying

$$\|\partial_t u_n\|_{L^2(Q_n)}^2 + \sum_{i_1,i_2,\ldots,i_N=0}^2 \|\partial_{x_1}^{i_1}\partial_{x_2}^{i_2}\ldots\partial_{x_N}^{i_N} u_n\|_{L^2(Q_n)}^2 \leq C \|f_n\|_{L^2(Q_n)} \leq C \|f\|_{L^2(Q)}.$$

This ends the proof of Proposition 3.1. □

4. Main results

We are now able to prove the main results of the paper.

4.1. Local in time result.

Theorem 4.1. Assume that the functions of parametrization $\varphi_i, i = 1, 2$ and the coefficients $\beta_i, i = 1, 2$ fulfill conditions (2), (3) and (4). Then, for $T$ small enough, the heat operator $L = \partial_t - \Delta$ is an isomorphism from $H_{1,2}^0(Q)$ into $L^2(Q)$.

Proof. 1) Injectivity of the operator $L$: Let us consider $u \in H_{1,2}^0(Q)$ a solution of the problem (1) with a null right-hand side term. So,

$$\partial_t u - \Delta u = 0 \text{ in } Q.$$

In addition $u$ fulfills the boundary conditions

$$u|_{\partial Q \setminus (\Sigma_1 \cup \Sigma_T)} = 0 \text{ and } \partial_{x_i} u + \beta_i u|_{\Sigma_i} = 0, i = 1, 2.$$

Using Green formula, we have

$$\int_Q (\partial_t u - \Delta u) u \, dt \, dx = \int_{\partial Q} \left( \frac{1}{2} |u|^2 \nu_t - \sum_{k=1}^N \partial_{x_k} u \cdot w \nu_k \right) \, d\sigma + \int_Q \sum_{k=1}^N |\partial_{x_k} u|^2 \, dt \, dx$$

where $\nu_t, \nu_{x_1}, \ldots, \nu_{x_N}$ are the components of the unit outward normal vector at $\partial Q$. We shall rewrite the boundary integral making use of the boundary conditions. On the parts of the boundary of $Q$ where $t = 0, x_k = 0, k = 2, \ldots, N$ and $x_k = b_{k-1}, k = 2, \ldots, N$ we have $u = 0$ and consequently the corresponding boundary integral vanishes. On the part
of the boundary where \( t = T \), we have \( \nu_{x_1} = \nu_{x_2} = \ldots = \nu_{x_N} = 0 \) and \( \nu_l = 1 \). Accordingly the corresponding boundary integral

\[
A = \frac{1}{2} \int_0^{b_{N-1}} \ldots \int_0^{b_1} \int_{\varphi_1(T)}^{\varphi_2(T)} |u|^2 (T, x) \, dx
\]

is nonnegative. On the part of the boundary where \( x_1 = \varphi_i (t), i = 1, 2 \), we have

\[
\nu_l = \frac{(-1)^{i+1} \varphi'_i (t)}{\sqrt{1 + (\varphi'_i (t))^2}}, \quad \nu_{x_1} = \frac{(-1)^i}{\sqrt{1 + (\varphi'_i (t))^2}}, \quad \nu_{x_k} = 0, \quad k = 2, \ldots, N
\]

and

\[
\partial_{x_1} u (t, \varphi_i (t), x') + \beta_i u (t, \varphi_i (t), x') = 0, \quad i = 1, 2.
\]

Consequently the corresponding boundary integral is

\[
\sum_{i=1}^{2} \int_0^{b_{N-1}} \ldots \int_0^{b_1} \int_0^T (-1)^i \left( \beta_i - \frac{\varphi'_i (t)}{2} \right) u^2 (t, \varphi_i (t), x') \, dt \, dx'.
\]

Then, we obtain

\[
\int_Q (\partial_t u - \Delta u) u \, dt \, dx = \sum_{i=1}^{2} \int_0^{b_{N-1}} \ldots \int_0^{b_1} \int_0^T (-1)^i \left( \beta_i - \frac{\varphi'_i (t)}{2} \right) u^2 (t, \varphi_i (t), x') \, dt \, dx' + \frac{1}{2} \int_0^{b_{N-1}} \ldots \int_0^{b_1} \int_{\varphi_2(T)}^{\varphi_1(T)} u^2 (T, x) \, dx + \int_Q \sum_{k=1}^{N} |\partial_{x_k} u|^2 \, dt \, dx.
\]

Consequently \( \int_Q (\partial_t u - \Delta u) u \, dt \, dx = 0 \) yields the equality \( \int_Q \sum_{k=1}^{N} |\partial_{x_k} u|^2 \, dt \, dx = 0 \), because

\[
\sum_{i=1}^{2} \int_0^{b_{N-1}} \ldots \int_0^{b_1} \int_0^T (-1)^i \left( \beta_i - \frac{\varphi'_i (t)}{2} \right) u^2 (t, \varphi_i (t), x') \, dt \, dx' \geq 0
\]

thanks to the hypothesis (4). This implies that \( \sum_{k=1}^{N} |\partial_{x_k} u|^2 = 0 \) and consequently \( \Delta u = 0 \). Then, the hypothesis \( \partial_t u - \Delta u = 0 \) gives \( \partial_t u = 0 \). Thus, \( u \) is constant. The boundary conditions and the fact that \( \beta_i \neq 0, i = 1, 2 \) imply that \( u = 0 \).

\textbf{2) Surjectivity of the operator} \( L \): Choose a sequence \( Q_n, n = 1, 2, \ldots \) of reference domains (see section 2). Then we have \( Q_n \to Q \), as \( n \to \infty \).

Consider the solution \( u_n \in H^{1,2} (Q_n) \) of the Robin problem (5) in \( Q_n \). Such a solution \( u_n \) exists by Theorem 2.1. Let \( \tilde{u}_n \) the 0—extension of \( u_n \) to \( Q \). Then, in virtue of Theorem 3.1, we know that there exists a constant \( C \) such that

\[
\| \tilde{u}_n \|_{L^2 (Q)} + \| \partial_t \tilde{u}_n \|_{L^2 (Q)} + \sum_{1 \leq i_1 + i_2 + \ldots + i_N \leq 2} \| \partial_{x_1}^{i_1} \partial_{x_2}^{i_2} \ldots \partial_{x_N}^{i_N} u_n \|_{L^2 (Q_n)}^2 \leq C \| f \|_{L^2 (Q)}.
\]

This means that \( \tilde{u}_n, \partial_t \tilde{u}_n, \partial_{x_1}^{i_1} \partial_{x_2}^{i_2} \ldots \partial_{x_N}^{i_N} u_n \) for \( 1 \leq i_1 + i_2 + \ldots + i_N \leq 2 \) are bounded functions in \( L^2 (Q) \). So for a suitable increasing sequence of integers \( n_k, k = 1, 2, \ldots \), there exist functions

\[
u, v \text{ and } v_{i_1, i_2, \ldots, i_N} \quad 1 \leq i_1 + i_2 + \ldots + i_N \leq 2
\]

in \( L^2 (Q) \) with \( 1 \leq i_1 + i_2 + \ldots + i_N \leq 2 \) such that

\[
\tilde{u}_n \to u, \quad \partial_t \tilde{u}_n \to v, \quad \partial_{x_1}^{i_1} \partial_{x_2}^{i_2} \ldots \partial_{x_N}^{i_N} u_n \to v_{i_1, i_2, \ldots, i_N},
\]

weakly in \( L^2 (Q) \) as \( k \to \infty \). Clearly,

\[
v = \partial_t u, \quad v_{i_1, i_2, \ldots, i_N} = \partial_{x_1}^{i_1} \partial_{x_2}^{i_2} \ldots \partial_{x_N}^{i_N} u, \quad 1 \leq i_1 + i_2 + \ldots + i_N \leq 2
\]
Proposition 4.1. If $u \in H^{1,2} (Q)$ and $\partial_t u - \Delta u = f$ in $Q$. On the other hand, the solution $u$ satisfies the boundary conditions

$$u|_{\partial Q \setminus (\Sigma_{i} \cup \Sigma_{T})} = 0 \quad \text{and} \quad \partial_{x_{i}} u + \beta_{i} u|_{\Sigma_{i}} = 0, \quad i = 1, 2,$$

since

$$\forall n \in \mathbb{N}^{*}, u|_{Q_{n}} = u_{n}.$$

This proves the existence of solution to Problem (1) and ends the proof of Theorem 4.1. \hfill \Box

4.1.1. Global in time result. In the case where $T$ is not in the neighborhood of zero, we set $Q = D_{1} \cup D_{2} \cup \Sigma_{T_{1}}$ ($T_{1}$ small enough) where

$$D_{1} = \{(t, x) \in Q : 0 < t < T_{1}\}, \quad D_{2} = \{(t, x) \in Q : T_{1} < t < T\},$$

$$\Sigma_{T_{1}} = \{(T_{1}, x_{1}) \in \mathbb{R}^{2} : \varphi_{1}(T_{1}) < x_{1} < \varphi_{2}(T_{1})\} \times \prod_{i=1}^{N-1} [0, b_{i}].$$

In the sequel, $f$ stands for an arbitrary fixed element of $L^{2} (Q)$ and $f_{i} = f|_{D_{i}}, \ i = 1, 2$. Theorem 4.1 applied to the non-regular domain $D_{1}$, shows that there exists a unique solution $v_{1} \in H^{1,2} (D_{1})$ of the problem

$$\begin{aligned}
\partial_{t} v_{1} - \Delta v_{1} &= f_{1} \in L^{2} (D_{1}), \\
\partial_{x_{i}} v_{1} + \beta_{i} v_{1}|_{\Sigma_{i,1}} &= 0, \quad i = 1, 2, \\
v_{1}|_{\partial D_{1} \setminus (\Sigma_{i,1} \cup \Sigma_{T_{1}})} &= 0, \quad i = 1, 2,
\end{aligned} \tag{15}$$

$\Sigma_{i,1}$ are the parts of the boundary of $D_{1}$ where $x_{1} = \varphi_{i}(t), \ i = 1, 2$.

Lemma 4.1. If $u \in H^{1,2} \left( \left[ 0, T \right[ \times \left[ 0, 1 \right[ \times \prod_{i=1}^{N-1} [0, b_{i}] \right) \right)$, then $u|_{t=0} \in H^{1} (\gamma_{0}), \ u|_{x_{1}=0} \in H^{3/2} (\gamma_{1})$ and $u|_{x_{1}=1} \in H^{3/2} (\gamma_{2})$, where $\gamma_{0} = \{0\} \times [0, 1] \times \prod_{i=1}^{N-1} [0, b_{i}], \ \gamma_{1} = [0, T] \times \{0\} \times \prod_{i=1}^{N-1} [0, b_{i}]$, and $\gamma_{2} = [0, T] \times \{1\} \times \prod_{i=1}^{N-1} [0, b_{i}]$. The above lemma is a particular case of [15, Theorem 2.1, Vol.2]. The transformation $(t, x) \mapsto (t, y) = (t, \varphi(t)x_{1} + \varphi_{1}(t), x')$, leads to the following lemma:

Lemma 4.2. If $u \in H^{1,2} (D_{2})$, then $u|_{\Sigma_{T_{1}}} \in H^{1} (\Sigma_{T_{1}}), \ u|_{x_{1}=\varphi_{i}(t)} \in H^{3/2} (\Sigma_{i,2})$, where $\Sigma_{i,2}, \ i = 1, 2$ are the parts of the boundary of $D_{2}$ where $x_{1} = \varphi_{i}(t)$.

Hereafter, we denote the trace $v_{1}|_{\Sigma_{T_{1}}}$ by $\psi$ which is in the Sobolev space $H^{1} (\Sigma_{T_{1}})$ because $v_{1} \in H^{1,2} (D_{1})$ (see Lemma 4.2). Now, consider the following problem in $D_{2}$

$$\begin{aligned}
\partial_{t} v_{2} - \Delta v_{2} &= f_{2} \in L^{2} (Q_{2}), \\
v_{2}|_{\Sigma_{T_{1}}} &= \psi, \\
\partial_{x_{i}} v_{2} + \beta_{i} v_{2}|_{\Sigma_{i,2}} &= 0, \quad i = 1, 2, \\
v_{2}|_{\partial D_{2} \setminus (\Sigma_{i,2} \cup \Sigma_{T_{2}})} &= 0, \quad i = 1, 2,
\end{aligned} \tag{16}$$

$\Sigma_{i,2}$ are the parts of the boundary of $D_{2}$ where $x_{1} = \varphi_{i}(t), \ i = 1, 2$. We use the following result, which is a consequence of [15, Theorem 4.3, Vol.2] to solve Problem (16).

Proposition 4.1. Let $R$ be the cylinder $\left[ 0, T \right[ \times \left[ 0, 1 \right[ \times \prod_{i=1}^{N-1} [0, b_{i}], \ f \in L^{2} (R)$ and $\psi \in H^{1} (\gamma_{0})$. Then, the problem

$$\begin{aligned}
\partial_{t} u - \Delta u &= f \quad \text{in} \ R, \\
u|_{\gamma_{0}} &= \psi, \\
\partial_{x_{i}} u + \beta_{i} u|_{\gamma_{i}} &= 0, \quad i = 1, 2, \\
u|_{\partial R \setminus (\gamma_{0} \cup \gamma_{i})} &= 0, \quad i = 1, 2,
\end{aligned}$$

satisfies the boundary conditions.
where $\gamma_0 = \{0\} \times [0,1[ \times \prod_{i=1}^{N-1} [0,b_i[$, $\gamma_1 = [0,T[ \times \{0\} \times \prod_{i=1}^{N-1} [0,b_i[$ and $\gamma_2 = ]0,T[ \times \{0\} \times \prod_{i=1}^{N-1} [0,b_i[$, admits a (unique) solution $u \in H^{1,2}(R)$.

**Remark 4.1.** In the application of [15, Theorem 4.3, Vol.2], we can observe that there are not compatibility conditions to satisfy because $\partial_{x_1}\psi$ is only in $L^2(\gamma_0)$.

Thanks to the transformation $(t,x) \mapsto (t,y) = (t,\varphi(t)x_1 + \varphi_1(t),x')$, we deduce the following result:

**Proposition 4.2.** Problem (16) admits a (unique) solution $v_2 \in H^{1,2}(D_2)$.

So, the function $u$ defined by

$$u = \begin{cases} v_1 & \text{in } D_1, \\ v_2 & \text{in } D_2, \end{cases}$$

is the (unique) solution of Problem (1) for an arbitrary $T$. Our second main result is

**Theorem 4.2.** Under the assumptions (2), (3) and (4) on the functions of parametrization $\varphi_i$ and the coefficients $\beta_i, i = 1,2$, Problem (1) admits a (unique) solution $u \in H^{1,2}(Q)$.

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**References**
