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EXISTENCE AND MULTIPLE POSITIVE SOLUTIONS TO SYSTEMS OF DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

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Abstract. We show under some conditions the existence and multiplicity of positive solutions for a system of differential equations of fractional order, subject to two-point boundary conditions by applying the fixed point index theory in cones.

Keywords: systems of differential equations of fractional order, the fixed point index theory, positive solutions, Greens' function, cone.

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1. Introduction

Fractional calculus is a very old concept dating back to 17th century; it involves fractional integration and fractional differentiation. At the first stage, fractional calculus theory is mainly focused on pure mathematical fields. In the last few decades, fractional differential equations and fractional integration equations have found many applications in various fields, such as science and engineering, physics, chemistry, biology, economics, and signal and image processing. In recent years, fractional differential equations have attracted increasing interests for their extensive applications, which leads to intensive development of the theory of fractional calculus.

Recently, much interest has been created in establishing positive solutions and multiple positive solutions for two-point and multi-point boundary value problems (BVPs) associated with ordinary and fractional order differential equations by using different methods such as fixed point theorems in cones, the Leray-Schauder continuation theorem and its nonlinear alternatives and the coincidence degree theory. To mention the related papers along these lines, we refer to Agarwal and O'regan [2], Henderson and Luca [12, 14, 15], Henderson and Ntouyas [16], Henderson, Ntouyas and Purnaras [17], Prasasd, Kameswararao and Nageswararao [18], Zhou and Xu [27] for ordinary differential equations. Agarwal, Zhou snd He [3], Ahmad and Ntouyas [4], Bai [5], Bai and Lu [6], Henderson and Luca [13], Khan, Rehman and Henderson [18], Kauffman and Mboumi [20], Liang and Zhang [22] for frcational order differential equations.

In this paper, we consider the existence and multiplicity of positive solutions to system of nonlinear differential equations of fractional order having the form

$$\begin{aligned} D_{0^+}^{\nu_1} u(t) + f(t, v(t)) = 0,\\ D_{0^+}^{\nu_2} v(t) + g(t, u(t)) = 0, \end{aligned} \tag{1}$$

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where $t \in (0,1)$, $\nu_1, \nu_2 \in (n-1, n]$ for $n > 3$ and $n \in \mathbb{N}$, subject to a couple of boundary conditions

$$\begin{aligned} u^{(i)}(0) =& 0 = v^{(i)}(0),\ 0 \le i \le n-2, \\ [D_{0+}^{\alpha} u(t)]_{t=1} =& 0 = [D_{0+}^{\alpha} v(t)]_{t=1},\ 1 \le \alpha \le n-2, \end{aligned} \tag{2}$$

where $f, g \in C([0,1] \times [0,\infty), [0,\infty))$.

Under the sufficient conditions on functions f and g, we study the existence and multiplicity of positive solutions of problem (1)-(2) by using the fixed point index theorems [1, 27]. By a positive solution of problem (1)-(2) we mean a pair of functions $(u, v) \in C[0,1] \times C[0,1]$ satisfying (1) and (2) with $u(t) > 0, v(t) > 0$ for all $t \in [0,1]$ and $\sup_{t\in[0,1]} u(t) > 0, \sup_{t\in[0,1]} v(t) > 0$.

The rest of this paper is organized as follows. In Section 2, we present the necessary definitions and properties from the fractional calculus theory and we give the Greens function for the homogeneous BVP and estimate the bounds for the Greens function. In Section 3, we shall prove some existence and multiplicity results for positive solutions with respect to a cone for our problem (1)-(2), which are based on the three fixed point index theorems. Finally, in Section 5, we shall provide some numerical examples which shall explicate the applicability of our results.

2. Preliminaries

For the convenience of the reader, we present here some definitions, lemmas, and basic results that will be used in the proofs of our theorems.

Definition 2.1. *Let $\nu > 0$ with $\nu \in \mathbb{R}$. Suppose that $y : [a, +\infty) \to \mathbb{R}$. Then the νth Riemann-Liouville fractional integral is defined to be*

$$D_{a^+}^{-\nu} y(t) = \frac{1}{\Gamma(\nu)} \int_a^t y(s)(t-s)^{\nu-1} ds,$$

whenever the right-hand side is defined. Similarly, with $\nu > 0$ and $\nu \in \mathbb{R}$, we define the νth Riemann-Liouville fractional derivative to be

$$D_{a^+}^{\nu} y(t) = \frac{1}{\Gamma(n-\nu)} \frac{d^n}{dt^n} \int_a^t \frac{y(s)}{(t-s)^{\nu+1-n}} ds,$$

whenever $n \in \mathbb{N}$ is the unique positive integer satisfying $n - 1 \le \nu < n$ and $t > a$.

Remark 2.1. *In the sequel, we shall usually suppress the explicit dependence of $D_{a^+}^{-\nu}$ on a. It will be clear from the context. In fact, in this paper $a = 0$ throughout.*

Lemma 2.1. *Let $\alpha \in \mathbb{R}$. Then $D^n D^{\alpha} y(t) = D^{n+\alpha} y(t)$, for each $n \in \mathbb{N}_0$, where $y(t)$ is assumed to be sufficiently regular so that both sides of equality are well-defined. Moreover, if $\beta \in (-\infty, 0]$ and $\gamma \in [0, +\infty)$, then $D^{\gamma} D^{\beta} y(t) = D^{\gamma+\beta} y(t)$.*

Lemma 2.2. *The general solution $D^{\nu} y(t) = 0$, where $n - 1 < \nu \le n$ and $\nu > 0$, is the function $y(t) = c_1 t^{\nu-1} + c_2 t^{\nu-2} + \cdots + c_n t^{\nu-n}$ where $c_i \in \mathbb{R}$ for each i.*

Lemma 2.3. [9] *Let $h(t) \in C^n[0,1]$ be given. Then the unique solution to the problem*

$$-D_{0+}^{\nu} u(t) = h(t) \tag{3}$$

together with the boundary conditions

$$u^{(i)}(0) = 0 = [D_{0+}^{\alpha} u(t)]_{t=1} \tag{4}$$

where $1 \leq \alpha \leq n-2$ *and* $0 \leq i \leq n-2$ *is*

$$u(t) = \int_0^1 G_1(t,s)h(s)ds, \tag{5}$$

where

$$G_1(t,s) = \begin{cases} \frac{t^{\nu-1}(1-s)^{\nu-\alpha-1}-(t-s)^{\nu-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1 \\ \frac{t^{\nu-1}(1-s)^{\nu-\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1 \end{cases} \tag{6}$$

is the Green's function for this problem.

Lemma 2.4. [9] *Let* $G_1(t,s)$ *be as in the statement of Lemma 2.3. Then we find that:*
(i) $G_1(t,s)$ *is a continuous function on the unit square* $[0,1] \times [0,1]$*;*
(ii) $G_1(t,s) \geq 0$*, for each* $(t,s) \in [0,1] \times [0,1]$*;*
(iii) $\max_{t\in[0,1]} G_1(t,s) = G_1(1,s)$*, for each* $s \in [0,1]$.

Lemma 2.5. [9] *Let* $G_1(t,s)$ *be as given in the statement of Lemma 2.3. Then there exists a constant* $\gamma_1 \in (0,1)$ *such that*

$$\min_{t\in[(1/2),1]} G_1(t,s) \geq \gamma_1 \max_{t\in[0,1]} G_1(t,s) = \gamma_1 G_1(1,s). \tag{7}$$

We can also formulate similar results as Lemma 2.4-2.5 above, for the fractional differential equation

$$-D_{0^+}^{\nu} v(t) = h(t) \tag{8}$$

$$v^{(i)}(0) = 0 = [D_{0^+}^{\alpha} v(t)]_{t=1}. \tag{9}$$

We denote by G_2 and γ_2 the corresponding Green's function and constant for the problem (8)-(9) defined in a similar manner as G_1 and γ_1 respectively.

We present now the Fixed point index theorems [1, 27] that we will use in the proofs of main results.

Fixed point index theorems: Let E be a real Banach space, $P \subset E$ a cone, " $\leq$ " the partial ordering defined by P and θ the zero element in E. For $\rho > 0$, let $B_\rho = \{u \in E : \|u\| < \rho\}$. The proofs of our results are based upon on the application of the following fixed point index theorems.

Theorem 2.1. [1] *Let* $A : \overline{B}_\rho \cap P \to P$ *be a completely continuous operator which has no fixed point on* $\partial B_\rho \cap P$*. If* $\|Au\| \leq \|u\|,\ \forall u \in \partial B_\rho \cap P$*. Then* $i(A, B_\rho \cap P, P) = 1$.

Theorem 2.2. [1] *Let* $A : \overline{B}_\rho \cap P \to P$ *be a completely continuous operator. If there exists* $u_0 \in P \setminus \{\theta\}$ *such that*

$$u - Au \neq \lambda u_0,\ \forall \lambda \geq 0,\ u \in \partial B_\rho \cap P.$$

Then $i(A, B_\rho \cap P, P) = 0$.

Theorem 2.3. [27] *Let* $A : \overline{B}_\rho \cap P \to P$ *be a completely continuous operator which has no fixed point on* $\partial B_\rho \cap P$*. If there exists a linear operator* $L : P \to P$ *and* $u_0 \in P \setminus \{\theta\}$ *such that*

$$(i)\ u_o \leq Lu_0,\ (ii)\ Lu \leq Au,\ \forall u \in \partial B_\rho \cap P.$$

Then $i(A, B_\rho \cap P, P) = 0$.

3. Main Results

In this section, we shall investigate the existence and multiplicity of positive solutions for our problem (1)-(2) under various assumptions on f and g.

We present the assumptions that we shall use in the sequel:

(A1) The functions $f, g \in C([0,1]\times[0,\infty),[0,\infty))$ and $f(t,0)\equiv 0, g(t,0)\equiv 0$ for all $t\in[0,1]$.

(A2) There exists a positive constant $p\in(0,1]$ such that

$$(i)\ f_\infty^i = \lim_{u\to\infty}\inf_{t\in[(1/2),1]}\frac{f(t,u)}{u^p}\in(0,\infty];\ (ii)\ g_\infty^i = \lim_{u\to\infty}\inf_{t\in[(1/2),1]}\frac{g(t,u)}{u^{1/p}}=\infty.$$

(A3) There exists a positive constant $q\in(0,\infty)$ such that

$$(i)\ f_0^s = \lim_{u\to0^+}\sup_{t\in[0,1]}\frac{f(t,u)}{u^q}\in[0,\infty);\ (ii)\ g_0^s = \lim_{u\to0^+}\sup_{t\in[0,1]}\frac{g(t,u)}{u^{1/q}}=0.$$

(A4) There exists a positive constant $r\in(0,\infty)$ such that

$$(i)\ f_\infty^s = \lim_{u\to\infty}\sup_{[0,1]}\frac{f(t,u)}{u^r}\in[0,\infty);\ (ii)\ g_\infty^s = \lim_{u\to\infty}\sup_{t\in[0,1]}\frac{g(t,u)}{u^{1/r}}=0.$$

(A5) The following conditions are satisfied

$$f_0^i = \lim_{u\to0^+}\inf_{t\in[(1/2),1]}\frac{f(t,u)}{u}\in(0,\infty];\ g_0^i = \lim_{u\to0^+}\inf_{t\in[(1/2),1]}\frac{g(t,u)}{u}=\infty.$$

(A6) $f(t,u)$, $g(t,u)$ are all nondecreasing with respect to u and there exists a constant $N>0$ such that

$$f\Big(t, m_0\int_0^1 g(s,N)ds\Big) < \frac{N}{m_0},\ \forall t\in[0,1],$$

where $m_0=\max\{K_1,K_2\}$, $K_1=\max_{s\in[0,1]}G_1(1,s)ds$ and $K_2=\max_{s\in[0,1]}G_2(1,s)ds$.

A pair of functions $(u,v)\in C[0,1]\times C[0,1]$ is a solution of the problem (1)-(2) if and only if $(u,v)\in C[0,1]\times C[0,1]$ is a solution of the following system of nonlinear integral equations:

$$u(t)=\int_0^1 G_1(t,s)f(s,v(s)ds,$$
$$v(t)=\int_0^1 G_2(t,s)g(s,u(s))ds.$$

Moreover, the above system can be written as the nonlinear integral equation

$$u(t)=\int_0^1 G_1(t,s)f\Big(s,\int_0^1 G_2(s,\tau)g(\tau,u(\tau))d\tau\Big)ds.$$

We consider the Banach space $X=C[0,1]$ with supremum norm $\|\cdot\|$ and define the cone $P\subset X$ by $P=\Big\{u\in X: u(t)\ge 0, \forall t\in[0,1]\Big\}$.

We also define the operator $A:P\to X$ by

$$(Au)(t)=\int_0^1 G_1(t,s)f\Big(s,\int_0^1 G_2(s,\tau)g(\tau,u(\tau))d\tau\Big)ds,$$

and $B:P\to X$, $C:P\to X$ by

$$(Bu)(t)=\int_0^1 G_1(t,s)u(s)ds,\ (Cu)(t)=\int_0^1 G_2(t,s)u(s)ds.$$

Under the assumption (A1) and Lemmas 2.4 and 2.5, it is easy to see that A, B and C are completely continuous from P to P. Thus, the existence and multiplicity of positive solutions of the system (1)-(2) are equivalent to the existence and multiplicity of fixed points of the operator A.

Theorem 3.1. *Assume (A1)-(A3) hold. Then the problem* (1)-(2) *has at least one positive solution* $(u(t), v(t))$ *with* $u(t) > 0, v(t) > 0,\ \forall t \in (0, 1)$.

Proof. From assumption (i) of (A2), we know that there exist constants $C_1 > 0, C_2 > 0$ such that

$$f(t, u) \geq C_1 u^p - C_2,\ \forall (t, u) \in [0, 1] \times [0, \infty). \tag{10}$$

Hence, for $u \in P$, by using (10), the reverse form of Holder's inequality and Lemma 2.5, we have for $p \in (0, 1]$,

$$\begin{aligned}
(Au)(t) &= \int_0^1 G_1(t, s) f\Big(s, \int_0^1 G_2(s, \tau) g(\tau, u(\tau)) d\tau\Big) ds \\
&\geq \int_0^1 G_1(t, s)\Big[C_1\Big(\int_0^1 G_2(s, \tau) g(\tau, u(\tau) d\tau\Big)^p ds - C_2\Big] ds \\
&\geq C_1 \int_0^1 G_1(t, s)\Big[\int_0^1 \big(G_2(s, \tau)\big)^p (g(\tau, u(\tau)))^p d\tau\Big] ds - C_2 \int_0^1 G_1(t, s) ds \\
&\geq C_1 \int_0^1 G_1(t, s)\Big[\int_0^1 \big(G_2(s, \tau)\big)^p (g(\tau, u(\tau)))^p d\tau\Big] ds - C_3,\ t \in [0, 1],
\end{aligned}$$

where $C_3 = C_2 \int_0^1 G_1(1, s) ds$. Therefore, for $u \in P$, we have

$$(Au)(t) \geq C_1 \int_{1/2}^1 G_1(t, s)\Big(\int_0^1 \big(G_2(s, \tau)\big)^p (g(\tau, u(\tau)))^p d\tau\Big) ds - C_3,\ \forall t \in [0, 1]. \tag{11}$$

We define the cone

$$P_0 = \Big\{u \in P : \inf_{[(1/2), 1]} u(t) \geq \gamma \|u\|\Big\},$$

where $\gamma = \{\gamma_1, \gamma_2\}$. From our assumptions and Lemma 2.5, it can be shown that for any $y \in P$, the functions $u(t) = (By)(t)$ and $v(t) = (Cy)(t)$ satisfy the inequalities

$$\inf_{t \in [(1/2), 1]} u(t) \geq \gamma_1 \|u\| \geq \gamma \|u\|, \quad \inf_{t \in [(1/2), 1]} v(t) \geq \gamma_2 \|v\| \geq \gamma \|v\|.$$

So $u = By \in P_0$, $v = Cy \in P_0$. Therefore, we deduce that $B(P) \subset P_0$, $C(P) \subset P_0$.

Now we consider the function $u_0(t), t \in [0, 1]$, the solution of problem (3)-(4) with $y = y_0$, where $y_0(t) = 1$ for all $t \in [0, 1]$. Then $u_0(t) = \int_0^1 G_1(t, s) ds = (By_0)(t),\ t \in [0, 1]$. Obliviously, we have $u_0(t) \geq 0$ for all $t \in [0, 1]$. Let

$$M = \Big\{u \in P : u = Au + \lambda u_0,\ \lambda \geq 0\Big\}.$$

In the following we show that $M \subset P_0$ and M is a bounded subset of X. If $u \in M$, then there exists $\lambda \geq 0$ such that $u(t) = (Au)(t) + \lambda u_0(t), t \in [0, 1]$. From the definition of u_0, we have

$$u(t) = (Au)(t) + \lambda (By_0)(t) = B(Fu(t)) + \lambda (By_0)(t) = B\big(Fu(t) + \lambda y_0(t)\big) \in P_0,$$

where $F : P \to P$ is defined by

$$(Fu)(t) = f\Big(t, \int_0^1 G_2(t, s) g(s, u(s)) ds\Big),$$

hence, $M \subset P_0$ and from the definition of P_0, we have

$$\|u\| \leq \frac{1}{\gamma} \cdot \min_{[(1/2),1]} u(t),\ \forall u \in M. \tag{12}$$

From (ii) of assumption (A2), we conclude that for $\epsilon_0 = \left(\frac{2}{C_1 m_1 m_2 \gamma_1 \gamma_2^p}\right)^{1/p} > 0$, there exists a constant $C_4 > 0$ such that

$$\big(g(t,u)\big)^p \geq \epsilon_0^p u - C_4,\ \forall (t,u) \in [0,1] \times [0,\infty), \tag{13}$$

where

$$m_1 = \int_{1/2}^1 G_1(t,s)ds > 0,\ m_2 = \int_{1/2}^1 (G_2(t,\tau))^p d\tau > 0.$$

For $u \in M$ and $t \in [(1/2),1]$, by using Lemma 2.5 and the relations (11) and (12), it follows that

$$\begin{aligned}
u(t) &= (Au)(t) + \lambda u_0(t) \geq (Au)(t)\\
&\geq C_1 \int_{1/2}^1 G_1(t,s)\left[\int_{1/2}^1 \big(G_2(s,\tau)\big)^p (g(\tau,u(\tau)))^p d\tau\right] ds - C_3\\
&\geq C_1\gamma_1 \int_{1/2}^1 G_1(1,s)\left(\int_{1/2}^1 (\gamma_2 G_2(1,s))^p(\epsilon_0^p u - C_4)d\tau\right) ds - C_3\\
&\geq C_1\gamma_1\gamma_2^p\epsilon_0^p\left(\int_{1/2}^1 G_1(1,s)ds\right)\left(\int_{1/2}^1 \big(G_2(1,s)\big)^p u(\tau)d\tau\right) - C_5\\
&\geq C_1\gamma_1\gamma_2^p\epsilon_0^p\left(\int_{1/2}^1 G_1(1,s)ds\right)\left(\int_{1/2}^1 \big(G_2(1,s)\big)^p d\tau\right) \inf_{\tau\in[(1/2),1]} u(\tau) - C_5\\
&= 2 \inf_{\tau\in[(1/2),1]} u(\tau) - C_5,
\end{aligned}$$

where $C_5 = C_3 + C_1 C_4 m_1 m_2 \gamma_1 \gamma_2^p > 0$ is a constant. Since $Bu \in P_0$, we have

$$\inf_{t\in[(1/2),1]} u(t) \geq 2 \inf_{t\in[(1/2),1]} u(t) - C_5,$$

and so

$$\inf_{t\in[(1/2),1]} u(t) \leq C_5,\ \forall u \in M. \tag{14}$$

Now the relation (12) and (14), it can be shown that

$$\|u\| \leq \frac{1}{\gamma} \inf_{t\in[(1/2),1]} u(t) \leq \frac{C_5}{\gamma},$$

for all $u \in M$, that is, M is a bounded subset of X.

Besides, there exists a sufficiently large $L > 0$ such that $u(t) \neq (Au)(t) + \lambda u_0,\ \forall u \in \partial B_L \cap P,\ \lambda \geq 0$. From Theorem 2.2, we deduce that

$$i(A, B_L \cap P, P) = 0. \tag{15}$$

Next, from (i) of assumption (A3), we conclude that there exists $M_0 > 0$ such that

$$f(t,u) \leq M_0 u^q,\ \forall (t,u) \in [0,1] \times [0,1]. \tag{16}$$

From (ii) of assumption (A3) and (A1), it can be shown that for
$\epsilon_1 = \min\left\{\frac{1}{M_2}, \left(\frac{1}{2M_0 M_1 M_2^q}\right)^{1/q}\right\} > 0$, there exists $\delta_1 \in (0,1)$ such that

$$g(t,u) \leq \epsilon_1 u^{1/q},\ \forall (t,u) \in [0,1] \times [0,\delta_1],$$

where $M_1 = \int_0^1 G_1(1,s)ds > 0$, $M_2 = \int_0^1 G_2(1,s)ds > 0$. Hence, we have

$$\begin{aligned}\int_0^1 G_2(t,s)g(s,u(s))ds &\le \epsilon_1 \int_0^1 G_2(t,s)(u(s))^{1/q}ds \\ &\le \epsilon_1 M_2 \|u\|^{1/q} \le 1,\ \forall u \in \overline{B}_{\delta_1} \cap P,\ t \in [0,1].\end{aligned} \tag{17}$$

Therefore, by (16) and (17) we deduce

$$\begin{aligned}(Au)(t) &= \int_0^1 G_1(t,s) f\Big(s, \int_0^1 G_2(s,\tau)g(\tau,u(\tau))d\tau\Big)ds \\ &\le M_0 \int_0^1 G_1(1,s)\Big(\int_0^1 G_2(s,\tau)g(\tau,u(\tau))d\tau\Big)^q ds \\ &\le M_0 \epsilon_1^q M_2^q \|u\| \int_0^1 G_1(1,s)ds \\ &= M_0 \epsilon_1^q M_1 M_2^q \|u\| \le \frac{1}{2}\|u\|,\ \forall u \in \overline{B}_{\delta_1} \cap P,\ t \in [0,1].\end{aligned}$$

This implies that $\|Au\| \le \frac{1}{2}\|u\|$, $\forall u \in \partial B_{\delta_1} \cap P$. From Theorem 2.1, we have

$$i(A, B_{\delta_1} \cap P, P) = 1. \tag{18}$$

Combining (15) and (18), we have

$$i(A, (B_L \setminus \overline{B}_{\delta_1}) \cap P, P) = i(A, B_L \cap P, P) - i(A, B_{\delta_1} \cap P, P) = -1.$$

We conclude that A has at least one fixed point $u_1 \in (B_L \setminus \overline{B}_{\delta_1}) \cap P$, that is $\delta_1 < \|u_1\| < L$.

Let

$$v_1(t) = \int_0^1 G_2(t,s)g(s,u_1(s))ds,$$

then $(u_1, v_1) \in P \times P$ is a solution of the problem (1)-(2). In addition $\|v_1\| > 0$. Indeed, if we suppose that $v_1(t) = 0$, for all $t \in [0,1]$, then by using (A1) we have $f(s, v_1(s)) = f(s,0) \equiv 0$, for all $s \in [0,1]$. This implies that $u_1(t) = \int_0^1 G_1(t,s)f(s,v_1(s))ds = 0$ for all $t \in [0,1]$, which contradicts $\|u_1\| > 0$. The proof of Theorem 3.1 is completed. □

Theorem 3.2. *Assume (A1), (A4) and (A5) hold. Then the problem* (1)-(2) *has at least one positive solution* $(u(t), v(t)), t \in [0,1]$.

Proof. From assumption (i) of (A4), we know that there exist constants $C_6 > 0, C_7 > 0$ such that

$$f(t,u) \le C_6 u^r + C_7,\ \forall (t,u) \in [0,1] \times [0,\infty). \tag{19}$$

From (ii) of (A4), we conclude that for $\epsilon_2 = \left(\frac{1}{2C_6 M_1 M_2^r}\right)^{1/r}$ there exists $C_8 > 0$ such that

$$g(t,u) \le \epsilon_2 u^{1/r} + C_8,\ \forall (t,u) \in [0,1] \times [0,\infty). \tag{20}$$

Hence, for $u \in P$, by using (19) and (20), we obtain

$$\begin{aligned}(Au)(t) &= \int_0^1 G_1(t,s) f\Big(s, \int_0^1 G_2(s,\tau) g(\tau, u(\tau)) d\tau\Big) ds \\ &\le \int_0^1 G_1(t,s)\Big[C_6\Big(\int_0^1 G_2(s,\tau) g(\tau, u(\tau) d\tau\Big)^r + C_7\Big] ds \\ &\le C_6 \int_0^1 G_1(t,s)\Big[\int_0^1 G_2(s,\tau)\Big(\epsilon_2 (u(\tau))^{1/r} + C_8\Big) d\tau\Big]^r ds + M_1 C_7 \\ &\le C_6\Big(\epsilon_2 \|u\|^{1/r} + C_8\Big)^r \Big(\int_0^1 G_1(1,s) ds\Big)\Big(\int_0^1 G_2(1,\tau) d\tau\Big)^r + M_1 C_7.\end{aligned}$$

Therefore, we have

$$(Au)(t) \le C_6 M_1 M_2^r \Big(\epsilon_2 \|u\|^{1/r} + C_8\Big)^r + M_1 C_7, \ \forall t \in [0,1]. \tag{21}$$

After some assumptions, it can be shown that

$$\lim_{\|u\| \to \infty} \frac{C_6 M_1 M_2^r \Big(\epsilon_2 \|u\|^{1/r} + C_8\Big)^r + M_1 C_7}{\|u\|} = \frac{1}{2},$$

so, there exists a sufficiently large $R > 0$ such that

$$C_6 M_1 M_2^r \Big[\epsilon_2 \|u\|^{1/r} + C_8\Big]^r + M_1 C_7 \le \frac{3}{4}\|u\|, \ \forall u \in P \text{ with } \|u\| \ge R. \tag{22}$$

Hence, from (21) and (22), we have $\|Au\| \le \frac{3}{4}\|u\| < \|u\|$, $\forall \in \partial B_R \cap P$, and from Theorem 2.1, we have

$$i(A, B_R \cap P, P) = 1. \tag{23}$$

On the other hand, from (i) of assumption (A5), we know that there exist constants $C_9 > 0$ and $\widetilde{u}_1 > 0$ such that

$$f(t,u) \ge C_9 u, \ \forall (t,u) \in [0,1] \times [0, \widetilde{u}_1].$$

From (ii) of assumption (A5), for $\epsilon = C_0/C_9 > 0$ with $C_0 = \frac{1}{\gamma_1 \gamma_2 m_1 m_2} > 0$ and $m_3 = \int_{1/2}^1 G_2(1,\tau) d\tau > 0$, we conclude that there exist $\widetilde{\widetilde{u}}_1 > 0$ such that $g(t,u) \ge \frac{C_0}{C_9} u$ for all $(t,u) \in [0,1] \times [0, \widetilde{\widetilde{u}}_1]$. We consider $u_1 = \min\{\widetilde{u}_1, \widetilde{\widetilde{u}}_1\}$ and then we obtain

$$f(t,u) \ge C_9 u, \ g(t,u) \ge \frac{C_0}{C_9} u, \ \forall (t,u) \in [0,1] \times [0, u_1]. \tag{24}$$

From the fact $g(t,0) \equiv 0$ and the continuity of $g(t,u)$, we know that there exists a sufficiently small $\delta_2 \in (0, u_1)$ such that

$$g(t,u) \le \frac{u_1}{M_2}, \ \forall (t,u) \in [0,1] \times [0, \delta_2].$$

Hence,

$$\int_0^1 G_2(s,\tau) g(\tau, u(\tau)) d\tau \le \int_0^1 G_2(1,s) g(\tau, u(\tau)) d\tau \le u_1, \ \forall u \in \overline{B}_{\delta_2} \cap P, \ s \in [0,1]. \tag{25}$$

From (24), (25) and Lemma 2.5, we have

$$\begin{aligned}(Au)(t) &= \int_0^1 G_1(t,s)f\Big(s,\int_0^1 G_2(s,\tau)g(\tau,u(\tau))d\tau\Big)ds\\ &\geq C_9\int_0^1 G_1(t,s)\Big(\int_0^1 G_2(s,\tau)g(\tau,u(\tau))d\tau\Big)ds\\ &\geq C_0\int_0^1 G_1(t,s)\Big(\int_0^1 G_2(s,\tau)u(\tau)d\tau\Big)ds\\ &\geq C_0\int_{1/2}^1 G_1(t,s)\Big(\int_0^1 G_2(s,\tau)u(\tau)d\tau\Big)ds\\ &\geq C_0\gamma_2\int_{1/2}^1 G_1(t,s)\Big(\int_0^1 G_2(1,\tau)u(\tau)d\tau\Big)ds\\ &= (Lu)(t),\ \forall u\in\partial B_{\delta_2}\cap P,\ t\in[0,1]\end{aligned}$$

where the linear operator $L: P\to P$ is defined by

$$(Lu)(t) = C_0\gamma_2\Big(\int_0^1 G_1(1,\tau)u(\tau)d\tau\Big)\Big(\int_{1/2}^1 G_1(t,s)ds\Big).$$

Hence, we obtain

$$Au\geq Lu,\ \forall u\in\partial B_{\delta_2}\cap P. \tag{26}$$

For $w_0(t)=\int_{1/2}^1 G_1(t,s)ds,\ t\in[0,1]$, we have $w_0\in P\setminus\{\theta_0\}$ and

$$\begin{aligned}(Lw_0)(t) &= C_0\gamma_2\Big[\int_0^1 G_2(1,\tau)\Big(\int_{1/2}^1 G_1(\tau,s)d\tau\Big)\Big]\Big(\int_{1/2}^1 G_1(t,s)ds\Big)\\ &\geq C_0\gamma_1\gamma_2\Big(\int_{1/2}^1 G_2(1,\tau)d\tau\Big)\Big(\int_{1/2}^1 G_2(1,\tau)d\tau\Big)\Big(\int_{1/2}^1 G_1(t,s)ds\Big)\\ &= C_0\gamma_1\gamma_2 m_1 m_3\int_{1/2}^1 G_1(t,s)ds = \int_{1/2}^1 G_1(t,s)ds = w_0(t),\ t\in[0,1].\end{aligned}$$

Therefore

$$Lw_0\geq w_0. \tag{27}$$

We may suppose that A has no fixed point in $\partial B_{\delta_2}\cap P$ (otherwise, the proof is finished). From (26), (27) and Theorem 2.3 (with $u_0=w_0$), we have

$$i(A,B_{\delta_2}\cap P,P)=0. \tag{28}$$

Hence, from (23) and (28), we have

$$i(A,(B_R\setminus\overline{B}_{\delta_2})\cap P,P) = i(A,B_R\cap P,P) - i(A,B_{\delta_2}\cap P,P) = 1.$$

We conclude that A has at least one fixed point $(B_R\setminus\overline{B}_{\delta_2})\cap P$, thus the problem (1)-(2) has at least one positive solution $(u,v)\in P\times P$ with $u(t)>0, v(t)>0,\ \forall t\in[0,1]$ and $\|u\|>0, \|v\|>0$. Thus completes the proof of Theorem 3.2 □

Theorem 3.3. *Assume that (A1), (A2), (A5) and (A6) hold. Then the problem* (1)-(2) *has at least two positive solutions* $(u_1(t),v_1(t)),(u_2(t),v_2(t)),\ t\in[0,1]$.

Proof. From (iii) of Lemma 2.4, we have $G_1(t,s) \le G_1(1,s) \le K_1$ and $G_2(t,s) \le G_2(1,s) \le K_2, \forall\ (t,s) \in [0,1] \times [0,1]$. Hence, from (A6), we have

$$\begin{aligned}(Au)(t) &= \int_0^1 G_1(t,s) f\left(s, \int_0^1 G_2(s,\tau) g(\tau, u(\tau)) d\tau\right) ds \\ &\le \int_0^1 G_1(t,s) f\left(s, K_2 \int_0^1 g(\tau, u(\tau)) d\tau\right) ds \\ &\le \int_0^1 G_1(t,s) f\left(s, m_0 \int_0^1 g(\tau, N) d\tau\right) ds \\ &< \frac{N}{m_0} \int_0^1 G_1(t,s) ds \le \frac{N}{m_0} K_1 \le N, \forall u \in \partial B_N \cap P,\ t \in [0,1],\end{aligned}$$

so, $\|Au\| < \|u\|$, for all $u \in \partial B_N \cap P$. By Theorem 2.1, we conclude that

$$i(A, B_N \cap P, P) = 1. \tag{29}$$

On the other hand, from (A2), (A5) and the proofs of Theorem 3.1 and Theorem 3.2, we know that there exists a sufficiently large $L > N$ and a sufficiently small δ_2 with $0 < \delta_2 < N$ such that

$$i(A, B_L \cap P, P) = 0,\ i(A, B_{\delta_2} \cap P, P) = 0. \tag{30}$$

From the relations (29) and (30), we obtain

$$\begin{aligned} i(A, (B_L \setminus \overline{B}_N) \cap P, P) &= i(A, B_L \cap P, P) - i(A, B_N \cap P, P) = -1, \\ i(A, (B_N \setminus \overline{B}_{\delta_2}) \cap P, P) &= i(A, B_N \cap P, P) - i(A, B_{\delta_2} \cap P, P) = 1. \end{aligned}$$

Then A has at least one fixed point u_1 in $(B_L \setminus \overline{B}_N) \cap P$ and has one fixed point u_2 in $(B_N \setminus \overline{B}_{\delta_2}) \cap P$ respectively. Therefore, the problem (1)-(2) has two distinct positive solutions $(u_1, v_1), (u_2, v_2) \in P \times P$ with $u_i(t) > 0, v_i(t) > 0$ for all $t \in (0,1)$ and $\|u_i\| > 0, \|v_i\| > 0, i = 1, 2$. The proof of Theorem 3.3 is completed. □

4. **Numerical Examples**

We now present three numerical examples illustrating, respectively, Theorems 3.1, 3.2 and 3.3.

Let $\nu_1 = 5.2, \nu_2 = 5.95$ and $\alpha = 1.5$.

We consider the system of fractional differential equations

$$\begin{aligned} D_{0^+}^{5.2} u(t) + f(t, v(t)) = 0,\ t \in (0,1) \\ D_{0^+}^{5.95} v(t) + f(t, u(t)) = 0,\ t \in (0,1) \end{aligned} \tag{31}$$

subject to the boundary conditions

$$\begin{aligned} u^{(i)}(0) = 0 = v^{(i)}(0),\ 0 \le i \le 4, \\ \text{and } \left[D_{0^+}^{1.5} u(t)\right]_{t=1} = 0 = \left[D_{0^+}^{1.5} v(t)\right]_{t=1} \end{aligned} \tag{32}$$

The Green's functions $G_1(t,s)$ and $G_2(t,s)$ of corresponding homogeneous BVPs are given by

$$G_1(t,s) = \begin{cases} \frac{t^{4.2}(1-s)^{2.7} - (t-s)^{4.2}}{\Gamma(5.2)}, & 0 \le s \le t \le 1 \\ \frac{t^{4.2}(1-s)^{2.7}}{\Gamma(5.2)}, & 0 \le t \le s \le 1 \end{cases}$$

and

$$G_2(t,s) = \begin{cases} \frac{t^{4.95}(1-s)^{3.45}-(t-s)^{4.95}}{\Gamma(5.95)}, & 0 \leq s \leq t \leq 1 \\ \frac{t^{4.95}(1-s)^{3.45}}{\Gamma(5.95)}, & 0 \leq t \leq s \leq 1. \end{cases}$$

We deduce that $\gamma_1 = 0.0544, \gamma_2 = 0.03235$ and $\gamma = 0.03235$.

4.1. **Example.** Let $f(t,v) = v^{1/2}, g(t,u) = u^3, p = \frac{1}{2}, q = \frac{1}{2}$. Then the assumptions (A2) and (A3) are satisfied; indeed, we have $f_\infty^i = 1, g_\infty^i = \infty, f_0^s = 1, g_0^s = 0$. Hence by Theorem 3.1, we deduce that the problem (31)-(32) has at least one positive solution.

4.2. **Example.** Let $f(t,v) = v^{1/2}, g(t,u) = u^{1/2}, q = \frac{1}{2}, r = \frac{1}{2}$. Then the assumptions (A4) and (A5) are satisfied; indeed, we have $f_\infty^s = 1, g_\infty^s = 0, f_0^i = \infty, g_0^i = \infty$. Hence by Theorem 3.2, we deduce that the problem (31)-(32) has at least one positive solution.

4.3. **Example.** Let $f(t,v) = v^2 + v^{1/2}$, $g(t,u) = u^3 + u^{1/2}$. We have $K_1 = 0.0307$, $K_2 = 0.0091$ then $m_0 = \max\{K_1, K_2\} = 0.0091$. The functions $f(t,v)$ and $g(t,u)$ are nondecreasing with respect to u, for any $t \in [0,1]$, and for $p = \frac{1}{2}$, the assumptions (A2) and (A5) are satisfied; indeed we obtain $f_\infty^i = \infty, g_\infty^i = \infty, f_0^i = \infty, g_0^i = \infty$. Take $N = 1$ then the assumption (A6) is satisfied. Hence, by Theorem 3.3, we deduce that the problem (31)-(32) has at least two positive solutions.

References

[1] Amann,H., (1976), Fixed point equation and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev., 18, pp.620-709.

[2] Agarwal,R.P. and O'Regan, D., (2000), Multiple solutions for a couple system of boundary value problems, Dyn. Contin. Discrete Impuls. Syst., 7, pp.97-106.

[3] Agarwal,R.P., Zhou,Y., and He,Y., (2010), Existence of fractional neutral functional differential equations, Comput. Math. Appl., 59, pp.1095-3554.

[4] Ahmad,B. and Ntouyas,S.K., (2012), A note on fractional differential differential equations with fractional separated boundary conditions, Abstr. Appl. Anal., Article ID 818703, pp.1-11.

[5] Bai,Z., (2010), On positive solutions of a nonlocal fractional boundary value problem, Nonlinear Anal., 72, pp.916-924.

[6] Bai,Z. and Lu,H., (2005), Positive solutions for boundary value problems of nonlinear fractional differential equations, J. Math. Anal. Appl., 311, pp.495-505.

[7] Balenu, D., Dielheelm,K., Scalas,E., and Trujillo,J.J., (2012), Fractional Calculus Models and Numerical Methods, (2008), Series on Complexity, Nonlinearity and Chaos, World Scientific, Boston.

[8] Das,S., Functional Fractional Calculus for System Identification and Controls, Springer, New York.

[9] Goodrich,C.S., (2010), Existence of a positive solution to a class of fractional differential equations, Appl. Math. Lett., 23, pp.1050-1055.

[10] Goodrich,C.S., (2011), Existence of a positive solution to systems of differential equations of fractional order, Comp. Math. Appl., 62, pp.1251-1268.

[11] Guo,D.J. and Lakshmikantham,V., (1988), Nonlinear Problems in Abstract Cones, Academic Press, New York.

[12] Henderson,J. and Luca,R., (2011), Positive solutions for a system of higher-order multipoint boundary value problems, Comput. Math. Appl., 62, pp.3920-3932.

[13] Henderson,J. and Luca,R., (2013), Positive solutions for a system of nonlinear fractional boundary value problems, Fract. Calc. Appl. Anal., 16(4), pp.985-1008.

[14] Henderson,J. and Luca,R., (2013), Existence and multiplicity for positive solutions of a system of higher-order multi-point boundary value problems, Nonlinear Differ. Equn. Appl., 20(3), pp.1035-1054.

[15] Henderson,J. and Luca,R., (2012), Positive solutions for a system of second-order multi-point boundary value problems, Appl. Math. Comput., 218, pp.6083-6094.

[16] Henderson,J. and Ntouyas,S.K., (2007), Positive solutions for system of nth order three-point nonlocal boundary value problems, Electron. J. Qual. Theory Differ. Equn., 18, pp.1-12.

[17] Henderson,J., Ntouyas,S.K. and Purnaras,I.K., (2008), Positive solutions for systems of second order four-point nonlinear boundary value problems, Comm. Appl. Anal., 12, pp.29-40.

[18] Khan,R.A., Rehman,M., and Henderson,J., (2011), Existence and uniqueness of solutions for nonlinear fractional differential equations with integral boundary conditions, Fractional. Diff. Calculus., 1, pp.29-43.

[19] Kilbas,A.A., Srivastava,H.M., and Trujillo,J.J., (2006), Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, 204, Elsevier Science B. V., Amsterdam.

[20] Kauffman, E.R. and Mboumi,E., (2008), Positive soltions of a boundary value problem for a nonlinear fractional differential equation, Electron. J. Qual. Theory Differ. Equn., 3, pp.1-11.

[21] Lakshmikantham,V., Leela, S. and Vasundara Devi, J., (2009), Theory of Fractional Dynamic Systems, Cambridge Scientific Publishers, Cambridge.

[22] Liang,S. and Zhang,J., (2009), Positive solutions for boundary value problems of nonlinear fractional differential equation, Nonlinear Anal., 71, pp.5545-5550.

[23] Miller,K.S. and Ross,B., (1993), An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York.

[24] Podlubny,I., (1999), Fractional Differential Equations., Academic Press, San Diego

[25] Prasad,K.R., Kameswararao,A., and Nageswararao,S., (2012), Existence of positive solutions for the system of higher order two-point boundary value problems, Proc. Indian Acad. Sci., 122(1), pp.139-152.

[26] Zhai,C. and Hao,M., (2014), Multiple positive solution to nonlinear bpundary value problems of a system for fractional differential equations, Scientific. World. J., ID 817542.

[27] Zhou,Y. and Xu,Y., (2006), Positive solutions of three-point boundary value problems for systems of nonlinear second order ordinary differential equations, J. Math. Anal. Appl., 320(2), pp.578-590.

Allaka Kameswara Rao for the photography and short autobiography, see TWMS J. App. Eng. Math., V.6, N.2, 2016.