ADAPTIVE METHODS FOR SOLVING OPERATOR EQUATIONS BY USING FRAMES OF SUBSPACES

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Abstract. In this paper, using a frame of subspaces we transform an operator equation to an equivalent $\ell_2$-problem. Then, we propose an adaptive algorithm to solve the problem and investigate the optimality and complexity properties of the algorithm.

Keywords: Hilbert space, dual space, frame of subspaces, best $N$-term approximation, adaptive algorithm.

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1. Introduction and preliminaries

The aim of the paper is to study the application of frames of subspaces in designing adaptive iterative methods for solving operator equations. Usually these operators are defined on a bounded domain or a closed manifold where a wavelet basis with specific properties is needed to be constructed. Most importantly, during the approach some serious drawbacks such as stability may not be avoided. Therefore, it is suggested to use a slightly weaker concept, namely frame. In [7, 8, 5, 6] some adaptive numerical methods for elliptic operator equations have been developed by using wavelet bases and frames. One of the advantages of frame of subspaces is that they facilitate the construction of frames for special applications and meanwhile it is easier to construct or choose already known frames for smaller spaces.

The main focus of the paper is to find $u \in H$ such that

$$Lu = f,$$

where $L : H \to H$ is a bounded, invertible and self adjoint linear operator on a separable Hilbert space $H$. In general, it is impossible to find the exact solution of the problem (1), because the separable Hilbert space $H$ is infinite dimensional. A natural approach to construct an approximate solution is to solve a finite dimensional counterpart of the problem (1). First, we briefly recall the definitions and basic properties of frames and frames of subspaces.

Assume that $H$ is a separable Hilbert space, $\Lambda$ is a countable set of indices and $\Psi = (\psi_\lambda)_{\lambda \in \Lambda} \subset H$ is a frame for $H$. This means that there exist constants $0 < A_\Psi \leq B_\Psi < \infty$ such that

$$A_\Psi \|f\|_H^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \psi_\lambda \rangle|^2 \leq B_\Psi \|f\|_H^2, \quad \forall f \in H.$$ (2)
For the frame $\Psi$, the frame operator $S : H \to H$ is defined by $S(f) = \sum_{\lambda \in \Lambda} \langle f, \psi_\lambda \rangle \psi_\lambda$. It was shown that $S$ is a positive definite and invertible operator satisfying $A_\Psi I_H \leq S \leq B_\Psi I_H$. Also, the sequence $\tilde{\Psi} = (\tilde{\psi}_\lambda)_{\lambda \in \Lambda} = (S^{-1} \psi_\lambda)_{\lambda \in \Lambda}$ is a frame (called the canonical dual frame) for $H$ with bounds $B_\Psi^{-1}, A_\Psi^{-1}$ and every $f \in H$ has the expansion
\[
f = \sum_{\lambda \in \Lambda} \langle f, \psi_\lambda \rangle \tilde{\psi}_\lambda = \sum_{\lambda \in \Lambda} \langle f, \tilde{\psi}_\lambda \rangle \psi_\lambda. \tag{3}\]

For an index set $\tilde{\Lambda} \subset \Lambda$, $(\tilde{\psi}_\lambda)_{\lambda \in \tilde{\Lambda}}$ is called a frame sequence, if it is a frame for its closed span. For more details see [1, 3].

For an index set $\Lambda$ and a family of weights $\{v_\lambda\}_{\lambda \in \Lambda}$, i.e., $v_\lambda > 0$ for all $\lambda \in \Lambda$, a family of subspaces $\{H_\lambda\}_{\lambda \in \Lambda}$ of a Hilbert space $H$ is called a frame of subspaces with respect to $\{v_\lambda\}_{\lambda \in \Lambda}$ for $H$, if there exist constants $0 < A \leq B < \infty$ such that
\[A \|f\|^2 \leq \sum_{\lambda \in \Lambda} v_\lambda^2 \|\pi_{H_\lambda}(f)\|^2 \leq B \|f\|^2 \quad \forall f \in H, \tag{4}\]
where $\pi_{H_\lambda}$ denotes the orthogonal projection onto the subspace $H_\lambda$.

The constants $A$ and $B$ is called the frame bounds of the frame of subspaces. If $A = B$ then the frame $\{H_\lambda\}_{\lambda \in \Lambda}$ with respect to $\{v_\lambda\}_{\lambda \in \Lambda}$, is called a $A$-tight frame of subspaces. It is clear, the family $\{H_\lambda\}_{\lambda \in \Lambda}$ of the frame of subspaces is complete, in the sense that $\overline{\text{span}}_{\lambda \in \Lambda}\{H_\lambda\} = H$.

The following theorem [2], shows how we are able to string together frames for each of the subspaces $H_\lambda$ to get a frame for $H$.

**Theorem 1.1.** Let $\Lambda$ be an index set, $v_\lambda > 0$ for each $\lambda \in \Lambda$, and $\{\psi_\lambda\}_{\lambda \in \Lambda}$ be a frame sequence in $H$ with frame bounds $A_\Lambda$ and $B_\Lambda$. Define $H_\lambda = \overline{\text{span}}_{\lambda \in \Lambda}\{\psi_\lambda\}$ for all $\lambda \in \Lambda$, and suppose that $0 < A = \inf_{\lambda \in \Lambda} A_\lambda \leq B = \sup_{\lambda \in \Lambda} B_\lambda < \infty$. Then $\{v_\lambda \psi_\lambda\}_{\lambda \in \Lambda, \lambda \in \Lambda}$ is a frame for $H$ if and only if $\{H_\lambda\}_{\lambda \in \Lambda}$ is a frame of subspaces with respect to $\{v_\lambda\}_{\lambda \in \Lambda}$ for $H$.

For a frame of subspaces $\{H_\lambda\}_{\lambda \in \Lambda}$ with respect to $\{v_\lambda\}_{\lambda \in \Lambda}$ define
\[
\left(\sum_{\lambda \in \Lambda} \oplus H_\lambda\right)_{\ell_2} = \left\{\{\psi_\lambda\}_{\lambda \in \Lambda} | \psi_\lambda \in H_\lambda, \sum_{\lambda \in \Lambda} \|\psi_\lambda\|^2 < \infty\right\}
\]
with inner product given by $\langle\{\psi_\lambda\}_{\lambda \in \Lambda}, \{\varphi_\lambda\}_{\lambda \in \Lambda}\rangle = \sum_{\lambda \in \Lambda} \langle\psi_\lambda, \varphi_\lambda\rangle$. Now the synthesis operator $T_{H,v} : (\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{\ell_2} \to H$ for the frame of subspace $\{H_\lambda\}_{\lambda \in \Lambda}$ with respect to $\{v_\lambda\}_{\lambda \in \Lambda}$ is defined by
\[
T_{H,v}(f) = \sum_{\lambda \in \Lambda} v_\lambda f_\lambda \quad \forall f = \{f_\lambda\}_{\lambda \in \Lambda} \in (\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{\ell_2}.
\]

Also, the adjoint $T_{H,v}^* \colon H \to (\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{\ell_2}$ is given by $T_{H,v}^*(f) = \{v_\lambda \pi_{H_\lambda}(f)\}_{\lambda \in \Lambda}$. It is proved that the synthesis operator $T_{H,v}$ is bounded, linear and onto [2]. Also, the analysis operator $T_{H,v}^*$ is an (possibly into) isomorphism. As in the well known frame situation, the frame operator $S_{H,v}$ for $\{H_\lambda\}_{\lambda \in \Lambda}$ and $\{v_\lambda\}_{\lambda \in \Lambda}$ is defined by
\[
S_{H,v}(f) = T_{H,v} T_{H,v}^*(f) = T_{H,v}(\{v_\lambda \pi_{H_\lambda}(f)\}_{\lambda \in \Lambda}) = \sum_{\lambda \in \Lambda} v_\lambda^2 \pi_{H_\lambda}(f).
\]

The frame operator $S_{H,v}$ for $\{H_\lambda\}_{\lambda \in \Lambda}$ and $\{v_\lambda\}_{\lambda \in \Lambda}$ is self-adjoint, invertible on $H$ with $AI \leq S_{H,v} \leq BI$, where $A$ and $B$ are the bounds of the frame of subspaces. Furthermore,
the following reconstruction formula satisfies:

\[ f = \sum_{\lambda \in \Lambda} v_\lambda^2 S_{H,v}^{-1} \pi H_{\lambda}(f) \quad \forall f \in H. \]

It is proved that \( \{S_{H,v}^{-1} H_{\lambda}\}_{\lambda \in \Lambda} \) is a frame of subspaces with respect to \( \{v_\lambda\}_{\lambda \in \Lambda} \). [2].

**Proposition 1.1.** Let \( \{H_\lambda\}_{\lambda \in \Lambda} \) be a frame of subspaces with respect to \( \{v_\lambda\}_{\lambda \in \Lambda} \), and let \( L : H \to H \) be a bounded invertible operator on \( H \). Then \( \{L(H_\lambda)\}_{\lambda \in \Lambda} \) is a frame of subspaces with respect to \( \{v_\lambda\}_{\lambda \in \Lambda} \).

**Proof.** See [2]. \( \square \)

### 2. Preconditioning by using frames of subspaces

The most straightforward approach to an iterative solution of a linear system is to rewrite the equation (1) as a linear fixed-point iteration. One way to do this is the Richardson iteration. The abstract method reads as follows:

write \( Lu = f \) as \( u = (I - L)u + f \). For given \( u_0 \in H \), define for \( n \geq 0 \),

\[ u_{n+1} = (I - L)u_n + f. \tag{5} \]

Since \( Lu - f = 0 \),

\[
\begin{align*}
  u_{n+1} - u &= (I - L)u_n + f - u - (f - Lu) \\
  &= (I - L)(u_n - u) \\
  &= (I - L)(u_n - u).
\end{align*}
\]

Hence \( \|u_{n+1} - u\|_H \leq \|I - L\|_{H \to H}\|u_n - u\|_H \), so that (5) converges if \( \|I - L\|_{H \to H} < 1 \). It is sometimes possible to precondition (1) by multiplying both sides by a matrix \( B \),

\[ BLu = Bf, \]

so that convergence of iterative methods is improved. This is a very effective technique for solving differential equations, integral equations, and related problems [2, 3]. We shall apply this technique by using frames of subspaces.

Let \( \{H_\lambda\}_{\lambda \in \Lambda} \) be a frame of subspaces with respect to \( \{v_\lambda\}_{\lambda \in \Lambda} \) for a separable Hilbert space \( H \) with the frame operator \( S_{H,v} \). By Proposition 1.1, \( \{L(H_\lambda)\}_{\lambda \in \Lambda} \) also is a frame with respect to \( \{v_\lambda\}_{\lambda \in \Lambda} \). We denote the frame operator for \( \{L(H_\lambda)\}_{\lambda \in \Lambda} \) and \( \{v_\lambda\}_{\lambda \in \Lambda} \), by \( S'_{H,v} \) and we note that \( S'_{H,v} f = \sum_{\lambda \in \Lambda} v_\lambda^2 L \pi H_{\lambda} L^{-1} f = L \sum_{\lambda \in \Lambda} v_\lambda^2 \pi H_{\lambda} L^{-1} f = L S_{H,v} L^{-1} f \), that means, \( S'_{H,v} = L S_{H,v} L^{-1} \).

Also since \( L \) is bounded invertible then there exist two positive constants \( c_1 \) and \( c_2 \) such that

\[ c_1\|u\|_H \leq \|Lu\|_H \leq c_2\|u\|_H, \quad \forall u \in H. \tag{6} \]

Now we design an algorithm in order to approximate the solution \( u \) of the equation (1). The convergence rate of the algorithm depends on the values of the bounds of the frames.

**Theorem 2.1.** Let \( \{H_\lambda\}_{\lambda \in \Lambda} \) be a frame of subspaces with respect to \( \{v_\lambda\}_{\lambda \in \Lambda} \) for \( H \) with frame operator \( S_{H,v} \), and let \( L \) be as in (1). Let \( u_0 = 0 \) and for \( k \geq 1 \),

\[ u_k = u_{k-1} + \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v}(f - Lu_{k-1}), \]

where \( S'_{H,v} \) is the frame operator for the frame of subspaces \( \{L(H_\lambda)\}_{\lambda \in \Lambda} \) with respect to \( \{v_\lambda\}_{\lambda \in \Lambda} \) with bounds \( A, B \), and \( c_1, c_2 \) as in (6). Then

\[ \|u - u_k\|_H \leq \left( \frac{2B - c_1^2 A}{c_1^2 A + c_2^2 B} \right)^k \|u\|_H. \]
In particular the vectors $u_k$ converges to $u$ as $k \to \infty$.

Proof. By definition of $u_k$ we obtain

$$u - u_k = u - u_{k-1} + \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v}(f - L u_{k-1})$$

$$= (I - \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v} L) (u - u_{k-1})$$

$$= (I - \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v} L)^2 (u - u_{k-2})$$

$$= \cdots = (I - \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v} L)^k (u - u_0),$$

therefore

$$\|u - u_k\|_H \leq \|I - \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v} L\|^k \|u\|_H. \quad (7)$$

But for every $v \in H$ we have

$$\langle (I - \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v} L), v \rangle = \|v\|_H^2 - \frac{2}{c_1^2 A + c_2^2 B} \langle S'_{H,v} L, v \rangle$$

$$= \|v\|_H^2 - \frac{2}{c_1^2 A + c_2^2 B} \left( \sum_{\lambda \in A} \|c_2^2 \pi_{L,H}(Lv)\|_H^2 \right)$$

$$\leq \|v\|_H^2 - \frac{2A}{c_1^2 A + c_2^2 B} \|Lv\|_H^2$$

$$\leq \|v\|_H^2 - \frac{2A}{c_1^2 A + c_2^2 B} \|v\|_H^2$$

$$= \left( \frac{c_2^2 B - c_1^2 A}{c_1^2 A + c_2^2 B} \right) \|v\|_H^2,$$

where in the first inequality we used the property of the lower bound of the frame of subspaces and in the second inequality we used the property of $c_1$ in (6). Similarly we have

$$-\left( \frac{c_2^2 B - c_1^2 A}{c_1^2 A + c_2^2 B} \right) \|v\|_H^2 \leq \langle (I - \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v} L), v \rangle,$$

and so we conclude that

$$\|I - \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v} L\| \leq \frac{c_2^2 B - c_1^2 A}{c_1^2 A + c_2^2 B}. \quad (8)$$

Combining this inequality with (7) gives the result. \hfill \square

Now, let $u$ be the solution of the equation (1) and $T_{H,v}$ be the synthesis operator of the frame of subspaces $\{H_\lambda\}_{\lambda \in A}$ for $H$. Since $T_{H,v}$ is onto then there exists $U \in \sum_{\lambda \in A} \oplus H_\lambda$ such that $u = T_{H,v} U$, so the equation (1) is equivalent to $L T_{H,v} U = f$, or $T_{H,v}^* LT_{H,v} U = T_{H,v}^* f$, where $T_{H,v}$ is the analysis operator of the frame of subspaces. Therefore finding the solution $u$ of the equation (1) is equivalent to finding the solution $U$ of the equation

$$MU = F,$$  \hspace{1cm} (9)

where $M := T_{H,v}^* LT_{H,v}$ and $F := T_{H,v}^* f$.

Note that we can consider the equation (9) as a matrix equation from $(\sum_{\lambda \in A} \oplus H_\lambda)_{\ell_2}$ to
We note that \((\sum_{\lambda \in \Lambda} + H_{\lambda})_{\ell_2} = \text{Ran}T^*_{H,v'} \oplus \text{Ker}T_{H,v},\) and the following lemma holds.

**Lemma 2.1.** The orthogonal projector onto \(\text{Ran}T^*_{H,v}\) is \(Q = T^*_{H,v}S^{-1}_{H,v}T_{H,v}^*\).

**Proof.** For \(x \in T^*_{H,v}f\),

\[
Qx = Q(\{v_{\lambda}\pi_{H_{\lambda}} \} \lambda \in \Lambda) = T^*_{H,v}S^{-1}_{H,v}T_{H,v}(\{v_{\lambda}\pi_{H_{\lambda}} \} \lambda \in \Lambda) = T^*_{H,v}(\sum_{\lambda \in \Lambda} v_{\lambda}^2\pi_{H_{\lambda}} f) = \{v_{\lambda}\pi_{H_{\lambda}}(\sum_{\lambda \in \Lambda} S^{-1}_{H,v}v_{\lambda}^2\pi_{H_{\lambda}} f)\} \lambda \in \Lambda = \{v_{\lambda}\pi_{H_{\lambda}}\} \lambda \in \Lambda.
\]

That is \(Q = id\) on \(\text{Ran}T^*_{H,v}\) and \(Q = 0\) on \(\text{Ker}T_{H,v}\). \(\square\)

Therefore, \(M|_{\text{Ran}T^*} : \text{Ran}T^*_{H,v} \to \text{Ran}T^*_{H,v}\) is boundedly invertible and we have \(\|M\| \leq B\|L\|\) and \(\|M^{-1}|_{\text{Ran}T^*_{H,v}}\| \leq A^{-1}\|L^{-1}\|\).

### 3. An adaptive algorithm based on a frame of subspaces

In this section, we construct an adaptive algorithm in order to give an approximate solution to the exact solution \(U\) of the equation (9). In order to analyze adaptive methods, we compare them with the best \(N\)-term approximation. The aim is to balance between the accuracy and computational complexity at the same time.

For \(N \in \mathbb{N}\), define

\[
\sum_{N} := \{V \in (\sum_{\lambda \in \Lambda} + H_{\lambda})_{\ell_2} : \#\text{supp } V \leq N\},
\]

and the corresponding error

\[
\rho_{N}(V) := \inf_{V_N \in \sum_{N}} \|V - V_N\|_{(\sum_{\lambda \in \Lambda} + H_{\lambda})_{\ell_2}}, \quad V \in (\sum_{\lambda \in \Lambda} + H_{\lambda})_{\ell_2},
\]

where \#\text{supp } \(V\) is the number of nonzero entries of \(V\).

A best approximation to \(V\) from \(\sum_{N}\) (called the best \(N\)-term approximation to \(V\)) is obtained by taking a set \(\Lambda_{N} \subset \Lambda\) with \#\(\Lambda_{N} \leq N\) on which \(v_{\lambda}\) takes its \(N\) largest values. Note that the set \(\Lambda_{N}\) is not unique.

Given a sequence \(V = (v_{\lambda})_{\lambda} \in (\sum_{\lambda \in \Lambda} + H_{\lambda})_{\ell_2}\), for each \(n \geq 1\) let \(v_{n}\) be the \(n\)-th largest of the values \(\|v_{\lambda}\|\) and define the decreasing rearrangement \(V^*\) of \(V\) by \(V^* := (v_{n}^\infty)_{n=1}^\infty\). For each \(0 < \tau < 2\) we let \(\ell_{\tau}^*(\Lambda)\) denote the collection of all vectors \(V \in (\sum_{\lambda \in \Lambda} + H_{\lambda})_{\ell_2}\) for which \(|V|_{\ell_{\tau}^*(\Lambda)} := \sup_{a \geq 1} n^{\frac{\tau}{2}} v_{n}^*\) is finite. This expression defines a quasi norm for \(\ell_{\tau}^*(\Lambda)\). A corresponding norm is defined by \(\|V\|_{\ell_{\tau}^*(\Lambda)} := |V|_{\ell_{\tau}^*(\Lambda)} + \|V\|_{(\sum_{\lambda \in \Lambda} + H_{\lambda})_{\ell_2}}\). Also there exists a constant \(C_{\tau}\) such that

\[
|V + W|_{\ell_{\tau}^*(\Lambda)} \leq (|V|_{\ell_{\tau}^*(\Lambda)} + |W|_{\ell_{\tau}^*(\Lambda)}), \quad (10)
\]
where $a \leq b$ means that, there is a constant $c$ such that $a \leq cb$. Now let $V_N$ be the best $N$-term approximation of $V$ such that $\|V - V_N\|_{(\sum_{\lambda \in A} \oplus H_\lambda)} \leq \epsilon$. If for some $s > 0$

$$\|V - V_N\|_{(\sum_{\lambda \in A} \oplus H_\lambda)} \leq N^{-s},$$

(11)

then $N \leq \epsilon^\frac{-1}{s}$. For $\tau = (\frac{1}{2} + s)^{-1}$, (11) means that $V \in \ell^w_\tau(\Lambda)$ and for $0 < \tau < 2$,

$$\sup_{N \in \mathbb{N}} \{N^s \|V - V_N\|_{(\sum_{\lambda \in A} \oplus H_\lambda)} \} \simeq \|V\|_{\ell^w_\tau(\Lambda)}.$$  

(12)

One can see [4, 9] for further details on the quasi-Banach spaces $\ell^w_\tau(\Lambda)$.

**Proposition 3.1.** Let $s > 0$ and $\tau = (s + \frac{1}{2})^{-1}$. If $V \in \ell^w_\tau(\Lambda)$, then

$$\rho_N(V) \simeq N^{-s} \|V\|_{\ell^w_\tau(\Lambda)},$$

(13)

with a constant only depending on $\tau$ for $\tau \searrow 0$.

**Proof.** See [4].

**Assumption.** We assume that the matrix $M$ is $s^*$-compressible, in the sense that for $0 < s < s^*$ there exists a sequence $\alpha = (\alpha_j)_j \in \ell_1(\Lambda)$, and a matrix $M_j$ having at most $\alpha_j 2^j$ nonzero entries per row and column, and a positive constant $C_M$ such that

$$\|M - M_j\| \leq C_M \alpha_j 2^{-js},$$

(14)

for all $j \in \mathbb{N}$. ($\|M - M_j\|$ is the spectral norm of $(M - M_j)$.) Such a matrix maps $\ell^w_\tau(\Lambda)$ boundedly into itself for $\tau = (\frac{1}{2} + s)^{-1}$. [4].

**Remark 3.1.** If $M$ is $s^*$ compressible then $M$ maps $\ell^w_\tau(\Lambda)$ boundedly into itself for every $\tau = (\frac{1}{2} + s)^{-1}$. [4].

For a finite support vector $V$, $N := (\#supp(V)) < \infty$, we denote the best $2^j$-term approximation to $V$ in $(\sum_{\lambda \in A} \oplus H_\lambda)_2$ by $V[j]$, for $j = 1, 2, \ldots, \lfloor \log N \rfloor$, and let $V[j] = V$, for $j > \log N$. For a given $K \in \mathbb{N}$ define

$$W_K := M_K V[0] + \sum_{j=0}^{K-1} M_j (V[K-j] - V[K-j-1]),$$

where $M_j$ is as (14). In this case

$$MV - W_K = MV - M_K V[0] - \sum_{j=0}^{K-1} M_j (V[K-j] - V[K-j-1])$$

$$= MV - M_K V[0] - M_{K-1} (V[1] - V[0]) - \ldots - M_0 (V[K] - V[K-1])$$

$$= M(V - V[K]) + (M - M_0)(V[K] - V[K-1]) + \ldots + (M - M_K)V[0],$$

and since $M$ is $s^*$-compressible then

$$\|MV - W_K\|_{(\sum_{\lambda \in A} \oplus H_\lambda)_2} \leq$$

$$\|M\| \|V - V[K]\|_{(\sum_{\lambda \in A} \oplus H_\lambda)_2} + \|M - M_0\| \|V[K] - V[K-1]\|_{(\sum_{\lambda \in A} \oplus H_\lambda)_2}$$

$$+ \ldots + \|M - M_K\| \|V[0]\|_{(\sum_{\lambda \in A} \oplus H_\lambda)_2} \leq$$

$$+ C_M (\alpha_0 \|V[K] - V[K-1]\|_{(\sum_{\lambda \in A} \oplus H_\lambda)_2} + \ldots + \alpha_K 2^{-J_s} \|V[0]\|_{(\sum_{\lambda \in A} \oplus H_\lambda)_2}).$$

In the other words there exists a constant $C_3$ such that

$$\|MV - W_K\|_{(\sum_{\lambda \in A} \oplus H_\lambda)_2} \leq$$

$$C_3 (\|V - V[K]\|_{(\sum_{\lambda \in A} \oplus H_\lambda)_2} +$$

(15)
The inequality (17) comes from the inequality (16) and the definition of \( \text{Lemma 3.1.} \)

\begin{align*}
\alpha_0 \| V[K] - V[K-1] \| ((\sum_{\lambda \in \Lambda} \oplus H_\lambda) \ell_2) + \ldots + \alpha_K 2^{-K_s} \| V[0] \| ((\sum_{\lambda \in \Lambda} \oplus H_\lambda) \ell_2). \end{align*}

Now by proposition 3.1 and the definition of \( V[j] \) there exists a constant \( C_4 \) such that

\begin{align*}
\| V - V[j] \| ((\sum_{\lambda \in \Lambda} \oplus H_\lambda) \ell_2) = \rho_{2j}(V) \leq C_4 2^{-j} \| V \| _{\ell^2(\Lambda)},
\end{align*}

de \( j = 1, \ldots, \lfloor \log N \rfloor \).

Applying the above inequalities and the fact that \( \| V[0] \| ((\sum_{\lambda \in \Lambda} \oplus H_\lambda) \ell_2) \leq \| V[0] \| _{\ell^2(\Lambda)} \), the inequality (15) induces a constant \( C \) such that

\begin{align*}
\| MV - W_K \| ((\sum_{\lambda \in \Lambda} \oplus H_\lambda) \ell_2) \leq C 2^{-K_s} \| V \| _{\ell^2(\Lambda)}. \tag{16}
\end{align*}

Now we are ready to design our algorithm. First, following [4], we introduce the following routine.

\textbf{APPLY} \([M, V, \epsilon]: \)

\begin{enumerate}
\item Compute \( V[0], \ V[j] - V[j-1], \ j = 1, \ldots, \lfloor \log N \rfloor \) and define \( V[j] := V \) for \( j > \log N \).
\item Compute \( K \) as the smallest integer such that \( 2^K \geq C_4^2 \epsilon^{-\frac{1}{2}} \| V \| _{\ell^2(\Lambda)} \).
\item For \( k = 1 \) to \( K \) compute

\begin{align*}
R_k := \| M \| \| V - V[k] \| ((\sum_{\lambda \in \Lambda} \oplus H_\lambda) \ell_2) + C_A (\alpha_0 \| V[k] - V[k-1] \| ((\sum_{\lambda \in \Lambda} \oplus H_\lambda) \ell_2) + \ldots + \alpha_k 2^{-k_s} \| V[0] \| ((\sum_{\lambda \in \Lambda} \oplus H_\lambda) \ell_2). \end{align*}

\item If \( R_k \leq \epsilon \) then exit.
\item \( W_K := M_K V[0] + \sum_{j=0}^{K-1} M_j (V[k-j] - V[k-j-1]) \).
\end{enumerate}

\textbf{Remark 3.2.} Since \( K \) is the smallest integer such that \( 2^K \geq C_4^2 \epsilon^{-\frac{1}{2}} \| V \| _{\ell^2(\Lambda)} \), then

\begin{align*}
2^{K-1} < C_4^2 \epsilon^{-\frac{1}{2}} \| V \| _{\ell^2(\Lambda)} \cdot
\end{align*}

Thus \( 2^K < 2C_4^2 \epsilon^{-\frac{1}{2}} \| V \| _{\ell^2(\Lambda)} \), which \( 2^K \leq \epsilon^{-\frac{1}{2}} \| V \| _{\ell^2(\Lambda)} \).

\textbf{Lemma 3.1.} Let \( V \in \ell^2(\Lambda) \) with \( \tau = (s + \frac{1}{2})^{-1} \). For a given accuracy \( \epsilon > 0 \), the output \( W_K \) of \textbf{APPLY} \([M, V, \epsilon]\) satisfies

\begin{align*}
\| MV - W_K \| ((\sum_{\lambda \in \Lambda} \oplus H_\lambda) \ell_2) \leq \epsilon, \tag{17}
\end{align*}

and

\begin{align*}
\# \text{supp}(W_K) \leq \epsilon^{-\frac{1}{2}} \| V \| _{\ell^2(\Lambda)}^{\frac{1}{2}}. \tag{18}
\end{align*}

Also the number of arithmetic operations to compute \( W_K \) is at most a multiple of \( \epsilon^{-\frac{1}{2}} \| V \| _{\ell^2(\Lambda)}^{\frac{1}{2}} \).

\textbf{Proof.} The inequality (17) comes from the inequality (16) and the definition of \( K \) in \textbf{APPLY}. Recalling the number of nonzero entries in \( V[j] \), and definition of \( M_j \) we conclude

\begin{align*}
\# \text{supp}(W_K) & \leq \# \text{rows}(M_K) + \# \text{rows}(M_K-1) + \ldots + \# \text{rows}(M_0) \\
& \leq \alpha_K 2^K + \alpha_{K-1} 2^{K-1} + \ldots + \alpha_0 \leq (|\alpha_K| + |\alpha_{K-1}| + \ldots + |\alpha_0|) 2^K \leq 2^K,
\end{align*}

where the last inequality is induced by \( (\alpha_j) \in \ell_1(\Lambda) \). Now by using remark 3.2, we obtain the inequality (18). Also if \( N_K \) denotes the number of arithmetic operation needed to compute \( W_K \) we have

\begin{align*}
N_K \leq \# \text{rows}(M_K) \# \text{supp}(V[0]) + \# \text{rows}(M_K-1) \# \text{supp}(V[1] - V[0]) \\
+ \ldots + \# \text{rows}(M_0) \# \text{supp}(V[K] - V[K-1])
\end{align*}
\[ \leq \alpha K 2^K + \alpha K^{-1} 2^{K-1} + \ldots + \alpha_0 2^{K-1} \leq 2^K \leq \epsilon^{-2} \| V \|^2_{\ell^2(\Lambda)}. \]

Also, as in [4], for an accuracy \( \epsilon > 0 \) and a finitely supported vector \( W \) with \( N = \#(\text{support}(W)) \), we introduce the following basic numerical ingredient that we will use in our algorithm.

**COARSE** \( [W, \epsilon] \to (\Lambda, \bar{W}) \)

(i) Sort the nonzero entries of \( W \) into decreasing order in modulus and obtain the vector \( \lambda^* := (\lambda_1, ..., \lambda_N) \) of indices which gives the decreasing rearrangement \( W^* = (\|W_{\lambda_1}\|, ..., \|W_{\lambda_N}\|) \) of nonzero entries of \( W \); then compute \( \|W\|^2_{(\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{\ell^2}} = \sum_{i=1}^N \|W_{\lambda_i}\|^2_{H_i} \).

(ii) Find the smallest \( K \in \mathbb{N} \) such that \( \sum_{i=1}^K \|W_{\lambda_i}\|^2 \) exceeds \( \|W\|^2_{(\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{\ell^2}} - \epsilon^2 \). For this \( K \) define \( \Lambda := \{\lambda_i : i = 1, ..., K\} \) and \( \bar{W} \) by \( \bar{W}_{\lambda} = W_{\lambda} \) for \( \lambda \in \Lambda \) and \( \bar{W}_{\lambda} = 0 \) for \( \lambda \notin \Lambda \).

Now, let \( 0 < \epsilon < \|V\| \|W\| \|\sum_{\lambda \in \Lambda} \oplus H_\lambda\|_{\ell^2} \) and \( W \) be a finitely supported approximation to \( V \) such that \( \|V - W\|_{(\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{\ell^2}} \leq d\epsilon \) for some \( d < 1 \), then it is obvious that the COARSE \( [W, (1-d)\epsilon] \) produces \( \bar{W} \) supported on \( \Lambda \) where \( \|V - \bar{W}\|_{(\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{\ell^2}} \leq \epsilon \). (Note that the output \( \bar{W} \) of COARSE, by construction, satisfies \( \|V - \bar{W}\|_{(\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{\ell^2}} \leq \epsilon \)). Moreover, we have the following lemma [4].

**Lemma 3.2.** If \( V \in \ell^2_{r'}(\Lambda), \tau = (s + \frac{1}{2})^{-1}, \) for some \( s > 0 \) then the outputs \( \bar{W}, \Lambda \) of COARSE \( [W, (1-d)\epsilon] \) requires at most \( 2N \) arithmetic operations and \( N \log N \) sorts, where \( N = \#\text{supp}(W) \). Moreover,

\[ |\bar{W}|_{\ell^2(\Lambda)} \leq |V|_{\ell^2(\Lambda)}, \quad (19) \]

and \( \#\Lambda \) (the cardinality of \( \text{supp}(\bar{W}) \)) satisfies

\[ \#(\Lambda) \leq |V|_{\ell^2(\Lambda)}^{s/2}. \quad (20) \]

Also for \( F = T^+_{H,\delta,f} \) we assume that the routine:

**RHS** \( [\epsilon, F] \to F_\epsilon \)

determines a finitely supported vector \( F_\epsilon \in (\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{\ell^2} \) satisfying

\[ \|F - F_\epsilon\|_{(\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{\ell^2}} \leq \epsilon. \]

Assuming \( Q \) is bounded on \( \ell^2_{r'}(\Lambda) \) for \( r' = (\frac{1}{2} + s')^{-1}, 0 < s' < s \) (hence \( Q \) is bounded on \( \ell^2_{r'} \)); [9]) we construct our algorithm for the target accuracy \( \epsilon > 0 \). At first, for some \( 0 < d < \frac{1}{3} \) and \( \rho := \|I - \alpha M\| < 1 \) (since \( M \) is a positive definite matrix this real number \( \alpha \) exists) set \( K := \min \{k \in \mathbb{N} : 3 \rho^k < d \min\{1, |C_1 C_2| I - Q|_{\ell^2_{r'} 2} \}^{2s' - 1} \} \), where \( C_1, C_2 \) are two constants induced from (10) and (19) for \( r' \).

**SOLVE** \( [\epsilon, M, F] \to (U, \Lambda_\epsilon) \)

(i) Set \( \lambda_0 = 0, \Lambda_0 = 0, \epsilon_0 := \|M\|_{\text{ran}(T^+)}^{-1} \|F\|_{(\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{\ell^2}} \).

(ii) If \( \epsilon_i \leq \epsilon \) stop and set \( U_\epsilon := U_i \), otherwise

(ii.1) \( i := i + 1, \quad \epsilon_i := 3 \rho^{K} \epsilon_{i-1} d. \)

(ii.2) \( F^i := \text{RHS} \ [F, \frac{d\epsilon_i}{\rho^K}]. \)

(ii.3) \( V^{(i,0)} := U_{i-1}. \)
(ii.4) For \( j = 1, \ldots, K \) compute

1. \( W_j^{j-1} := \text{APPLY } [M, V^{(i,j-1)}, \frac{d\epsilon_i}{6\alpha}] \).
2. \( V^{(i,j)} := V^{(i,j-1)} + \alpha(F^i - W_j^{j-1}). \)

(iii) \( U^i := \text{COARSE } [V^{(i,K)}, (1 - d)\epsilon] \) and go to (ii).

Remark 3.3. By definition of \( V^{(i,j)} \), \( F^i \) in \textit{SOLVE} and lemma (3.1) and since \( MQU = F \),

\[
\|QU - V^{(i,1)} - (I - \alpha M)(QU - U^{i-1})\|_{(\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{T_2}} \\
= \|QU - (V^{(i,0)} - \alpha(\text{APPLY } [M, V^{(i,0)}, \frac{d\epsilon_i}{6\alpha}]) - F^i)\) \\
- (I - \alpha M)(QU - U^{i-1})\|_{(\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{T_2}} \\
= \|QU - U^{i-1} + \alpha(\text{APPLY } [M, V^{(i,0)}, \frac{d\epsilon_i}{6\alpha}] - \alpha F^i) \\
- QU + U^{i-1} + \alpha MU^{i-1} - \alpha MU^{i-1}\|_{(\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{T_2}} \\
= \|\alpha \text{APPLY } [M, V^{(i,0)}, \frac{d\epsilon_i}{6\alpha}] - \alpha F^i + \alpha F - \alpha MU^{i-1}\|_{(\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{T_2}} \\
= \|\alpha(\text{APPLY } [M, V^{(i,0)}, \frac{d\epsilon_i}{6\alpha}] - MU^{i-1}) + \alpha(F - F^i)\|_{(\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{T_2}} \\
\leq \alpha \text{APPLY } [M, V^{(i,0)}, \frac{d\epsilon_i}{6\alpha}] - MU^{i-1}\|_{(\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{T_2}} + \alpha\|F - F^i\|_{(\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{T_2}} \\
\leq \alpha \frac{d\epsilon_i}{6\alpha} + \alpha \frac{d\epsilon_i}{6\alpha} = \frac{d\epsilon_i}{3}.
\]

Similarly we can prove

\[
\|QU - V^{(i,K)} - (I - \alpha M)^K(QU - U^{i-1})\|_{(\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{T_2}} \leq \frac{d\epsilon_i}{3}. \quad (21)
\]

Theorem 3.1. If \( U \) is a solution for (9) then the following inequalities hold for the algorithm \textit{SOLVE}:

\[
\|Q(U - U_i)\|_{(\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{T_2}} \leq \epsilon_i \\
\|QU + (I - Q)U^{i-1} - V^{(i,K)}\|_{(\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{T_2}} \leq \frac{2}{3}d\epsilon_i, \quad (i \geq 1) \quad (22)
\]

Proof. In order to prove the first inequality, it is enough to prove \( \|Q(U - U_i)\|_{(\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{T_2}} \leq \epsilon_i \) for each \( i \geq 0 \). For \( i = 0 \), since \( QU = M|_{\text{Ran} T^{1}_{H,v}} F \), then

\[
\|Q(U - U^0)\|_{(\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{T_2}} = \|QU\|_{(\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{T_2}} = \|M|_{\text{Ran} T^{1}_{H}} F\|_{(\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{T_2}} \\
\leq \|M|_{\text{Ran} T^{1}_{H}} F\|_{(\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{T_2}} = \epsilon_0.
\]

Now for an \( i \geq 1 \), let \( \|Q(U - U^{i-1})\|_{(\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{T_2}} \leq \epsilon_{i-1} \). Since \( MQU = F \) and

\[
(I - \alpha M)^K(QU - U^{i-1}) = (I - \alpha M)^K(QU - U^{i-1}) - (I - Q)U^{i-1},
\]

by using the inequality (21)

\[
\|QU + (I - Q)U^{i-1} - V^{(i,K)}\|_{(\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{T_2}} \\
= \|QU + (I - Q)^K(QU - U^{i-1}) - (I - Q)^K(QU - U^{i-1}) - V^{(i,K)}\|_{(\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{T_2}}
\]

\[ \leq \|(I - \alpha M)^K Q(U - U^{i-1})\|_{(\sum_{\lambda \in \mathcal{A}} \oplus H_\lambda)^2} + \|QU - V^{(i,K)} - (I - \alpha M)^K (QU - U^{i-1})\|_{(\sum_{\lambda \in \mathcal{A}} \oplus H_\lambda)^2} \]

\[ \leq \rho^K \|Q(U - U^{i-1})\|_{(\sum_{\lambda \in \mathcal{A}} \oplus H_\lambda)^2} + \frac{d\epsilon_i}{3} \]

\[ \leq \rho^K \epsilon_{i-1} + \frac{d\epsilon_i}{3} = \frac{2}{3}d\epsilon_i, \]

as we desired in (22).

Now by using (22) and the definition of \( U^i \) in SOLVE we obtain,

\[ ||QU + (I - Q)U^{i-1} - U^i||_{(\sum_{\lambda \in \mathcal{A}} \oplus H_\lambda)^2} \]

\[ \leq ||QU + (I - Q)U^{i-1} - V^{(i,K)} + V^{(i,K)} - U^i||_{(\sum_{\lambda \in \mathcal{A}} \oplus H_\lambda)^2} \]

\[ \leq (2d^3 + (1 - d))\epsilon_i = (1 - \frac{d}{3})\epsilon_i. \]

Therefore

\[ ||Q(U - U^i)||_{(\sum_{\lambda \in \mathcal{A}} \oplus H_\lambda)^2} \leq ||Q(U - U^i)||_{(\sum_{\lambda \in \mathcal{A}} \oplus H_\lambda)^2} + \]

\[ ||(I - Q)(U^{i-1} - U^i)||_{(\sum_{\lambda \in \mathcal{A}} \oplus H_\lambda)^2} \]

\[ = ||Q(U - U^i) + (I - Q)(U^{i-1} - U^i)||_{(\sum_{\lambda \in \mathcal{A}} \oplus H_\lambda)^2} \]

\[ \leq (1 - \frac{d}{3})\epsilon_i^2 \leq \epsilon_i^2, \]

and so

\[ ||Q(U - U^i)||_{(\sum_{\lambda \in \mathcal{A}} \oplus H_\lambda)^2} \leq \epsilon. \]

In the following theorems, we investigate the optimal computational complexity of the algorithm SOLVE as it recovers an approximate solution with desired accuracy at a computational expense that stays proportional to the number of terms in a corresponding wavelet-best \( N \)-term approximation.

**Theorem 3.2.** Assume that the solution \( U \) of (9) belongs to \( \ell^s_{\mathcal{F}} \). Then

\[ \#(\text{supp}(U_i)) \leq \epsilon_i^\frac{1}{2} |U|_{\ell^s_{\mathcal{F}}}^{-\frac{1}{2}}. \]

**Proof.** Let \( (QU)_N \) be the best \( N_s \)-term approximation for \( QU \) such that

\[ ||QU - (QU)_N||_{(\sum_{\lambda \in \mathcal{A}} \oplus H_\lambda)^2} \leq \frac{d\epsilon_i}{3}, \]

where \( 0 < d < \frac{1}{3} \). Since \( U \in \ell^s_{\mathcal{F}} \), by (11) we have

\[ N_s \leq \epsilon_i^{-\frac{1}{2}} |QU|_{\ell^s_{\mathcal{F}}}^{-\frac{1}{2}} \leq \epsilon_i^{-\frac{1}{2}} |U|_{\ell^s_{\mathcal{F}}}^{-\frac{1}{2}}. \]

Since \( s' < s \), then for a vector \( V \) with \#supp(\( V \)) = \( N \)

\[ |V|_{\ell^s_{\mathcal{F}}} \leq N^{s-s'} |V|_{\ell^{s'}_{\mathcal{F}}}, \]

combination (25) and (24) gives

\[ \epsilon_i^{1-\frac{s'}{s}} ||(QU)_N||_{\ell^{s'}_{\mathcal{F}}} \leq \epsilon_i^{1-\frac{s'}{s}} N_s^{-s'} |(QU)_N|_{\ell^{s'}_{\mathcal{F}}} \leq \]
\[ \epsilon_i^{1 - s'} (\epsilon_i^{-\frac{1}{2}} (s - s')(U_{c_i}^{\frac{1}{2}})^{(s - s')}((QU)_{N_i}|c_i \leq |U|^{\frac{1}{2}} (s - s')) |QU|_{c_i} \leq |U|^{\frac{1}{2}}, \]

therefore

\[ \epsilon_i^{1 - s'} ((QU)_{N_i}|c_i \leq |U|^{\frac{1}{2}}. \tag{26} \]

Using (22) and (23),

\[ \| (QU)_{N_i} + (I - Q)U^{i} - V K \| (\sum_{\lambda \in \Lambda} \oplus H_{\lambda})_{c_2} \leq \]

\[ \| (QU)_{N_i} - QU \| (\sum_{\lambda \in \Lambda} \oplus H_{\lambda})_{c_2} + |QU + (I - Q)U^{i} - V K \| (\sum_{\lambda \in \Lambda} \oplus H_{\lambda})_{c_2} \leq \]

\[ \frac{d \epsilon_i}{3} + \frac{2d \epsilon_i}{3} = d \epsilon_i, \]

then by lemma 3.2 and 10 we have

\[ |U|^i |c_i| \leq C_2 |(QU)_{N_i} + (I - Q)U^{i} - V K \| (\sum_{\lambda \in \Lambda} \oplus H_{\lambda})_{c_2} \leq \]

\[ |C_1 C_2| |c_i| + C_1 C_2 |I - Q| |c_i| \leq |U|^{i} - V K \| (\sum_{\lambda \in \Lambda} \oplus H_{\lambda})_{c_2} \leq \]

by (26) and since \( \epsilon_i = \frac{3pKd_{i-1}}{d} \)

\[ \epsilon_i^{1 - s'} |U|^i |c_i| \leq C |U|^{i - s'} + C_1 C_2 |I - Q| |c_i| \leq (3pKd_{i-1})^{1 - s'} (\epsilon_i^{1 - s'} |U|^i), \tag{27} \]

for some constant \( C \). Now by the properties of \( K \) we obtain

\[ \epsilon_i^{1 - s'} |U|^i |c_i| \leq |U|^{i - s'}. \tag{28} \]

Finally by (28), (26) and lemma 3.2 we conclude

\[ \#(supp(U^i)) \leq \epsilon_i^{1 - s'} |(QU)_{N_i} + (I - Q)U^{i} - V K \| (\sum_{\lambda \in \Lambda} \oplus H_{\lambda})_{c_2} \leq \]

\[ \leq \epsilon_i^{1 - s'} |(QU)_{N_i}| |c_i| + |I - Q| |c_i| \leq |U|^{i} - V K \| (\sum_{\lambda \in \Lambda} \oplus H_{\lambda})_{c_2} \leq \]

\[ \leq \epsilon_i^{1 - s'} |U|^{i - s'}, \]

and it is obvious that this proves our request. \( \square \)

**Theorem 3.3.** Assume that the solution \( U \) of (9) belongs to \( \ell_{p}^{s'} \). Then the number of arithmetic operations needed to compute \( U_{c} \) is bounded by a multiple of \( \epsilon^{1 - s'} |U|^{\frac{1}{2}} \).

**Proof.** Since \( MU = F \) and \( M \) is bounded on \( \ell_{p}^{s'} \) then \( |F|_{c} \leq |U|_c \), and therefore by lemma 3.2

\[ \#(supp(U^i)) \leq \epsilon_i^{1 - s'} |U|^i, \tag{29} \]

and

\[ |F^i|_{c_i} \leq |U|_{c_i}. \tag{30} \]

Now by (25), (29) and (30) we obtain

\[ |F^i|_{c_i} \leq \#(supp(F^i))^{s - s'} |F^i|_{c_i} \]

\[ \leq (\epsilon_i^{\frac{1}{2}} |U|^{\frac{1}{2}})^{s - s'} |U|_{c_i} \leq \epsilon_i^{s' - 1} |U|^{\frac{1}{2}}. \]

Thus

\[ (\epsilon_i)^{1 - s'} |F^i|_{c_i} \leq |U|^{\frac{1}{2}}. \tag{31} \]
Lemma 3.1 together with (28) and (31) give \((\epsilon_i)^{1-\frac{j}{2}}|V(i,j)|_{L^2} \leq |U|_{L^2}^{-\frac{j}{2}}, \quad 0 \leq j \leq K\), therefore by lemma 3.1 (for \(s\)), 
\[\#supp(W^{j-1}_{i}) \leq \epsilon_i^{-\frac{1}{2}}|U|_{L^2}^{-\frac{1}{2}}, \quad \#supp(V_{i,j}^{(i,j-1)}) \leq \epsilon_i^{-\frac{1}{2}}|U|_{L^2}^{-\frac{1}{2}}.\] Also by the previous theorem 
\[\#supp(U^{j-1}_{i}) = \#supp(V_{i,0}^{(i,j-1)}) \leq \epsilon_i^{-\frac{1}{2}}|U|_{L^2}^{-\frac{1}{2}}, \quad \#supp(V_{i,j}^{(i,j)}) \leq \#supp(V_{i,j-1}^{(i,j-1)}) + \#supp(F^i) + \#supp(W^{j-1}_{i}).\]
Therefore we conclude
\[\#supp(V_{i,j}^{(i,j)}) \leq \epsilon_i^\frac{1}{2}|U|_{L^2}^{-\frac{1}{2}}.\] (32)

Now by lemmas 3.2 and 3.1 together with (32), the number of arithmetic operations needed from \(i\) to \(i+1\) is at most a multiple of \(\epsilon_i^\frac{1}{2}|U|_{L^2}^{-\frac{1}{2}}\), which is the desired result. 

\[\square\]

References


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