SOLUTIONS FOR A DISCRETE BOUNDARY VALUE PROBLEM INVOLVING KIRCHHOFF TYPE EQUATION VIA VARIATIONAL METHODS

ZEHRA YÜCEDAĞ

Abstract. In this paper, Mountain Pass theorem is applied together with Ekeland variational principle, and we show the existence of nontrivial solutions for a discrete boundary value problem of \( p(k) \)-Kirchhoff-type in a finite dimensional Hilbert space.

Keywords: Kirchhoff type equation, discrete boundary value problem, Variational methods, Ekeland variational principle, Mountain Pass theorem.

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1. Introduction

We are concerned with the following problem

\[
\begin{aligned}
- M (A(k-1, \Delta u(k-1))) \Delta (a(k-1, \Delta u(k-1))) &= \lambda f(k, u(k)), \\
u(0) &= u(T+1) = 0, \quad k \in \mathbb{Z}[1,T],
\end{aligned}
\]  

where \( T \geq 2 \) is a positive integer, \( \mathbb{Z}[a,b] \) denotes the discrete interval \( \{a, a+1, \ldots, b\} \) with \( a \) and \( b \) are integers such that \( 0 < a < b \); \( \Delta u(k) = u(k+1) - u(k) \) is the forward difference operator; \( f: \mathbb{Z}[1,T] \times \mathbb{R} \to (0, +\infty) \) is a continuous function; \( \lambda \) is a positive constant.

Moreover, we assume that the function \( a(k, \xi): \mathbb{Z}[1,T] \times \mathbb{R} \to \mathbb{R} \) is continuous derivative with respect to \( \xi \) of the mapping \( A: \mathbb{Z}[1,T] \times \mathbb{R} \to \mathbb{R} \), \( A = A(k, \xi) \), i.e. \( a(k, \xi) = \nabla_\xi A(k, \xi) \); \( p: \mathbb{Z}[0,T] \to [2, \infty) \) satisfies

\[
p^- = \min_{k \in \mathbb{Z}[0,T]} p(k) \leq p^+ = \max_{k \in \mathbb{Z}[0,T]} p(k) < \infty.
\]

We suppose that \( f, M, a \) and \( A \) satisfy the following conditions:

\((M_1)\) \( M: (0, +\infty) \to (0, +\infty) \) is continuous function such that

\[
(1-\eta) s^{\alpha-1} \leq M(s) \leq (1+\eta) s^{\alpha-1},
\]

for all \( s > 0 \), \( 0 \leq \eta < 1 \) and \( \alpha \geq 1 \).

\((f_0)\) There exists a function \( q(k): \mathbb{Z}[1,T] \to [2, \infty) \) such that

\[
|f(k,t)| \leq c_0 \left(1 + |t|^{q(k)-1}\right),
\]

for all \( (k,t) \in \mathbb{Z}[1,T] \times \mathbb{R} \), where \( c_0 \) is pozitif constant.

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1. Dicle University, Faculty of Science, Department of Mathematics, 21280 Diyarbakir, Turkey. 
e-mail: zyucedag@dicle.edu.tr; ORCID: http://orcid.org/0000-0003-1950-0163. 
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(A1) There exists a constant $c_0$ such that

$$|a(k, \xi)| \leq c_0(1 + |\xi|^{p(k)-1}),$$

for all $(k, \xi) \in \mathbb{Z}[0, T] \times \mathbb{R}$.

(A2) The following inequality holds true

$$|\xi|^{p(k)} \leq a(k, \xi) \cdot \xi \leq p(k) A(k, \xi),$$

for all $(k, \xi) \in \mathbb{Z}[0, T] \times \mathbb{R}$.

(A3) The following inequality holds true

$$(a(k, \xi) - a(k, \eta)) \cdot (\xi - \eta) > 0,$$

for all $k \in \mathbb{Z}[0, T]$ and $\xi, \eta \in \mathbb{R}$ such that $\xi \neq \eta$.

(A4) $A(k, 0) = 0$, for all $k \in \mathbb{Z}[0, T]$.

Problem $(P)$ is related to the stationary version of a model, the so-called Kirchhoff equation, introduced by Kirchhoff [20]. To be more precise, Kirchhoff established a model given by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

where $\rho$, $P_0$, $h$, $E$, $L$ are constants, which extends the classical D’Alambert’s wave equation, by considering the effects of the changes in the length of the strings during the vibrations.

Discrete boundary value problems have been extensively studied by applying variational methods in the last few years because they can be used to models various phenomena arising from the study of elastic mechanics [30], electrorheological fluids [26] and image restoration [9]. We refer to the recent results of involving the discrete $p$–Laplacian operator and $p(k)$ – Laplacian operator [6, 7, 8, 9, 19, 22, 29]. The discrete problems of type $(P)$ involving anisotropic exponents have first been discussed by [24, 21]. Moreover, Koné and Guiro studied a more general operator in [21, 13]. In [24], by using critical point theory, the authors showed the existence of a continuous spectrum of eigenvalues for a discrete anisotropic problem. In [21], using minimization method, Koné and Ouaro obtained the existence and uniqueness of weak solutions for anisotropic discrete boundary value problems. Then, the authors studied the existence and multiplicity of positive solutions for a discrete anisotropic equation by variational methods and a critical point theory in [11, 28]. Moreover, many interesting results are obtained see for examples, in [2, 3, 4, 5, 12, 15, 16, 17, 18, 25].

In this paper, applying Mountain-Pass theorem together with Ekeland variational principle of Ambrosetti-Rabinovitz’s (see [24]), we obtain the existence of at least one nontrivial weak solution of an anisotropic discrete boundary value problem of $p(k)$-Kirchhoff type.

This paper is organized as follows. In Section 2, we present some necessary preliminary results. In Section 3, we get some existence results for $(P)$.

2. Preliminaries

Let us define the function space

$$W = \{ u : \mathbb{Z}[0, 1 + T] \to \mathbb{R}; \text{ such that } u(0) = u(T + 1) = 0 \}.$$
Then, $W$ is a $T$-dimensional Hilbert space $[1]$ with the inner product
\[ \langle u, v \rangle = \sum_{k=1}^{T+1} \Delta u(k-1) \Delta v(k-1), \forall u, v \in W. \]

Then, the associated norm is defined by
\[ \|u\| = \left( \sum_{k=1}^{T+1} |\Delta u(k-1)|^2 \right)^{1/2}. \]

On the other hand, it is useful to introduce other equivalent norms on $W$, namely
\[ |u|_m = \left( \sum_{k=1}^{T} |u(k)|^m \right)^{1/m}, \forall u \in W \text{ and } m \geq 2. \]

It can be verified $[7, 24]$ that
\[ T^{2-m/(2m)} |u|_2 \leq |u|_m \leq T^{1/m} |u|_2, \forall u \in W \text{ and } m \geq 2. \tag{2.1} \]

Lemma 2.1$[24]$
\[ (i) \text{ For every } u \in W \text{ with } \|u\| > 1, \text{we have} \]
\[ \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq T^{(2-p^-)/2} \|u\|^{p^-} - (T + 1). \]

\[ (ii) \text{ For every } u \in W \text{ with } \|u\| \leq 1, \text{we have} \]
\[ \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq T^{(p^+ - 2)/2} \|u\|^{p^+}. \]

\[ (iii) \text{ For any } m \geq 2 \text{ there exist a positive constant } c_m \text{ such that} \]
\[ \sum_{k=1}^{T} |u(k)|^m \leq c_m \sum_{k=1}^{T+1} |\Delta u(k-1)|^m, \forall u \in W. \]

Furthermore, from (2.1) and Lemma 2.1(iii), it reads
\[ |u|_m \leq T |u|_2 \leq c_m T \left( \sum_{k=1}^{T+1} |\Delta u(k-1)|^2 \right)^{\frac{m}{2}} = c_m T \|u\|_m^m. \tag{2.2} \]

Definition 2.2 Let $X$ be a real Banach spaces and let $I_\lambda$ be a functional such that $I_\lambda \in C(X, \mathbb{R})$. We say that $I_\lambda$ satisfies Palais-Smale condition ("(PS) condition for short") if any sequence $\{u_n\}$ in $X$ such that $\{I_\lambda(u_n)\}$ is bounded and $I_\lambda'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence.

Theorem 2.3 (Mountain-Pass lemma)(see $[27]$) Let $X$ be a Banach space and $I \in C^1(X, \mathbb{R})$ satisfy the Palais-Smale condition. Assume that $I(0) = 0$, and
\[ (i) \text{ There exist two positive real numbers } \gamma \text{ and } r \text{ such that } I(u) \geq r > 0, \text{ for all } u \in X \text{ with } \|u\| = \gamma. \]

\[ (ii) \text{ There exists } u \in X \text{ with } \|u\| > \gamma \text{ such that } I_\lambda(u) < 0. \]

Then, $I$ has a critical value $\beta \geq \alpha$. Moreover, $\beta$ can be characterized as
where \( G = \{ \varphi \in C([0, 1], E) : \varphi(0) = 0, \varphi(1) = u \} \).

3. The Main results

The energy functional corresponding to problem (P) is defined as

\[
I_\lambda(u) = \widetilde{M} \left( \sum_{k=1}^{T+1} A(k-1, \Delta u(k-1)) \right) - \lambda \sum_{k=1}^{T} F(k, u(k)),
\]

where \( k \in \mathbb{Z}[1, T] \), \( \widetilde{M}(t) = \frac{t}{\int_{0}^{t} M(s) \, ds} \) and \( F(k, t) = \frac{t}{\int_{0}^{t} f(k, \xi) \, d\xi} \).

A critical point to \( I_\lambda \) is a point \( u \in W \) such that

\[
M \left( \sum_{k=1}^{T+1} A(k-1, \Delta u(k-1)) \right) \sum_{k=1}^{T+1} a(k-1, \Delta u(k-1)) \Delta v(k-1)
\]

\[
- \lambda \sum_{k=1}^{T} f(k, u(k)) v(k), \forall k \in \mathbb{Z}[1, T],
\]

which in turn is a weak solution to (P) for any \( v \in W \). Since we work in a finite dimensional space, we see that any weak solution of (P) is in fact a strong, i.e. classical solution. Hence, in order to solve (P), we need to find critical points of \( I_\lambda \).

Lemma 3.1

(i) The functional \( I_\lambda \) is well defined on \( W \).

(ii) The functional \( I_\lambda \) is of class \( C^1(W, \mathbb{R}) \) and

\[
\left\langle I'_\lambda(u), v \right\rangle = M \left( \sum_{k=1}^{T+1} A(k-1, \Delta u(k-1)) \right) \sum_{k=1}^{T+1} a(k-1, \Delta u(k-1)) \Delta v(k-1)
\]

\[
- \lambda \sum_{k=1}^{T} f(k, u(k)) v(k), \forall k \in \mathbb{Z}[1, T],
\]

for all \( u, v \in W \).

Since the proof of Lemma 3.2 is very similar to that of the proof of Lemma 3.4 in [13], we omit it.

Lemma 3.2 [23] A verifies the following condition

\[
A(k, t\xi) \leq A(k, \xi) t^\rho(k), \text{ for all } (k, t) \in \mathbb{Z}[0, T] \times \mathbb{R}, t \geq 1.
\]

Theorem 3.3 Assume that \((M_1)\) and \((f_0)\) hold with \( q^+ < \alpha p^- \). Then, there exists \( \lambda_{**} > 0 \) such that for any \( \lambda \in (0, \lambda_{**}) \) the problem (P) has at least one nontrivial solution.

Proof. Let \( \|u\| > 1 \). Using the condition \((M_1)\), we have
\[ I_\lambda(u) = \frac{1}{\alpha} \left( \sum_{k=1}^{T+1} A(k-1, \Delta u(k-1)) \right)^\alpha - \frac{\lambda}{q} q^T \| u \|_{q^T} \quad \text{for all } u \in W, \lambda > 0. \]
Example 3.2 As an example of application of the Theorem 3.3, we consider the following: Let
\[ f(k, t) = |t|^{k+2}, \quad p(k - 1) = e^{k-1} + 5, q(k) = k + 3, M(t) = t, T = 2, \quad \alpha = 2, \quad c_q^+ = 1 \] and \( \eta = 1/2 \), then we have
\[ F(k, t) = \frac{1}{k+3} |t|^{k+3}, \quad p^- = 6, \quad p^+ = e + 5, \quad q^- = 4, \quad q^+ = 5 \] and \( \lambda_{ss} = 0, 134291561 \).
Thus, all the assumptions requested in Theorem 3.3 are provided.

Theorem 3.4 Assume that \((M_1), (f_0)\) hold. Suppose, additionally, that the following conditions hold:
\((f_1)\): \( f(k, t) = o(|t|^{\alpha p^+ - 1}), \quad t \to 0 \), with \( \alpha p^+ < q^- \) for \( \forall k \in \mathbb{Z}[1, T] \);
\((\text{AR})\): Ambrosetti-Rabinowitz’s condition holds, i.e. There exists constants \( \exists \epsilon > 0, 0 \leq \eta < 1 \)
\[ \theta > \frac{1 + \eta}{1 - \eta} \alpha p^+ \] such that
\[ 0 < \theta F(k, t) \leq f(k, t)t, \quad |t| \geq \epsilon, \quad \forall k \in \mathbb{Z}[1, T]. \]
Then, the problem \((P)\) has at least a nontrivial solution for any \( \lambda > 0 \).

Lemma 3.5 Assume that the assumptions of Theorem 3.4 hold. Then,
(i) There exist two positive real numbers \( \gamma \) and \( r \) such that \( I_\lambda(u) \geq r > 0, \quad u \in W \) with \( \|u\| = \gamma \).
(ii) There exists \( u \in W \) such that \( \|u\| > \gamma \) and \( I_\lambda(u) < 0 \).
Proof. (i) Let \( \|u\| < 1 \). Then, using condition \((M_1), (A2), (A4)\) and relation (2.2), we have
\[ I_\lambda(u) = M \left( \sum_{k=1}^{T+1} A(k - 1, \Delta u(k - 1)) \right) - \lambda \sum_{k=1}^{T} F(k, u(k)) \] \[ \geq \frac{(1 - \eta)}{\alpha (p^+)^\alpha} T^{(\frac{2 - p^-}{p^-})\alpha} \|u\|^\alpha p^+ - \lambda \sum_{k=1}^{T} F(k, u(k)) \] From \((f_0)\) and \((f_1)\), we can write the following inequality
\[ F(k, t) \leq \epsilon |t|^{\alpha p^+} + c_1 |t|^q(k), \] where \( c_1 \) is pozitif constant and \( t \in \mathbb{R} \). Let \( \epsilon > 0 \) be small enough such that \( \lambda \epsilon c_\alpha T \leq \frac{1 - \eta}{2\alpha (p^+)^\alpha} T^{(\frac{2 - p^-}{p^-})\alpha} \). Thus, considering also inequality (2.2), (3.2) and Lemma 2.1(ii) we obtain
\[ I_\lambda(u) \geq \frac{(1 - \eta)}{\alpha (p^+)^\alpha} T^{(\frac{2 - p^-}{p^-})\alpha} \|u\|^\alpha p^+ - \lambda \epsilon \sum_{k=1}^{T} |u(k)|^{\alpha p^+} - \lambda c_1 \sum_{k=1}^{T} |u(k)|^{q(k)} \] \[ \geq \frac{(1 - \eta)}{\alpha (p^+)^\alpha} T^{(\frac{2 - p^-}{p^-})\alpha} \|u\|^\alpha p^+ - \lambda \epsilon c_\alpha T \|u\|^\alpha p^+ - \lambda c_1 \left( c_q^+ T \|u\|^{q^+} + c_q^- T \|u\|^{q^-} \right) \] \[ \geq \frac{1 - \eta}{2\alpha (p^+)^\alpha} T^{(\frac{2 - p^-}{p^-})\alpha} \|u\|^\alpha p^+ - c_2 c_q^- T \|u\|^{q^-}. \]
Since \( \|u\| < 1 \) and \( \alpha p^+ < q^- \), there exist two positive real numbers \( \gamma \) and \( r \) such that \( I_\lambda (u) \geq r > 0, u \in W \) with \( \|u\| = \gamma \in (0, 1) \).

(ii) From \((AR)\) and for each \( t \geq 1 \), we can write

\[
F(k, tu) \geq t^\theta F(k, u), \quad \forall k \in \mathbb{Z}[1, T].
\]

Thus, for \( \psi \in W, \psi \neq 0 \) and \( t > 1 \), we have

\[
I_\lambda (t\psi) = M \left( \frac{\sum_{k=1}^{T+1} A(k-1, \Delta t\psi(k-1))}{\sum_{k=1}^{T} F(k, t\psi(k))} \right) - \lambda \sum_{k=1}^{T} F(k, t\psi(k)) \\
\leq \frac{1 + \eta}{\alpha (p^-)^{\frac{1}{2}}} t^{\frac{1}{2} + \eta} \theta^\alpha \left( \frac{\sum_{k=1}^{T+1} A(k-1, \Delta \psi(k-1))}{\sum_{k=1}^{T} A(k-1, \Delta \psi(k-1))} \right)^\alpha - \lambda c_3 t^\theta \sum_{k=1}^{T} F(k, \psi(k)).
\]

Since \( \theta > \frac{1 + \eta}{\alpha} p^+ \), it can be obtained that \( I_\lambda (t\psi) \to -\infty \) as \( t \to +\infty \).

**Lemma 3.6** Assume that the conditions \((M_1)\), \((f_0)\) and \((AR)\) hold with \( \alpha p^+ < q^- \). Then, for any \( \lambda > 0 \) the functional \( I_\lambda \) satisfies Palais-Smale condition.

**Proof.** First, we deduce the existence of a sequence \( \{u_n\} \subset W \) such that

\[
|I_\lambda (u_n)| \leq c \quad \text{and} \quad I'_\lambda (u_n) \to 0 \quad \text{as} \quad n \to \infty.
\]

We prove that \( \{u_n\} \) is bounded in \( W \). Arguing by contradiction and passing to a subsequence, we have \( \|u_n\| \to \infty \) as \( n \to \infty \). Thus, we may assume that for \( n \) large enough, we have \( \|u_n\| > 1 \). Moreover, using the conditions \((M_1)\), \((AR)\), \((f_0)\), and relations \((A2)\), \((A4)\), we can write

\[
c + \|u_n\| \geq I_\lambda (u_n) - \frac{1}{\theta} \langle I'_\lambda (u_n), u_n \rangle \\
\geq (1 - \eta) \int_0^T s^{\alpha-1} ds - \frac{(1 + \eta) p^+}{\theta} \left( \sum_{k=1}^{T+1} A(k-1, \Delta u_n(k-1)) \right)^\alpha \\
+ \lambda \left( \frac{1}{\theta} \sum_{k=1}^{T} f(k, u_n(k))u_n(k) - \sum_{k=1}^{T} F(k, u_n(k)) \right) \\
\geq \left( \frac{1 - \eta}{\alpha} - \frac{(1 + \eta) p^+}{\theta} \right) \left( \sum_{k=1}^{T+1} A(k-1, \Delta u_n(k-1)) \right)^\alpha,
\]

for \( n \) large enough. From \((A2)\), \((AR)\) and Lemma 2.1 (i), we obtain

\[
c + \|u_n\| \geq \frac{1}{(p^-)^\alpha} \left( \frac{1 - \eta}{\alpha} - \frac{(1 + \eta) p^+}{\theta} \right) T^{(2 - p^-)/2} \|u\|^{p^-} - K(\alpha, T) T^\alpha.
\]

Dividing the last inequality above by \( \|u\|^{p^-} \), and passing to the limit as \( n \to \infty \), we infer that \( \{\|u_n\|\} \) is bounded in \( W \). This information combined with the fact that \( W \) is a finite dimensional Hilbert space imply that there exists a subsequence, still denoted by \( \{u_n\} \), and \( u_0 \in W \) such that \( \{u_n\} \) converges to \( u_0 \) in \( W \). Thus, \( I_\lambda \) satisfies \((PS)\) condition.

**Proof of Theorem 3.4** From Lemma 3.5, Lemma 3.6 and the fact that \( I_\lambda (0) = 0 \), \( I_\lambda \) satisfies the conditions of Theorem 2.3. Therefore, the problem \((P)\) has at least one nontrivial solution.
Example 3.3 We consider the function
\[ f(k, t) = |t|^q(k-2) t \] for all \((k, t) \in \mathbb{Z} [1, T] \times \mathbb{R}.
From the above definition of \(f\), we get \(F(k, t) = \frac{1}{q(k)} |t|^q(k)\). If we take
\[ p(k - 1) = k + 1, \quad q(k) = 2(k + 1), \quad M(t) = 2t, \quad T = 2, \alpha = 1 \quad \text{and} \quad \eta = 0, 1316309013, \]
we obtain
\[ f(1, t) = t^3 \text{and} \quad f(2, t) = t^5, \]
we get
\[ p^- = 2, \quad p^+ = 3, \quad q^- = 4, \quad q^+ = 6, \quad F(1, t) = t^4/4 \text{and} \quad F(2, t) = t^6/6. \]
Hence, all the assumptions requested in Theorem 3.4 hold.

Theorem 3.7 Assume that \((M_1)\) and \((f_0)\) hold with \(q^- < \alpha p^-\). Then, there exists \(\lambda_* > 0\) such that for any \(\lambda \in (0, \lambda_*)\) the problem \((P)\) has at least one solution.

Lemma 3.8 Assume that the assumptions of Theorem 3.7 hold. Then, there exist \(\eta, a > 0\) and \(\lambda_* > 0\) such that for any \(\lambda \in (0, \lambda_*)\), we have
\[ I\_\lambda (u) \geq a > 0, \forall u \in W \quad \text{with} \quad \|u\| = \eta. \]

Proof. Let \(\|u\| < 1\). Using conditions \((M_1), (f_0)\) and relations \((A.2), (A.4), (2.2)\), we obtain that for \(u \in W\) with \(\|u\| = \eta\) the following inequalities hold true
\[ I\_\lambda (u) = \int_{k=1}^{T} A(k-1, \Delta u(k-1)) - \lambda \sum_{k=1}^{T} F(k, u(k)) \]
\[ \geq \frac{(1-\eta)}{\alpha (p^+)^\alpha} \|u\|^{\alpha p^+} - \lambda \frac{c_2 c^- q^-}{q^-} \|u\|^q \]
\[ = \left( c_3 \beta^{\alpha p^- - q^-} - \lambda c_4 \right) \eta^q, \]
where \(c_3\) and \(c_4\) are positive constants. If we use (3.3) and \(q^- < \alpha p^- \leq \alpha p^+\), and choose \(\lambda_*\) as
\[ \lambda_* = \frac{c_3 \eta^{\alpha p^- - q^-}}{2c_4}, \]
then for any \(\lambda \in (0, \lambda_*)\) and \(\forall u \in W\) with \(\|u\| = \eta\) there exists \(a = \frac{c_2 c^- q^-}{2}\) such that \(I\_\lambda (u) \geq a > 0\).

Lemma 3.9 Assume that \((M_1)\) and \((f_0)\) hold with \(q^- < \alpha p^-\). Then, there exists \(\varphi \in W\) such that \(\varphi \geq 0, \varphi \neq 0\) and \(I\_\lambda (t \varphi) < 0\), for \(t > 0\) small enough.

Proof. For any fixed \(\varphi \in W, \varphi \neq 0\) and each \(t \in (0, 1)\), using conditions \((M_1), (f_0)\) and Lemma 2.1(ii), we have
\[ I\_\lambda (t \varphi) = \int_{k=1}^{T} A(k-1, \Delta t \varphi(k-1)) - \lambda \sum_{k=1}^{T} F(k, t \varphi(k)) \]
\[ \leq \frac{1 + \eta}{\alpha} \left( \sum_{k=1}^{T} A(k-1, \Delta \varphi(k-1)) \right)^{\alpha} - \lambda \frac{c_1 t^{\alpha p^+}}{q^+} \sum_{k=1}^{T} \frac{|\varphi(k)|^{q(k)}}{m(k)} \]
\[ \leq \frac{1 + \eta}{\alpha} \left( \sum_{k=1}^{T} A(k-1, \Delta \varphi(k-1)) \right)^{\alpha} - \lambda \frac{c_1 t^{\alpha p^-}}{q^-} \sum_{k=1}^{T} |\varphi(k)|^{q(k)}. \]
Thus, \( I_{\lambda}(t\varphi) < 0 \), for all \( t < \delta \) with
\[
0 < \delta < \min \left\{ 1, \lambda \alpha c_1 \sum_{k=1}^{T} |\varphi(k)|^{m(k)} / \left( (1 + \eta) q^{-} \left( \sum_{k=1}^{T+1} A(k-1, \Delta \varphi(k-1)) \right)^{\alpha} \right) \right\}.
\]
The proof of Lemma 3.9 is complete.

**Proof of Theorem 3.7** By Lemma 3.8, we infer that there exists a ball centered at the origin \( B \subset W \), such that
\[
\inf_{\partial B} \rho(0) I_{\lambda} > 0.
\]
Moreover, from Lemma 3.9, there exists \( \varphi \in W \) such that \( I_{\lambda}(t\varphi) < 0 \), for all \( t > 0 \) small enough. Thus, by taking into account (3.3), we obtain the following
\[
-\infty < c := \inf_{B} \rho(0) I_{\lambda} < 0.
\]
Let choose \( \varepsilon > 0 \). Then, it follows
\[
0 < \varepsilon < \inf_{\partial B} \rho(0) I_{\lambda} - \inf_{B} \rho(0) I_{\lambda}.
\]
Applying Ekeland’s variational principle [10] to the functional \( I_{\lambda} : B_{\rho}(0) \to \mathbb{R} \), we can find \( u_{\varepsilon} \in B_{\rho}(0) \) such that
\[
I_{\lambda}(u_{\varepsilon}) < \inf_{B_{\rho}(0)} I_{\lambda} + \varepsilon \quad \text{and} \quad I_{\lambda}(u_{\varepsilon}) < I_{\lambda}(u) + \varepsilon \|u - u_{\varepsilon}\|, \quad u \neq u_{\varepsilon}.
\]
By the fact that
\[
I_{\lambda}(u_{\varepsilon}) < \inf_{B_{\rho}(0)} I_{\lambda} + \varepsilon < \inf_{\partial B_{\rho}(0)} I_{\lambda} + \varepsilon < \inf_{B_{\rho}(0)} I_{\lambda},
\]
we can infer that \( u_{\varepsilon} \in B_{\rho}(0) \). Now, we define \( \phi_{\lambda} : B_{\rho}(0) \to \mathbb{R} \) by \( \phi_{\lambda}(u) = I_{\lambda}(u) + \varepsilon \|u - u_{\varepsilon}\| \). It is clear that \( u_{\varepsilon} \) is a minimum point of \( \phi \), and thus
\[
\frac{\phi_{\lambda}(u_{\varepsilon} + tv) - \phi_{\lambda}(u_{\varepsilon})}{t} \geq 0,
\]
for \( t > 0 \) small enough and any \( v \in B_{1}(0) \). By the above relation, we have
\[
\frac{I_{\lambda}(u_{\varepsilon} + tv) - I_{\lambda}(u_{\varepsilon})}{t} + \varepsilon \|v\| \geq 0.
\]
Letting \( t \to 0 \), we have that \( \langle I_{\lambda}(u_{\varepsilon}), v \rangle + \varepsilon \|v\| > 0 \), and hence, we infer that \( \|I_{\lambda}'(u_{\varepsilon})\| \leq \varepsilon \). The information obtained so far shows that there exists a sequence \( \{u_{n}\} \subset B_{\rho}(0) \) such that
\[
I_{\lambda}(u_{n}) \to c = \inf_{B_{\rho}(0)} I_{\lambda} < 0 \quad \text{and} \quad I_{\lambda}'(u_{n}) \to 0.
\]
Since the sequence \( \{u_{n}\} \) is bounded in \( W \), there exists \( u \in W \) such that, up to a subsequence, \( \{u_{n}\} \) converges to \( u \) in \( W \). So, we conclude that \( I_{\lambda} \) has at least one nontrivial critical point, i.e., the problem \( (P) \) has a nontrivial solution.

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REFERENCES


Zehra Yucedag is an assistant professor of mathematics at Dicle University. His research interests are Variable Exponent Lebesgue and Sobolev Spaces; Differential equations with variable exponents; Calculus of variations. He has been doing research in the mentioned areas for over ten years.