ANTIMAGIC LABELING OF THE UNION OF SUBDIVIDED STARS

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**Abstract.** Enomoto et al. (1998) defined the concept of a super \((a, 0)\)-edge-antimagic total labeling and proposed the conjecture that every tree is a super \((a, 0)\)-edge-antimagic total labeling. In support of this conjecture, the present paper deals with different results on antimagicness of subdivided stars and their unions.

Keywords: super \((a, d)\)-EAT labeling, star and subdivision of stars.

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1. Introduction

All graphs in this paper are simple, finite and undirected. For a graph \(G\), \(V(G)\) and \(E(G)\) denote the vertex-set and the edge-set. A \((v, e)\)-graph \(G\) is a graph such that \(v = |V(G)|\) and \(e = |E(G)|\).

In this paper, the domain will be the set of all vertices and edges, and such a labeling is called a total labeling. Details on antimagic labeling can be seen in [7]. The subject of edge-magic total labeling of graphs has its origin in the works of Kotzig and Rosa [1, 2] on what they called magic valuations of graphs. The definition of \((a, d)\)-edge-antimagic total labeling was introduced by Simanjuntak, Bertault and Miller in [21] as a natural extension of edge-magic labeling defined by Kotzig and Rosa. Enomoto et al. also proposed the following conjecture:

**Conjecture 1.1** [6] *Every tree admits a super edge-magic total labeling.*

In favour of this conjecture, many authors have considered super edge-magic total labeling for particular classes of trees for example [3, 4, 5, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25]. Lee and Shah [22] verified this conjecture by a computer search for trees with at most 17 vertices. However, this conjecture is still open.

A star is a particular type of tree graph and many authors have proved the magicness for subdivided stars. Lu [24, 25] called the subdivided star \(T(m, n, k)\) as a three path trees and proved that it is super edge-magic if \(n\) and \(k\) are odd, \(k = n+1\) or \(n+2\). Ngurah et al. [5] proved that \(T(m, n, k)\) is also super edge-magic if \(k = n+3\) or \(n+4\). In [3], Salman et al. found the super edge-magic total labeling of a subdivision of a star \(S_n^m\) for \(m = 1, 2\). Javaid et al. [17] furnished super edge-magic total labeling on subdivided star \(K_{1,4}\) and w-trees.

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Definition 1.1 A graph $G$ is called $(a, d)$-edge-antimagic total $((a, d) – EAT)$ if there exist integers $a > 0$, $d \geq 0$ and a bijection

$$\lambda : V(G) \cup E(G) \rightarrow \{1, 2, ..., v + e\}$$

such that $W = \{w(rs) : rs \in E(G)\}$ forms an arithmetic progression starting from $a$ with the difference $d$, where $w(rs) = \lambda(r) + \lambda(s) + \lambda(rs)$ for any $rs \in E(G)$. $W$ is called the set of edge-weights of the graph $G$.

Definition 1.2 A $(a, d)$-edge-antimagic total labeling $\lambda$ is called super $(a, d)$-edge-antimagic total labeling if $\lambda(V(G)) = \{1, 2, ..., v\}$.

Definition 1.3 For $n_i \geq 1$ and $r \geq 3$, let $G \cong T(n_1, n_2, ..., n_r)$ be a graph obtained by inserting $n_i – 1$ vertices to each of the $i$-th edge of the star $K_{1,r}$, where $1 \leq i \leq r$.

Definition 1.4 Two graphs $G_1$ and $G_2$ are said to be isomorphic if their exist a bijective function $\lambda : V(G_1) \rightarrow V(G_2)$ such that for all $x, y \in V(G_1) : xy \in E(G_1)$ if and only if $\lambda(x)\lambda(y) \in E(G_2)$.

2. Main Results

We consider the following proposition which we will use frequently in the main results.

Proposition 2.1. [14] If a $(v, e)$-graph $G$ has a $(s, d)$-EAV labeling then

(i) $G$ has a super $(s + v + 1, d + 1)$-EAT labeling,
(ii) $G$ has a super $(s + v + e, d - 1)$-EAT labeling. \(\square\)

Theorem 2.1. For all $n \geq 1$, $G \cong T(n + 1, n, n + 2, n + 3, n_5, ..., n_p)$ admits super $(a, 0)$-edge-antimagic total labeling with $a = 2v + s - 1$ and super $(a, 2)$-edge-antimagic total labeling with $a = v + s + 1$ where $v = |V(G)|$, $s = 2(n + 3) + \sum_{m=5}^{p} [2^{m-5}(n + 2) + 1]$ and $n_p = 2^{r-4}(n + 2) + 1$.

Proof. We denote the vertices and edges of $G$ as follows:

$$V(G) = \{c \cup \{x_i^1 \mid 1 \leq i \leq r \mid 1 \leq l_i \leq n_i\},$$

$$E(G) = \{c x_i^1 \mid 1 \leq i \leq r\} \cup \{x_i^1 x_i^{l_i+1} \mid 1 \leq i \leq r \mid 1 \leq l_i \leq n_i - 1\}.$$ 

Therefore,

$$v = (4n + 7) + \sum_{m=5}^{p} [2^{m-4}(n + 2) + 1]$$

and

$$e = v - 1.$$ 

We define the labeling $\lambda : V(G) \rightarrow \{1, 2, ..., v\}$ as follows:

$$\lambda(c) = (3n + 5) + \sum_{m=5}^{p} [2^{m-5}(n + 2) + 1].$$

For odd $1 \leq l_i \leq n_i$, where $i = 1, 2, 3, 4$ and $5 \leq i \leq r$, we define

$$\lambda(u) = \begin{cases} 
\frac{l_i+1}{2}, & \text{for } u = x_i^{l_i}, \\
(n + 2) - \frac{l_i+1}{2}, & \text{for } u = x_i^{l_2}, \\
n + 1 + \frac{l_i+1}{2}, & \text{for } u = x_i^{l_3}, \\
(2n + 5) - \frac{l_i+1}{2}, & \text{for } u = x_i^{l_4}. 
\end{cases}$$
For even $1 \leq l_i \leq n_i$, and $\alpha = 2(n + 2) + \sum_{m=5}^{r} [2^{m-5}(n + 2) + 1]$

For $i = 1, 2, 3, 4$ and $5 \leq i \leq r$, we define

$$
\lambda(u) = \begin{cases}
\alpha + \frac{l_i}{2}, & \text{for } u = x_1^1, \\
(\alpha + n + 1) - \frac{l_i}{2}, & \text{for } u = x_2^2, \\
(\alpha + n + 1) + \frac{l_i}{2}, & \text{for } u = x_3^3, \\
(\alpha + 2n + 4) - \frac{l_i}{2}, & \text{for } u = x_4^4.
\end{cases}
$$

and

$$
\lambda(x_i^i) = (\alpha + 2n + 4) + \sum_{m=5}^{i} [2^{m-5}(n + 2)] - \frac{l_i}{2} \text{ respectively.}
$$

The set of all edge-sums generated by the above formula forms a set of consecutive integer sequence $s = \alpha + 2, \alpha + 3, \ldots, \alpha + 1 + e$. Therefore, by Lemma 2.1, $\lambda$ can be extended to a super $(a, 0)$-edge-antimagic total labeling and we obtain the magic constant

$$
a = v + e + s = (10n + 29) + \sum_{m=5}^{r} [2^{m-5}3(n + 2) + 3].$$

Similarly by Lemma 2.2, $\lambda$ can be extended to a super $(a, 2)$-edge-antimagic total labeling and we obtain the magic constant

$$
a = v + 1 + s = (6n + 14) + \sum_{m=5}^{p} [2^{m-5}3(n + 2) + 2].
$$

**Theorem 2.2.** For all $n \geq 1$ and $r \geq 5$, $G \cong T(n + 1, n, n + 2, n + 3, n_5, ..., n_p)$ admits super $(a, 1)$-edge-antimagic total labeling with $a = s + \frac{3v}{2}$ if $v$ is even, where $v = |V(G)|$,

$s = 2(n + 3) + \sum_{m=5}^{p} [2^{m-5}(n + 2) + 1]$ and $n_p = 2^{r-4}(n + 2) + 1$.

**Proof.** Let us consider the vertices and edges of $G$, as defined in Theorem 2.6. Now, we define the labeling $\lambda : V(G) \to \{1, 2, \ldots, v\}$ as in same theorem. It follows that the edge-weights of all edges of $G$ constitute an arithmetic sequence $s = \alpha + 2, \alpha + 3, \ldots, \alpha + 1 + e$ with common difference 1, where $\alpha = 2(n(2) + \sum_{m=5}^{p} [2^{m-5}(n + 2) + 1]$. We denote it by $A = \{a_i; 1 \leq i \leq e\}$. Now for $G$ we complete the edge labeling $\lambda$ for super $(a, 1)$-edge-antimagic total labeling with values in the arithmetic sequence $v + 1, v + 2, \ldots, v + e$ with common difference 1. Let us denote it by $B = \{b_j; 1 \leq j \leq e\}$. Define $C = \{a_{2i-1} + b_{e-i+1}; 1 \leq i \leq \frac{e+1}{2}\} \cup \{a_{2j} + b_{e-j+1}; 1 \leq j \leq \frac{e+1}{2} - 1\}$. It is easy to see that $C$ constitutes an arithmetic progression with $d = 1$ and $a = s + \frac{3(e)}{2} = \frac{1}{2}(16n + 33) + \frac{1}{2} \sum_{m=5}^{p} [2^{m-2}(n + 2) + 5]$

Consequently, $\lambda$ is a super $(a, 1)$-edge-antimagic total labeling.

**Theorem 2.3.** For all positive integers $n$, $G \cong 2T(n + 1, n, n, n + 1, n_5, n_6, ..., n_p)$ admits super $(a, 0)$-edge-antimagic total labeling with $a = 2v + s - 2$ and super $(a, 2)$-edge-antimagic total labeling with $a = v + s + 1$ where $n_i = 2^{i-1}(n + 1)$ for $i = 5, 6, ..., p - 1, n_p = 2^{p-4}(n + 1) - 1$ and $v = |V(G)|$. 

$$
\lambda(x_i^i) = (\alpha + 2n + 4) + \sum_{m=5}^{i} [2^{m-5}(n + 2)] - \frac{l_i}{2} \text{ respectively.}
$$

Proof. We suppose the vertex-set and the edge-set of $G$ as follows: $V(G) = \{c_j \mid 1 \leq j \leq 2\} \cup \{x_{ij}^l \mid 1 \leq i \leq p; 1 \leq l_i \leq n_i; 1 \leq j \leq 2\}$, $E(G) = \{c_jx_{ij}^1 \mid 1 \leq j \leq p; 1 \leq j \leq 2\} \cup \{x_{ij}^l x_{ij}^{l+1} \mid 1 \leq i \leq p; 1 \leq l_i \leq n_i - 1; 1 \leq j \leq 2\}$.

If $v = |V(G)|$ and $e = |E(G)|$ then

$$v = 4n + 2^{p-2}(n + 1)$$

and

$$e = 4n - 2 + 2^{p-2}(n + 1).$$

Now, we define the vertex labeling $\lambda : V(G) \to \{1, 2, ..., v\}$ as follows:

$$\lambda(c_j) = 3(n + 1) + 2^{p-3}(n + 1) + [(n - 1) + 2^{p-4}(n + 1)](j - 1), \ j = 1, 2.$$

For odd $l_i \ 1 \leq l_i \leq n_i$, we define

$$\lambda(u) = \begin{cases} 
\frac{l_i + 1}{2} + [(n + 1) + 2^{p-4}(n + 1)](j - 1), & \text{for } u = x_{1j}^l, \\
\frac{2n+3-l_i}{2} + [(n + 1) + 2^{p-4}(n + 1)](j - 1), & \text{for } u = x_{2j}^l, \\
\frac{(2n+3)+l_i}{2} + [(n + 1) + 2^{p-4}(n + 1)](j - 1), & \text{for } u = x_{3j}^l, \\
\frac{4n+5-l_i}{2} + [(n + 1) + 2^{p-4}(n + 1)](j - 1), & \text{for } u = x_{4j}^l, \\
(n + 1 + 2^{p-4}(n + 1))j, & \text{for } u = x_{pj}^l,\ \text{for } \ l_p = 1, \\
\frac{2n+3+2^{p-3}(n+1)-l_k}{2} + [(n + 1) + 2^{p-4}(n + 1)](j - 1), & \text{for } u = x_{kj}^l,\ \text{for } \ k = 5, 6, ..., p - 1, \\
\frac{2n+3+2^{p-3}(n+1)-l_k}{2} + [3n + 2^{p-4}(3n + 3)](j - 1), & \text{for } u = x_{pj}^l,\ \text{for } \ 4 \leq l_p \leq n_p.
\end{cases}$$
For even \( l_i \), \( 1 \leq l_i \leq n_i \), we define
\[
\lambda(u) = \begin{cases} 
\frac{4(n+1)+2p^2(n+1)+l_1}{2} + [(n - 1) + 2p^{-4}(n + 1)](j - 1), & \text{for } u = x_{i,j}^{l_1}, \\
\frac{6(n+1)+2p^2(n+1)-l_2}{2} + [(n - 1) + 2p^{-4}(n + 1)](j - 1), & \text{for } u = x_{i,j}^{l_2}, \\
\frac{6(n+1)+2p^2(n+1)+l_3}{2} + [(n - 1) + 2p^{-4}(n + 1)](j - 1), & \text{for } u = x_{i,j}^{l_3}, \\
\frac{6(n+1)+2p^2(2^k-3)(n+1)-l_k}{2} + [(n - 1) + 2p^{-4}(n + 1)](j - 1), & \text{for } u = x_{i,j}^{l_k}, \text{ for } k = 4, 5, ..., p - 1, \\
\frac{6(n+1)+2+2p^2-3(n+1)-l_p}{2} - [(n + 2p^{-4}(n + 1)](j - 1), & \text{for } u = x_{i,j}^{l_p}, \text{ for } 2 \leq l_p \leq n_p.
\end{cases}
\]

The set of all edge-sums generated by the above formula forms a set of consecutive integer sequence \( S = \{(2n + 3) + 2p^{-3}(n + 1) + 1, (2n + 3) + 2p^{-3}(n + 1) + 2, ..., (2n + 3) + 2p^{-3}(n + 1) + e\} \), where \( s = \min(S) \). Therefore, by Proposition 2.1, \( \lambda \) can be extended to a super \((a, 0)\)-edge-antimagic total labeling and we obtain the magic constant \( a = 2v + s - 2 = 2(5n + 1) + 5(n + 1)2^{-3} \). Similarly by Proposition 2.1, \( \lambda \) can be extended to a super \((a, 2)\)-edge-antimagic total labeling and we obtain the magic constant \( a = v + 1 + s = 6n + 5 + 3(n + 1)2^{-3} \).

**Theorem 2.4.** For all positive integers \( n \), \( G \cong 2T(n + 1, n, n, (n + 1), n_5, ..., n_p) \) admits super \((a, 1)\)-edge-antimagic total labeling with \( a = v + s + e \) and super \((a, 3)\)-edge-antimagic total labeling with \( a = v + s + 1 \) where \( v = |V(G)| \), \( s = 4 \), \( n_i = 2^{-4}(n + 1) \) for \( i = 5, 6, ..., n_p \) and \( n_p = 2^{i-3}(n + 1) - 1 \).

**Proof.** We suppose the vertex-set and the edge-set of \( G \) as follows: \( V(G) = \{c_j \mid 1 \leq j \leq 2\} \cup \{x_{i,j}^{l_1} \mid 1 \leq i \leq 5 \ ; \ 1 \leq l_i \leq n_i \ ; \ 1 \leq j \leq 2\}, \)
\[
E(G) = \{c_jx_{i,j}^{l_1} \mid 1 \leq i \leq 5 \ ; \ 1 \leq j \leq 2\} \cup \{x_{i,j}^{l_1}x_{i,j}^{l_1+1} \mid 1 \leq i \leq 5 \ ; \ 1 \leq l_i \leq n_i - 1 \ ; \ 1 \leq j \leq 2\}.
\]
If \( v = |V(G)| \) and \( e = |E(G)| \) then
\[
v = 2(2n + 1) + 2p^{-2}(n + 1)
\]
and
\[
e = 4n + 2p^{-2}(n + 1)
\]
Now, we define the vertex labeling \( \lambda : V(G) \to \{1, 2, ..., v\} \) as follows:
\[
\lambda(c_j) = 2(2n + 1) + j, \ j = 1, 2.
\]
For all \( l_i \ 1 \leq l_i \leq n_i \), we define

\[
\lambda(u) = \begin{cases} 
2(l_1 - 1) + j, & \text{for } u = x_{l_1}^{l_1}, \\
2(2n + 1) - 2l_2 + j, & \text{for } u = x_{l_2}^{l_2}, \\
2(2n + 3) - 2l_3 + j, & \text{for } u = x_{l_3}^{l_3}, \\
(10n + 12) + j - 2l_4, & \text{for } u = x_{l_4}^{l_4}, \\
2(4n + 3) + \sum_{m=5}^{i} [2^{m-3}(n+1)] - 2l_i + j, & \text{for } u = x_{l_i}^{l_i}, \quad i \geq 5.
\end{cases}
\]

The set of all edge-sums generated by the above formula forms a set of consecutive integer sequence \( s = \{4, 4+2, \ldots, 4+2(e-1)\} \), where \( s = \min(S) \). Therefore, by Proposition 2.1, \( \lambda \) can be extended to a super \((a,1)\)-edge-antimagic total labeling and we obtain the magic constant \( a = v + e + s = 2(4n + 3) + 2^{p-1}(n + 1) \). Similarly by Proposition 2.1, \( \lambda \) can be extended to a super \((a,3)\)-edge-antimagic total labeling and we obtain the magic constant \( a = v + 1 + s = 4n + 7 + 2^{p-2}(n + 1) \). \( \square \)

3. Conclusion

In this paper, we have shown that a subclass of trees, namely subdivided stars \( G \cong 2T(n + 1, n, n + 1, n_5, n_6, \ldots, n_p) \), admits super \((a,d)\)-edge-antimagic total labeling for \( d = 0, 1, 2, 3 \), for all positive integers \( n \). However the problem of the magicness is still open for different values of magic constant (minimum edge-weight \( a \)).

References


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