NEW CLASSES OF HARMONIC FUNCTIONS DEFINED BY FRACTIONAL OPERATOR

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ABSTRACT. In the present study, we introduce an investigation of new subclasses of harmonic functions which are defined by fractional operator. Firstly, using by fractional operator, we define new subclasses of harmonic functions. Later, we obtain main theorems of our study which contain sufficient and necessary coefficient bounds for functions related to the classes newly defined. Also, several particular characterization properties of these classes are given. Some of these properties involve extreme points, convex combination, distortion bounds. Finally, several corollaries of the main theorems are presented.

Keywords: Harmonic functions, fractional operator, coefficient estimates, univalent.

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1. INTRODUCTION

Let \( u, v \) be real harmonic functions in the simply connected domain \( \Omega \), then the continuous function \( f = u + iv \) defined in \( \Omega \) is said to be harmonic in \( \Omega \). In any simply connected domain \( \Omega \subset \mathbb{C} \) we can write \( f = h + g \), where \( h \) and \( g \) are analytic in \( \Omega \). A necessary and sufficient condition for \( f \) to be locally univalent and sense-preserving in \( \Omega \) is that \( |h'(z)| > |g'(z)| \) in \( \Omega \) (see [5]). Let \( H \) be the family of functions \( f = h + g \) which are harmonic univalent and sense-preserving in the open unit disk \( U = \{ z : |z| < 1 \} \) that \( f = h + \overline{g} \) normalized by \( f(0) = h(0) = f'(0) - 1 = 0 \). Let \( HW \) be the set harmonic univalent and sense-preserving functions in \( U \) [1] of the form \( f(z) = h(z) + g(z) \), where

\[
\begin{align*}
  h(z) &= z + \sum_{n=2}^{\infty} a_n z^n, \\
  g(z) &= \sum_{n=1}^{\infty} b_n z^n \quad |b_1| < 1
\end{align*}
\]  

are analytic in \( U \).

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Let \( HW \) be the subclass of \( HW \) consisting of functions of the form \( f(z) = h(z) + g(z) \), where

\[
h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^n \quad |b_1| < 1.
\]  

In 1984 Clunie and Sheil-Small [5] investigated the class \( SH \) as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several related papers on \( SH \) and its subclasses studied by Avcı and Zlotkiewicz [3], Silverman [11], Silverman and Silvia [13], Jahangiri [6] and Jahangiri et al. [7]. Furthermore, some other authors (e.g. see [2] and [14]) have recently studied the harmonic univalent functions using by certain operators. In current study, a new subclasses of harmonic functions has been investigated by using fractional operator especially.

A fairly complete treatment, with applications of the fractional calculus, is given in the books [10] by Oldham and Spanier, and [9] by Miller and Ross. We refer to [12] for more insight into the concept of the fractional calculus. For further details on the materials in this paper see [8].

For convenience, we shall remind some definitions of the fractional calculus (i.e., fractional integral and fractional derivative).

The fractional integral of order \( \lambda \) for analytical function \( f(z) \) in \( D \) is defined by

\[
D^{-\lambda}_z f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta, \quad (\lambda > 0)
\]  

where the multiplicity of \((z-\zeta)^{(-\lambda-1)}\) is removed by requiring \( \text{log}(z-\zeta) \) to be real when \((z-\zeta) > 0\).

The fractional derivative of order \( \lambda \) for an analytic function \( f(z) \) in \( D \) is defined by

\[
D^\lambda_z f(z) = \frac{d}{dz}(D^{-\lambda}_z f(z)) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta, \quad (0 \leq \lambda < 1),
\]  

where the multiplicity of \((z-\zeta)^{-\lambda}\) is removed by requiring \( \text{log}(z-\zeta) \) to be real when \((z-\zeta) > 0\).

Under the hypothesis of the fractional derivative, the fractional derivative of order \((n+\lambda)\) for an analytic function \( f(z) \) in \( D \) is defined by

\[
D^{n+\lambda}_z f(z) = \frac{d^n}{dz^n}(D^\lambda_z f(z)), \quad (0 \leq \lambda < 1, n \in N_0 = \{0, 1, 2, \ldots\}).
\]  

From the definitions of the fractional calculus, we see that

\[
D^{-\lambda}_z z^k = \frac{\Gamma(k+1)}{\Gamma(k+1+\lambda)} z^{k+\lambda}, \quad (\lambda > 0, k > 0)
\]  

\[
D^\lambda_z z^k = \frac{\Gamma(k+1)}{\Gamma(k+1+\lambda)} z^{k-\lambda}, \quad (0 \leq \lambda < 1, k > 0)
\]
and
\[ D_z^{n+\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\lambda-n)} z^{k-n-\lambda}, \quad (0 \leq \lambda < 1, k > 0, n \in N_0, k-n \neq -1, -2,...). \] (8)

Therefore, we see that for any real \( \lambda \),
\[ D_z^\lambda z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)} z^{k-\lambda}, \quad (k > 0, k-\lambda \neq -1, -2,...). \] (9)

Using the rule of the fractional derivative, which is mentioned in the preceding, Aydo˘gan et al. [4] defined the fractional operator as follows,
\[ f(z) = z + a_2 z^2 + ... + a_n z^n + ... \Rightarrow D_z^\lambda f(z) = D_z^\lambda (z + a_2 z^2 + ... + a_n z^n + ...), \]
\[ D_z^\lambda f(z) = \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z) = z + \sum_{n=2}^\infty \frac{\Gamma(2-\lambda) \Gamma(n+1)}{\Gamma(n+1-\lambda)} a_n z^n. \] (10)

From the definition of \( D_z^\lambda f(z) \) we have the following properties:

\( i.\) \( D_z^\lambda f(z) = D f(z) = \lim_{\lambda \to 1} D_z^\lambda f(z) = z f'(z); \)

\( ii.\) \( D_z^\lambda (D_z^\delta f(z)) = D_z^\delta (D_z^\lambda f(z)) = z + \sum_{n=2}^\infty \frac{\Gamma(2-\lambda) \Gamma(2-\delta) \Gamma(n+1)}{\Gamma(n+1-\lambda) \Gamma(n+1-\delta)} a_n z^n; \)

\( iii.\) \( D(D_z^\delta f(z)) = z + \sum_{n=2}^\infty \frac{\Gamma(2-\lambda) \Gamma(n+1)}{\Gamma(n+1-\lambda)} a_n z^n = z(D_z^\delta f(z))' = \Gamma(2-\lambda) z^\lambda (\lambda D_z^\lambda f(z) + z D_z^{\lambda+1} f(z)); \)

\( iv.\) \( \frac{D(D_z^\lambda f(z))}{D_z^\lambda f(z)} = \frac{z f'(z)}{f(z)}, \text{ for } \lambda = 0 \)
\[ = 1 + z \frac{f''(z)}{f'(z)}, \text{ for } \lambda = 1. \]

Thus, we define the following class of functions.

**Definition 1.1.** Let \( f(z) = h(z) + g(z) \), be the harmonic univalent function given by Eq.(1), then \( f \in HW(\lambda, \beta, t, \alpha) \) if and only if
\[ \Re \left\{ (1-\beta)(1-t) \frac{D_z^\lambda f(z)}{z} + (\beta + t) \frac{(D_z^\lambda f(z))'}{z'} + \beta t \frac{(D_z^\lambda f(z))''}{z''} - 2\beta t \right\} \geq \alpha \] (11)
where \( 0 \leq \alpha < 1, \beta \geq 0, \quad 0 \leq t \leq 1 \) and \( z = re^{i\theta} \in U. \)

We also let \( \overline{HW}(\lambda, \beta, t, \alpha) = HW(\lambda, \beta, t, \alpha) \cap \overline{H}. \)

When we take \( \beta = 0 \), the class of \( HW(\lambda, \beta, t, \alpha) \) reduces to the class of harmonic functions \( HG(\lambda, t, \alpha) \equiv HW(\lambda, 0, t, \alpha) \) as follows
\[ \Re \left\{ (1-t) \frac{D_z^\lambda f(z)}{z} + t \frac{(D_z^\lambda f(z))'}{z'} \right\} \geq \alpha. \] (12)

When we take \( \beta = 1 \), the class of \( HW(\lambda, \beta, t, \alpha) \) reduces to the class of harmonic functions \( HA(\lambda, t, \alpha) \equiv HW(\lambda, 1, t, \alpha) \) class of harmonic functions \( HA(\lambda, t, \alpha) \) as follows:
\[ \Re \left\{ (1+t) \left( \frac{D^\lambda f(z)}{z'} + t \frac{(D^\lambda f(z))'}{z''} - 2t \right) \right\} \geq \alpha. \] (13)

2. COEFFICIENT BOUNDS

Now, firstly we presented sufficient coefficient conditions for functions to be in the class of \( \mathcal{H}_W(\lambda, \beta, t, \alpha) \). Furthermore, it is shown that these conditions are also necessary for the functions of \( \mathcal{H}_W(\lambda, \beta, t, \alpha) \).

**Theorem 2.1.** Let \( f = h + \bar{g} \), \( h \) and \( g \) be given by (1) and

\[
\sum_{n=2}^{\infty} |(\beta + t)(n-1) + \beta t(1+n^2) + 1| \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} |a_n| \\
+ \sum_{n=1}^{\infty} |(\beta + t)(n+1) - \beta t(1+n^2) - 1| \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} |b_n| \leq 1 - \alpha.
\]

(14)

Then \( f(z) \in \mathcal{H}_W(\lambda, \beta, t, \alpha) \).

**Proof.** Using the fact that \( \Re w \geq \alpha \) if and only if \( |w + 1 - \alpha| \geq |w - 1 - \alpha| \)

\[
w = (1 - \beta)(1 - t) \frac{D^\lambda f(z)}{z'} + (\beta + t) \frac{(D^\lambda f(z))'}{z''} + \beta t \frac{(D^\lambda f(z))''}{z'''} - 2\beta t
\]

It is enough to show that \( |w + 1 - \alpha| - |w - 1 - \alpha| \geq 0. \)

Now, we have

\[
|w + 1 - \alpha| = \left| (1 - \beta)(1 - t) \left( 1 + \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} a_n z^{n-1} + \sum_{n=1}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} b_n(z)^n \right) \right|
\]

\[
+ (\beta + t) \left( 1 + \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} (na_n z^{n-1} - \sum_{n=1}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} nb_n(z)^n) \right)
\]

\[
+ \beta t \left( 1 + \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} na_n z^{n-1} + \sum_{n=1}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} (n(n-1)a_n z^{n-1}) \right)
\]

\[
+ \sum_{n=1}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} nb_n(z)^n - \sum_{n=1}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} (n(n-1)b_n(z)^n) \right) - 2\beta t + 1 - \alpha \right| \\
\geq 2 - \alpha - \sum_{n=2}^{\infty} |1 + (\beta + t)(n-1) + \beta t(1+n^2)| \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} |a_n||z^{n-1}|
\]

\[
- \sum_{n=1}^{\infty} |1 - (\beta + t)(n+1) + \beta t(1+n^2)| \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} |b_n||z^{n-1}|
\]

and

\[
|w - 1 - \alpha| = \left| (1 - \beta)(1 - t) \left( 1 + \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} a_n z^{n-1} + \sum_{n=1}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} b_n(z)^n \right) \right|
\]
Theorem 2.2. Let 
\[ + (\beta + t) \left( 1 + \sum_{n=2}^{\infty} \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} n a_n z^{n-1} - \sum_{n=1}^{\infty} \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} n b_n(\overline{z})^{n-1} \right) \]

\[ + \beta t \left( 1 + \sum_{n=2}^{\infty} \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} n a_n z^{n-1} + \sum_{n=2}^{\infty} \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} n (n - 1) a_n z^{n-1} \right) \]

\[ + \sum_{n=1}^{\infty} \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} n b_n(\overline{z})^{n-1} + \sum_{n=1}^{\infty} \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} n (n - 1) b_n(\overline{z})^{n-1} \right) \] - 2 \beta t - 1 - \alpha \right| \]

\[ \leq \alpha + \sum_{n=2}^{\infty} |1 + (\beta + t)(n - 1) + \beta t(1 + n^2)| \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} |a_n| |z^{n-1}| \]

\[ + \sum_{n=1}^{\infty} |1 - (\beta + t)(n + 1) + \beta t(1 + n^2)| \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} |b_n| |z^{n-1}|. \]

So by using (14) we have

\[ |w+1-\alpha|-|w-1-\alpha| \geq 2 \left[ 1 - \alpha - \sum_{n=2}^{\infty} |(\beta + t)(n - 1) + \beta t(1 + n^2) + 1| \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} |a_n| \right. \]

\[ - \sum_{n=1}^{\infty} |(\beta + t)(n + 1) - \beta t(1 + n^2) - 1| \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} |b_n| \right] \geq 0 \]

and so the proof is completed. \qed

Remark 1 The coefficient bound (14) in previous theorem is sharp for the function

\[ f(z) = z + \sum_{n=2}^{\infty} \frac{u_n}{|\lambda + (\beta + t)(n - 1) + \beta t(1 + n^2) + 1|} \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} z^n \]

\[ + \sum_{n=1}^{\infty} \frac{w_n}{|\lambda + (\beta + t)(n + 1) - \beta t(1 + n^2) - 1|} \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} (\overline{z})^n, \] (15)

where

\[ \frac{1}{1 - \alpha} \left( \sum_{n=2}^{\infty} |u_n| + \sum_{n=1}^{\infty} |w_n| \right) = 1. \]

Theorem 2.2. Let \( f = h + \overline{g} \in \overline{H}(\lambda, \beta, t, \alpha) \) if and only if

\[ \sum_{n=2}^{\infty} |(\beta + t)(n - 1) + \beta t(1 + n^2) + 1| \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} |a_n| \]

\[ + \sum_{n=1}^{\infty} |(\beta + t)(n + 1) - \beta t(1 + n^2) - 1| \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} |b_n| \leq 1 - \alpha. \] (16)
Proof. From Theorem 2.1 and since $\overline{HW}(\lambda, \beta, t, \alpha) \subset HW(\lambda, \beta, t, \alpha)$ we conclude the "if" part. For the "only if" part, assume that $f(z) \in \overline{HW}(\lambda, \beta, t, \alpha)$. Therefore, for $z = re^{i\theta} \in U$, we have

$$\Re \left\{ (1 - \beta)(1 - t) \frac{D^\lambda f(z)}{z} + (\beta + t) \left( \frac{D^\lambda f(z)}{z'} + \beta t \frac{(D^\lambda f(z))'}{z'} - 2\beta \right) \right\}$$

$$= \Re \left\{ (1 - \beta)(1 - t) \left( 1 + \sum_{n=2}^{\infty} \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} a_n z^{n-1} + \sum_{n=1}^{\infty} \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} b_n(z)^{n-1} \right) \right\}$$

$$+ (\beta + t) \left( 1 - \sum_{n=2}^{\infty} \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} n a_n z^{n-1} - \sum_{n=1}^{\infty} \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} n(n - 1) a_n z^{n-1} \right)$$

$$+ \sum_{n=1}^{\infty} \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} n b_n(z)^{n-1} - \sum_{n=1}^{\infty} \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} n(n - 1) b_n(z)^{n-1} \right) - 2\beta \right\}$$

$$= \Re \left\{ 1 - \sum_{n=2}^{\infty} [(1 - \beta)(1 - t) + (\beta + t)n + n\beta t + n(n - 1)\beta t] \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} a_n z^{n-1} \right\}$$

$$+ \sum_{n=1}^{\infty} [(1 - \beta)(1 - t) - (\beta + t)n + n\beta t + n(n - 1)\beta t] \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} b_n(z)^{n-1} \right\}$$

$$\geq 1 - \sum_{n=2}^{\infty} |(\beta + t)(n - 1) + \beta t(1 + n^2) + 1| \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} |a_n| r^{n-1}$$

$$- \sum_{n=1}^{\infty} |(\beta + t)(n + 1) - \beta t(1 + n^2) - 1| \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} |b_n| r^{n-1} \geq \alpha.$$
Corollary 2.2. If $f = h + g \in \overline{HA}(\lambda, t, \alpha) \equiv HA(\lambda, t, \alpha) \cap \overline{HW}$ if and only if
\[
\sum_{n=2}^{\infty} |n(1 + nt)| \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} |a_n| + \sum_{n=1}^{\infty} |n(1 + nt)| \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} |b_n| \leq 1 - \alpha.
\]

Especially for $\lambda = 0$
\[
\sum_{n=2}^{\infty} |n(1 + nt)||a_n| + \sum_{n=1}^{\infty} |n(1 + nt)||b_n| \leq 1 - \alpha.
\]

3. Extreme Points

In the following theorem, we represent extreme points of $\overline{HW}(\lambda, \beta, t, \alpha)$.

Theorem 3.1. If $f = h + g \in \overline{HW}(\lambda, \beta, t, \alpha)$ if and only if it can be expressed as
\[
f(z) = X_1z + \sum_{n=2}^{\infty} X_nh_n(z) + \sum_{n=1}^{\infty} Y_ng_n(z), z \in U \tag{17}
\]

where
\[
h_n(z) = z - \frac{1 - \alpha}{[(\beta + t)(n - 1) + \beta t(1 + n^2) + 1] \Gamma(2 - \lambda)\Gamma(n + 1)} z^n \quad (n = 2, 3, \ldots ) \tag{18}
\]
\[
g_n(z) = z + \frac{1 - \alpha}{[(\beta + t)(n + 1) - \beta t(1 + n^2) - 1] \Gamma(2 - \lambda)\Gamma(n + 1)} (\bar{z})^n \quad (n = 1, 2, \ldots ) \tag{19}
\]

$X_1 \geq 0, \ Y_1 \geq 0, \ \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n = 1 \quad X_n \geq 0, \ Y_n \geq 0 \quad \text{for} \quad n = 2, 3, \ldots$.

Proof. If $f(z)$ be given by (17), then
\[
f(z) = z - \sum_{n=2}^{\infty} \frac{1 - \alpha}{[(\beta + t)(n - 1) + \beta t(1 + n^2) + 1] \Gamma(2 - \lambda)\Gamma(n + 1)} X_n z^n + \sum_{n=1}^{\infty} \frac{1}{[(\beta + t)(n + 1) - \beta t(1 + n^2) - 1] \Gamma(2 - \lambda)\Gamma(n + 1)} Y_n (\bar{z})^n.
\]

Since by (16), we have
\[
\sum_{n=2}^{\infty} \frac{1}{[(\beta + t)(n - 1) + \beta t(1 + n^2) + 1]} \frac{1 - \alpha}{\Gamma(2 - \lambda)\Gamma(n + 1)} |X_n| + \sum_{n=1}^{\infty} \frac{1}{[(\beta + t)(n + 1) - \beta t(1 + n^2) - 1]} \frac{1 - \alpha}{\Gamma(2 - \lambda)\Gamma(n + 1)} |Y_n|
\]
\[
= (1 - \alpha) \left( \sum_{n=2}^{\infty} |X_n| + \sum_{n=1}^{\infty} |Y_n| \right) = (1 - \alpha)(1 - X_1) \leq 1 - \alpha.
\]
So \( f(z) \in \overline{HW}(\lambda, \beta, t, \alpha) \). Conversely, assume \( f(z) \in \overline{HW}(\lambda, \beta, t, \alpha) \) by setting

\[
X_1 = 1 - \left( \sum_{n=2}^{\infty} |X_n| + \sum_{n=1}^{\infty} |Y_n| \right), \quad \text{where} \quad X_1 \geq 0. \quad \text{Then}
\]

\[
X_n = \frac{\left| (\beta + t)(n - 1) + \beta t(1 + n^2) + 1 \right| \Gamma(2 - \lambda) \Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} |a_n|, \quad (n \geq 2)
\]

\[
Y_n = \frac{\left| (\beta + t)(n + 1) - \beta t(1 + n^2) - 1 \right| \Gamma(2 - \lambda) \Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} |b_n| \quad (n \geq 1),
\]

we obtain

\[
f(z) = z - \sum_{n=2}^{\infty} |a_n|z^n + \sum_{n=1}^{\infty} |b_n|(z)^n
\]

\[
= \sum_{n=2}^{\infty} \frac{(1 - \alpha)X_n}{\left| (\beta + t)(n - 1) + \beta t(1 + n^2) + 1 \right| \Gamma(2 - \lambda) \Gamma(n + 1)} z^n
\]

\[
+ \sum_{n=1}^{\infty} \frac{(1 - \alpha)Y_n}{\left| (\beta + t)(n + 1) - \beta t(1 + n^2) - 1 \right| \Gamma(2 - \lambda) \Gamma(n + 1)} (z)^n
\]

\[
= \left[ 1 - \left( \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n \right) \right] z + \sum_{n=2}^{\infty} h_n(z)X_n + \sum_{n=1}^{\infty} g_n(z)Y_n
\]

\[
= X_1 z + \sum_{n=2}^{\infty} X_n h_n(z) + \sum_{n=1}^{\infty} Y_n g_n(z),
\]

that is the required representation. \( \square \)

4. Convex Combination

Now we introduce \( \overline{HW}(\lambda, \beta, t, \alpha) \) is closed under convex combination.

**Theorem 4.1.** If \( f_n(i = 1, 2, \ldots) \) belongs to \( \overline{HW}(\lambda, \beta, t, \alpha) \), then the function

\[
\phi(z) = \sum_{i=1}^{\infty} \sigma_i f_n(z) \text{ is also in } \overline{HW}(\lambda, \beta, t, \alpha), \text{ where } f_n(z) \text{ is defined by}
\]

\[
f_n(z) = z - \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n(z)^n \quad (i = 1, 2, \ldots, \quad 0 \leq \sigma_i \leq 1, \quad \sum_{i=1}^{\infty} \sigma_i = 1). \quad (20)
\]

**Proof.** Since \( f_n(z) \in \overline{HW}(\lambda, \beta, t, \alpha) \), by (16) we have

\[
\sum_{n=2}^{\infty} \left| (\beta+t)(n-1)+\beta t(1+n^2)+1 \right| \Gamma(2 - \lambda) \Gamma(n + 1) \frac{\Gamma(n + 1 - \lambda)}{\Gamma(n + 1)} |a_n|.
\]
+ \sum_{n=1}^{\infty} |(\beta+t)(n+1)-\beta t(1+n^2) - 1| \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} |b_{n_i}| \leq 1-\alpha, \quad (i = 1, 2, \ldots).

Also

\phi(z) = \sum_{i=1}^{\infty} \sigma_i f_i(z) = z - \sum_{n=2}^{\infty} \left( \sum_{i=1}^{\infty} \sigma_i a_{n_i} \right) z^n + \sum_{n=1}^{\infty} \left( \sum_{i=1}^{\infty} \sigma_i b_{n_i} \right) (\overline{z})^n.

Now according to Theorem 2.2 we have

\sum_{n=2}^{\infty} |(\beta+t)(n-1)+\beta t(1+n^2) + 1| \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} \left| \sum_{i=1}^{\infty} \sigma_i a_{n_i} \right|

+ \sum_{n=1}^{\infty} |(\beta+t)(n+1)-\beta t(1+n^2) - 1| \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} \left| \sum_{i=1}^{\infty} \sigma_i b_{n_i} \right|

= \sum_{i=1}^{\infty} \sigma_i \left\{ \sum_{n=2}^{\infty} |(\beta+t)(n-1)+\beta t(1+n^2) + 1| \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} |a_{n_i}| \right.

+ \sum_{n=1}^{\infty} |(\beta+t)(n+1)-\beta t(1+n^2) - 1| \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} |b_{n_i}| \left. \right\}

\leq (1-\alpha) \sum_{i=1}^{\infty} \sigma_i = 1-\alpha.

Thus \(\phi(z) \in \overline{H}(\lambda, \beta, t, \alpha)\).

So, we note that \(\overline{H}(\lambda, \beta, t, \alpha)\) is a convex set.

\[\square\]

5. Distortion Bounds

In the following theorem we obtain distortion bounds for \(f \in \overline{H}(\lambda, \beta, t, \alpha)\).

**Theorem 5.1.** Let \(f = h + g \in \overline{H}(\lambda, \beta, t, \alpha)\), \(\beta \geq 1\), \(|z| = r < 1\). Then, we have

\[|f(z)| \geq (1 - |b_1|) r - \left( \frac{1 - \alpha}{1 + \beta + t(1+5\beta)} - \frac{-1 + 2(\beta + t - \beta t)}{1 + \beta + t(1+5\beta)} |b_1| \right) \frac{2 - \lambda}{2} r^2, \quad (21)\]

and

\[|f(z)| \leq (1 + |b_1|) r + \left( \frac{1 - \alpha}{1 + \beta + t(1+5\beta)} - \frac{-1 + 2(\beta + t - \beta t)}{1 + \beta + t(1+5\beta)} |b_1| \right) \frac{2 - \lambda}{2} r^2. \quad (22)\]
Proof. Assume $f(z) \in \overline{H}(\lambda, \beta, t, \alpha)$, then by (16), we have

$$f(z) = \left| z - \sum_{n=2}^{\infty} a_n |z^n| + \sum_{n=1}^{\infty} b_n |z^n| \right| = \left| z + |b_1|\overline{(z)} - \sum_{n=2}^{\infty} |a_n| z^n - |b_n| |z^n| \right|$$

$$\geq (1 - |b_1|) r - \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^2$$

$$\geq (1 - |b_1|) r - \frac{1 - \alpha}{1 + \beta + t(1 + 5\beta)} \frac{2 - \lambda}{2} \times$$

$$\sum_{n=2}^{\infty} \frac{1 + \beta + t(1 + 5\beta)}{1 - \alpha} \frac{2 - \lambda}{2} (|a_n| + |b_n|) r^2$$

$$\geq (1 - |b_1|) r - \frac{1 - \alpha}{1 + \beta + t(1 + 5\beta)} \frac{2 - \lambda}{2} \times$$

$$\left( \sum_{n=2}^{\infty} \frac{(\beta + t)(n - 1) + \beta t(1 + n^2) + 1}{1 - \alpha} \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} |a_n| \right. + \left. \frac{(\beta + t)(n + 1) - \beta t(1 + n^2) - 1}{1 - \alpha} \frac{\Gamma(2 - \lambda)\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} |b_n| \right) r^2$$

$$\geq (1 - |b_1|) r - \frac{1 - \alpha}{1 + \beta + t(1 + 5\beta)} \frac{2 - \lambda}{2} \times \left( 1 - \frac{-1 + 2(\beta + t - \beta t)}{1 - \alpha} |b_1| \right) r^2$$

$$= r - |b_1| r - \left( \frac{1 - \alpha}{1 + \beta + t + 5\beta t} - \frac{-1 + 2(\beta + t - \beta t)}{1 + \beta + t + 5\beta t} |b_1| \right) \frac{2 - \lambda}{2} r^2.$$
Corollary 5.2. If \( f \in \mathcal{HA}(\lambda, t, \alpha) \) then
\[
|f(z)| \geq (1 - |b_1|)r - \left( \frac{1 - \alpha}{2(1 + 3t)} - \frac{1}{2(1 + 3t)}|b_1| \right) \frac{2 - \lambda}{2} r^2
\]
and
\[
|f(z)| \leq (1 + |b_1|)r + \left( \frac{1 - \alpha}{2(1 + 3t)} - \frac{1}{2(1 + 3t)}|b_1| \right) \frac{2 - \lambda}{2} r^2.
\]

Especially for \( \lambda = 0 \), we obtain inequalities as follows
\[
|f(z)| \geq (1 - |b_1|)r - \left( \frac{1 - \alpha}{2(1 + 3t)} - \frac{1}{2(1 + 3t)}|b_1| \right) r^2
\]
and
\[
|f(z)| \leq (1 + |b_1|)r + \left( \frac{1 - \alpha}{2(1 + 3t)} - \frac{1}{2(1 + 3t)}|b_1| \right) r^2.
\]

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