SECOND HANKEL DETERMINANT PROBLEM FOR SEVERAL CLASSES OF ANALYTIC FUNCTIONS RELATED TO SHELL-LIKE CURVES CONNECTED WITH FIBONACCI NUMBERS

JANUSZ SOKÓŁ, SEDAT İLHAN AND H. ÖZLEM GÜNÉY

ABSTRACT. In this paper, we investigate upper bounds for the second Hankel determinant in several classes of analytic functions in the open unit disc, related to shell-like curves and connected with Fibonacci numbers.

Keywords: Analytic functions, shell-like curve, Fibonacci numbers, starlike functions, convex functions, Hankel determinant.

AMS Subject Classification: 30C45, 30C50

1. INTRODUCTION

Let $A$ denote the class of functions $f$ which are analytic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and let $S$ denote the class of functions in $A$ which are univalent in $U$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$ and are of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1)$$

We say that $f$ is subordinate to $F$ in $U$, written as $f \prec F$, if and only if $f(z) = F(w(z))$ for some analytic function $w$ such that $|w(z)| \leq |z|$ for all $z \in U$.

If $f \in A$ and

$$\frac{zf'(z)}{f(z)} \prec p(z) \quad \text{or} \quad 1 + \frac{zf''(z)}{f'(z)} \prec p(z) \quad \text{or} \quad (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) \prec p(z)$$

where $p(z) = \frac{1+z}{1-z}$, then we say that $f$ is starlike or convex or $\alpha$-convex function, respectively. These functions form known classes denoted by $S^*$, $C$ or $M(\alpha)$, respectively. These classes are very important subclasses of the class $S$ in geometric function theory.

In [14], Sokól introduced the class $SL$ of shell-like functions as the set of functions $f \in A$ which is described in the following definition:

---

1 University of Rzeszów, Faculty of Mathematics and Natural Sciences, ul. Prof. Pignia 1, 35-310 Rzeszów, Poland.
2 Dicle University, Faculty of Science, Department of Mathematics, Diyarbakır-Turkey.
3 Dicle University, Faculty of Science, Department of Mathematics, Diyarbakır-Turkey.

$^*$ Corresponding author

Manuscript received: September 8, 2017; accepted: January 16, 2018.

TWMS Journal of Applied and Engineering Mathematics, Vol.8, No.1a © Işık University, Department of Mathematics, 2018; all rights reserved.

220
Definition 1.1. The function \( f \in A \) belongs to the class \( SL \) if it satisfies the condition that
\[
\frac{zf'(z)}{f(z)} < \tilde{p}(z)
\]
with
\[
\tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2},
\]
where \( \tau = (1 - \sqrt{5})/2 \approx -0.618 \).

Later, Dziok et al. in [1] and [2] defined and introduced the class \( KSL \) and \( SLM_\alpha \) of convex and \( \alpha \)-convex functions related to a shell-like curve connected with Fibonacci numbers, respectively. These classes can be given in the following definitions.

Definition 1.2. The function \( f \in A \) belongs to the class \( KSL \) of convex shell-like functions if it satisfies the condition that
\[
1 + \frac{zf''(z)}{f'(z)} < \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2},
\]
where \( \tau = (1 - \sqrt{5})/2 \approx -0.618 \).

Definition 1.3. The function \( f \in A \) belongs to the class \( SLM_\alpha \), \( 0 \leq \alpha \leq 1 \) if it satisfies the condition that
\[
\alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) + (1 - \alpha) \frac{zf'(z)}{f(z)} < \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2},
\]
where \( \tau = (1 - \sqrt{5})/2 \approx -0.618 \).

The class \( SLM_\alpha \) is related to the class \( KSL \) only through the function \( \tilde{p} \) and \( SLM_\alpha \neq \ KSL \) for all \( \alpha \neq 1 \). It is easy to see that \( KSL = SLM_1 \). The function \( \tilde{p} \) is not univalent in \( U \), but it is univalent in the disc \( |z| < (3 - \sqrt{5})/2 \approx 0.38 \). For example, \( \tilde{p}(0) = \tilde{p}(-1/2\tau) = 1 \) and \( \tilde{p}(e^{\pm i \arccos(1/4)}) = \sqrt{5}/5 \), and it may also be noticed that
\[
\frac{1}{|\tau|} = \frac{|\tau|}{1 - |\tau|},
\]
which shows that the number \( |\tau| \) divides \([0, 1]\) such that it fulfils the golden section. The image of the unit circle \(|z| = 1\) under \( \tilde{p} \) is a curve described by the equation given by
\[
(10x - \sqrt{5})y^2 = (\sqrt{5} - 2x)(\sqrt{5}x - 1)^2,
\]
which is translated and revolved trisectrix of Maclaurin. The curve \( \tilde{p}(re^{it}) \) is a closed curve without any loops for \( 0 < r \leq r_0 = (3 - \sqrt{5})/2 \approx 0.38 \). For \( r_0 < r < 1 \), it has a loop, and for \( r = 1 \), it has a vertical asymptote. Since \( \tau \) satisfies the equation \( \tau^2 = 1 + \tau \), this expression can be used to obtain higher powers \( \tau^n \) as a linear function of lower powers, which in turn can be decomposed all the way down to a linear combination of \( \tau \) and 1. The resulting recurrence relationships yield Fibonacci numbers \( u_n \):
\[
\tau^n = u_n \tau + u_{n-1}.
\]
In 1976, Noonan and Thomas [10] stated the $s^{th}$ Hankel determinant for $s \geq 1$ and $k \geq 1$ as

$$H_s(k) = \begin{vmatrix}
    a_k & a_{k+1} & \cdots & a_{k+s-1} \\
    a_{k+1} & a_{k+2} & \cdots & \vdots \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{k+s-1} & \cdots & \cdots & a_{k+2(s-1)}
\end{vmatrix},$$

where $a_1 = 1$.

This determinant has also been considered by several authors. For example, Noor [11] determined the rate of growth of $H_s(k)$ as $k \to \infty$ for functions $f$ given by (1) with bounded boundary. Ehrenborg in [3] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [8]. Also, several authors considered the case $s = 2$. Especially, $H_2(1) = a_3 - a_2^2$ is known as Fekete-Szegő functional and this functional is generalized to $a_3 - \mu a_2^2$ where $\mu$ is some real number [4]. Estimating for an upper bound of $|a_3 - \mu a_2^2|$ is known as the Fekete-Szegő problem. In [13], Raina and Sokol considered Fekete-Szegő problem for the class $S_L$. In 1969, Keogh and Merkes [7] solved this problem for the classes $S^*$ and $C$.

The second Hankel determinant is $\{\text{SL} = 1\}$.

In 1969, Keogh and Merkes [7] solved this problem for the classes $S^*$ and $C$. The second Hankel determinant is $H_2(2) = a_2 a_4 - a_3^2$. Janteng [5] found the sharp upper bound for $|H_2(2)|$ for univalent functions whose derivative has positive real part. Also, in [6] Janteng et al. obtained the bounds for $|H_2(2)|$ for the classes $S^*$ and $C$.

Let $P(\beta)$, $0 \leq \beta < 1$, denote the class of analytic functions $p$ in $U$ with $p(0) = 1$ and $\text{Re}\{p(z)\} > \beta$. Especially, we will use $P$ instead of $P(0)$.

**Theorem 1.1.** ([2]) The function $p(z) = \frac{1 + r^2 + z^2}{1 - r^2 - 2rz}$ belongs to the class $P(\beta)$ with $\beta = \sqrt{5}/10 \approx 0.2236$.

Now we give the following lemmas which will use in proving.

**Lemma 1.1.** ([12]) Let $p \in P$ with $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$, then

$$|c_n| \leq 2, \quad \text{for} \quad n \geq 1. \quad (3)$$

If $|c_1| = 2$, then $p(z) \equiv p_1(z) \equiv (1 + z)/(1 - xz)$ with $x = c_1/2$. Conversely, if $p(z) \equiv p_1(z)$ for some $|x| = 1$, then $c_1 = 2x$. Furthermore, we have

$$\left|c_2 - \frac{c_1^2}{2}\right| \leq 2 - \frac{|c_1|^2}{2}. \quad (4)$$

If $|c_1| < 2$, and $|c_2 - \frac{c_1^2}{2}| = 2 - \frac{|c_1|^2}{2}$, then $p(z) \equiv p_2(z)$, where

$$p_2(z) = \frac{1 + \bar{x}wz + z(wz + x)}{1 + \bar{x}wz - z(wz + x)},$$

and $x = \frac{c_1}{2}$, $w = \frac{2c_2 - \frac{c_1^2}{2}}{4 - |c_1|^2}$ and $|c_2 - \frac{c_1^2}{2}| = 2 - \frac{|c_1|^2}{2}$.

**Lemma 1.2.** ([9]) Let $p \in P$ with coefficients $c_n$ as above, then

$$|c_3 - 2c_1 c_2 + c_1^3| \leq 2. \quad (5)$$

In this paper, we use ideas and techniques used in geometric function theory. The central problem considered here is the sharp upper bounds for the functional $|a_2 a_4 - a_3^2|$ of functions in the classes $S_L$, $KSL$ and $S_L M_a$, depicted by the Fibonacci numbers, respectively.
2. Main Results

In [13], Raina and Sokół proved the following result:

**Theorem 2.1.** If \( p(z) = 1 + p_1 z + p_2 z^2 + \cdots \), and \( p \prec \tilde{p} \), then
\[
|p_1| \leq |\tau| \tag{6}
\]
and
\[
|p_2| \leq 3\tau^2. \tag{7}
\]
The above estimates are sharp.

Now, we prove the following theorem as addition to Theorem 2.1.

**Theorem 2.2.** If \( p(z) = 1 + p_1 z + p_2 z^2 + \cdots \), and \( p \prec \tilde{p} \), then
\[
|p_3| \leq 4|\tau|^3. \tag{8}
\]
The result is sharp.

**Proof.** If \( p \prec \tilde{p} \), then there exists an analytic function \( w \) such that \( |w(z)| \leq |z| \) in \( \mathbb{U} \) and \( p(z) = \tilde{p}(w(z)) \). Therefore, the function
\[
h(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots \tag{9}
\]
is in the class \( \mathcal{P}(0) \). It follows that
\[
w(z) = \frac{c_1 z}{2} + \left( c_2 - \frac{c_1^2}{2} \right) \frac{z^2}{2} + \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \frac{z^3}{2} + \cdots \tag{10}
\]
and
\[
\tilde{p}(w(z)) = 1 + \tilde{p}_1 \left\{ \frac{c_1 z}{2} + \left( c_2 - \frac{c_1^2}{2} \right) \frac{z^2}{2} + \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \frac{z^3}{2} + \cdots \right\}
\]
\[
+ \tilde{p}_2 \left\{ \frac{c_1 z}{2} + \left( c_2 - \frac{c_1^2}{2} \right) \frac{z^2}{2} + \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \frac{z^3}{2} + \cdots \right\}^2
\]
\[
+ \tilde{p}_3 \left\{ \frac{c_1 z}{2} + \left( c_2 - \frac{c_1^2}{2} \right) \frac{z^2}{2} + \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \frac{z^3}{2} + \cdots \right\}^3 + \cdots
\]
\[
= 1 + \tilde{p}_1 \frac{c_1 z}{2} + \left\{ \frac{1}{2} \left( c_2 - \frac{c_1^2}{2} \right) \tilde{p}_1 + \frac{c_1^3}{4} \tilde{p}_2 \right\} z^2
\]
\[
+ \left\{ \frac{1}{2} \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \tilde{p}_1 + \frac{1}{2} c_1 \left( c_2 - \frac{c_1^2}{2} \right) \tilde{p}_2 + \frac{c_1^3}{8} \tilde{p}_3 \right\} z^3 + \cdots. \tag{11}
\]
To find the coefficients \( \tilde{p}_n \) of the function \( \tilde{p} \), on putting \( \tau z = t \), then we have
\[
\tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2} = \left( t + \frac{1}{t} \right) \frac{t}{1 - t - t^2}
\]
\[
= \frac{1}{\sqrt{5}} \left( t + \frac{1}{t} \right) \left( \frac{1}{1 - (1 - \tau)t} - \frac{1}{1 - \tau t} \right)
\]
\[
= \left( t + \frac{1}{t} \right) \sum_{n=1}^{\infty} \frac{(1 - \tau)^n - \tau^n}{\sqrt{5} \ t^n} t^n
\]
It is known that
\[
\sum_{n=1}^{\infty} u_n t^n = 1 + \sum_{n=1}^{\infty} (u_{n-1} + u_{n+1}) \tau^n z^n,
\]
where
\[
u_n = \frac{(1 - \tau)^n - \tau^n}{\sqrt{5}}, \tau = \frac{1 - \sqrt{5}}{2} \quad (n = 1, 2, \ldots).
\]
This shows that the relevant connection of \( \tilde{p} \) with the sequence of Fibonacci numbers \( u_n \), such that \( u_0 = 0, u_1 = 1, u_{n+2} = u_n + u_{n+1} \) for \( n = 0, 1, 2, \ldots \). Now using (11), we get
\[
\tilde{p}(z) = 1 + \sum_{n=1}^{\infty} \tilde{p}_n z^n
\]
\[
= 1 + (u_0 + u_2) \tau z + (u_1 + u_3) \tau^2 z^2 + \sum_{n=3}^{\infty} (u_{n-3} + u_{n-2} + u_{n-1} + u_n) \tau^n z^n
\]
\[
= 1 + \tau z + 3\tau^2 z^2 + 4\tau^3 z^3 + 7\tau^4 z^4 + 11\tau^5 z^5 + \cdots.
\]
Thus, \( \tilde{p}_1 = \tau, \tilde{p}_2 = 3\tau^2 \) and
\[
\tilde{p}_n = (u_{n-1} + u_{n+1}) \tau^n = (u_{n-3} + u_{n-2} + u_{n-1} + u_n) \tau^n = \tau \tilde{p}_{n-1} + \tau^2 \tilde{p}_{n-2} \quad (n = 3, 4, 5, \ldots).
\]
If \( p(z) = 1 + p_1 z + p_2 z^2 + \cdots \), then using (10) and (13), we have
\[
p_1 = \frac{c_1}{2} \tau,
\]
\[
p_2 = \frac{1}{2} \left( c_2 - \frac{c_1^2}{2} \right) \tau + \frac{3}{4} c_1^3 \tau^2,
\]
and
\[
p_3 = \frac{1}{2} \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \tau + \frac{3}{2} c_1 \left( c_2 - \frac{c_1^2}{2} \right) \tau^2 + \frac{1}{2} c_1^3 \tau^3.
\]
In [13], Raina and Sokol proved Theorem 2.1 and obtained sharp estimates for \( |p_1| \) and \( |p_2| \). Now we shall obtain sharp estimate for \( |p_3| \). Taking absolute value of (17) we can write
\[
|p_3| = \left| \frac{1}{2} \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \tau + \frac{3}{2} c_1 \left( c_2 - \frac{c_1^2}{2} \right) \tau^2 + \frac{1}{2} c_1^3 \tau^3 \right|
\]
\[
= \left| \frac{1}{2} \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \tau + \frac{3}{2} c_1 \left( c_2 - \frac{c_1^2}{2} \right) (\tau + 1) + \frac{1}{2} c_1^3 (2\tau + 1) \right|
\]
\[
= \left| \left\{ \frac{1}{2} \left( c_3 - 2c_1 c_2 + \frac{c_1^3}{2} \right) + \frac{c_1}{4} \left( c_2 - \frac{c_1^2}{2} \right) + \frac{7}{4} c_1 c_2 \right\} \tau + \left\{ \frac{3c_1}{2} \left( c_2 - \frac{c_1^2}{2} \right) + \frac{c_1^3}{2} \right\} \right|.
\]
It is known that
\[
\forall n \in \mathbb{N}, \quad \tau = \frac{\tau^n}{u_n} - x_n, \quad x_n = \frac{u_{n-1}}{u_n}, \quad \lim_{n \to \infty} \frac{u_{n-1}}{u_n} = |\tau| \approx 0.618.
\]
Therefore, we have
\[ |p_3| = \left\{ \frac{1}{2} \left( c_3 - 2c_1c_2 + c_1^3 \right) + \frac{1}{4}c_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{7}{4}c_1c_2 \right\} \frac{\tau^n}{u_n} \\
+ \left\{ \frac{1}{2} \left( c_3 - 2c_1c_2 + c_1^3 \right) x_n + \frac{2 - x_n}{4}c_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{4 - 7x_n}{4}c_1c_2 \right\} \left| \frac{\tau^n}{u_n} \right| \\
\leq \frac{1}{2} \left( c_3 - 2c_1c_2 + c_1^3 \right) + \frac{1}{4}c_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{7}{4}c_1c_2 \left| \frac{\tau^n}{u_n} \right| \\
+ \left\{ \frac{1}{2} \left( c_3 - 2c_1c_2 + c_1^3 \right) x_n + \frac{2 - x_n}{4}c_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{4 - 7x_n}{4}c_1c_2 \right\} \left| \frac{\tau^n}{u_n} \right| \\
\leq \left\{ \frac{1}{2} c_3 - 2c_1c_2 + c_1^3 \right\} + \frac{1}{4}c_1 \left| c_2 - c_1^2 \right| + \frac{7}{4}c_1c_2 \left| \frac{\tau^n}{u_n} \right| \\
+ \left\{ \frac{1}{2} c_3 - 2c_1c_2 + c_1^3 \right\} x_n + \frac{2 - x_n}{4}c_1 \left| c_2 - c_1^2 \right| + \frac{4 - 7x_n}{4}c_1c_2 \left| \frac{\tau^n}{u_n} \right| .
\]

By (19), for sufficiently large \( n \) we have \( |4 - 7x_n| = 7x_n - 4 \). Therefore, from (3), (4), and (5) we can write
\[ |p_3| \leq \left\{ 1 + \frac{1}{4}c_1 \left( 2 - \frac{|c_1|^2}{2} \right) + \frac{7}{2}c_1 \right\} \frac{|\tau^n|}{u_n} + \left\{ x_n + \frac{2 - x_n}{4}|c_1| \left( 2 - \frac{|c_1|^2}{2} \right) + \frac{7x_n - 4}{2}|c_1| \right\} \\
= \left\{ 1 + 4|c_1| - \frac{|c_1|^3}{8} \right\} \frac{|\tau^n|}{u_n} + \left\{ x_n + (3x_n - 1)|c_1| - \frac{2 - x_n}{8}|c_1|^3 \right\} .
\]

We have
\[ \max_{y \in [0, 2]} \left\{ 1 + 4y - \frac{y^3}{8} \right\} = 8 \text{ at } y = 2,
\]

since
\[ \lim_{n \to \infty} \left\{ 1 + 4|c_1| - \frac{|c_1|^3}{8} \right\} \frac{|\tau^n|}{u_n} = 0.
\]

Furthermore, for sufficiently large \( n \) we have
\[ \max_{y \in [0, 2]} \left\{ x_n + (3x_n - 1)y - \frac{2 - x_n}{8}y^3 \right\} = 8x_n - 4 \text{ at } y = 2,
\]

so
\[ \lim_{n \to \infty} \max_{y \in [0, 2]} \left\{ x_n + (3x_n - 1)y - \frac{2 - x_n}{8}y^3 \right\} = 8|\tau| - 4 = 4|\tau|^3.
\]

Therefore, we get
\[ \lim_{n \to \infty} \left\{ 1 + 4|c_1| - \frac{|c_1|^3}{8} \right\} \frac{|\tau^n|}{u_n} + \left\{ x_n + (3x_n - 1)|c_1| - \frac{2 - x_n}{8}|c_1|^3 \right\} \leq 4|\tau|^3
\]
which shows that
\[ |p_3| \leq 4|\tau|^3.
\]

If we take in (9)
\[ h(z) = \frac{1 + z}{1 - z} = 1 + 2z + 2z^2 + \ldots,
\]

then putting \( c_1 = c_2 = c_3 = 2 \) in (17) gives \( p_3 = 4\tau^3 \) and it shows that (8) is sharp. It completes the proof.
Conjecture 2.1. If \( p(z) = 1 + p_1 z + p_2 z^2 + \cdots \), and \( p \prec \tilde{p} \), then
\[
|p_n| \leq (u_{n-1} + u_{n+1})|\tau|^n, \quad n = 1, 2, 3, \ldots,
\]
where \( u_0 = 0, u_1 = 1, u_{n+2} = u_n + u_{n+1} \) for \( n = 0, 1, 2, \ldots \), is the Fibonacci sequence. This bound would be sharp for the function (14).

This conjecture has been just verified for \( n = 3 \) in last Theorem 2.2, while for \( n = 1, 2 \) it was proved in [13].

Theorem 2.3. If \( f(z) = z + a_2 z^2 + \cdots \) belongs to \( SL \), then
\[
|a_2 a_4 - a_3^2| \leq \frac{11}{3} \tau^4. \tag{20}
\]

Proof. For given \( f \in SL \), define \( p(z) = 1 + p_1 z + p_2 z^2 + \cdots \), by
\[
z f'(z) f(z) = p(z)
\]
where \( p \prec \tilde{p} \). Hence
\[
z f'(z) f(z) = 1 + a_2 z + (2a_3 - a_2^2) z^2 + (3a_4 - 3a_2 a_3 + a_2^3) z^3 + \cdots = 1 + p_1 z + p_2 z^2 + \cdots
\]
and
\[
a_2 = p_1, \quad a_3 = \frac{p_1^2 + p_2}{2}, \quad a_4 = \frac{p_1^3 + 3 p_1 p_2 + 2 p_3}{6}.
\]
Therefore,
\[
a_2 a_4 - a_3^2 = \frac{1}{12} (-p_1^4 + 4 p_1 p_3 - 3 p_2^2). \tag{21}
\]

Using Theorem 2.1 and Theorem 2.2, we obtain
\[
|a_2 a_4 - a_3^2| = \left| \frac{1}{12} (-p_1^4 + 4 p_1 p_3 - 3 p_2^2) \right|
\]
\[
\leq \frac{1}{12} (|p_1|^4 + |p_1| |p_3| + 3 |p_2|^2)
\]
\[
\leq \frac{1}{12} (|\tau|^4 + 4 |\tau| |\tau|^3 + 3 (3 \tau^2)^2)
\]
\[
= \frac{1}{12} (|\tau|^4 + 16 |\tau|^4 + 27 |\tau|^4)
\]
\[
= \frac{11}{3} \tau^4.
\]

\(\square\)

The bound in 20 is not sharp. So we give the following conjecture for sharpness.

Conjecture 2.2. If \( f(z) = z + a_2 z^2 + \cdots \) belongs to \( SL \), then
\[
|a_2 a_4 - a_3^2| \leq \tau^4. \tag{22}
\]

The bound is sharp.

Theorem 2.4. If \( f(z) = z + a_2 z^2 + \cdots \) belongs to \( KSL \), then
\[
|a_2 a_4 - a_3^2| \leq \frac{4}{9} \tau^4.
\]
Proof. For given \( f \in KSL \), define \( p(z) = 1 + p_1 z + p_2^2 z^2 + \cdots \), by

\[
1 + \frac{zf''(z)}{f'(z)} = p(z)
\]

where \( p \preceq \tilde{p} \) in \( U \). Hence

\[
1 + \frac{zf''(z)}{f'(z)} = 1 + 2a_2 z + (6a_3 - 4a_2^2) z^2 + (12a_4 - 18a_2 a_3 + 8a_2^3) z^3 + \cdots = 1 + p_1 z + p_2^2 z^2 + \cdots
\]

and

\[
a_2 = \frac{p_1}{2}, \quad a_3 = \frac{p_1^2 + p_2}{6}, \quad a_4 = \frac{p_1^3 + 3p_1 p_2 + 2p_3}{24}.
\]

Therefore, using Theorem 2.1 and Theorem 2.2, we obtain

\[
|a_2 a_4 - a_3^2| \leq \frac{4}{9} r^4.
\]

\( \square \)

Theorem 2.5. If \( f(z) = z + a_2 z^2 + \cdots \) belongs to \( SLM_\alpha \), then

\[
|a_2 a_4 - a_3^2| \leq \frac{145\alpha^5 + 625\alpha^4 + 1061\alpha^3 + 867\alpha^2 + 330\alpha + 44}{12(1 + \alpha)^4(1 + 2\alpha)^2(1 + 3\alpha)} r^4.
\]

Proof. For given \( f \in SLM_\alpha \), define \( p(z) = 1 + p_1 z + p_2^2 z^2 + \cdots \), by

\[
(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) = p(z)
\]

where \( p \preceq \tilde{p} \) in \( U \). Hence

\[
(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) = 1 + (1 + \alpha) a_2 z + [2(1 + 2\alpha) a_3 - (1 + 3\alpha) a_2^2] z^2 + [3(1 + 3\alpha) a_4 - 3(1 + 5\alpha) a_2 a_3 + (1 + 7\alpha) a_2^3] z^3 + \cdots = 1 + p_1 z + p_2^2 z^2 + \cdots
\]

and

\[
a_2 = \frac{p_1}{1 + \alpha}, \quad a_3 = \frac{(1 + 3\alpha) p_1^2 + (1 + \alpha)^2 p_2}{2(1 + \alpha)^2(1 + 2\alpha)},
\]

\[
a_4 = \frac{3(1 + 3\alpha)(1 + 5\alpha) p_1^3 + 3(1 + \alpha)^2(1 + 5\alpha) p_1 p_2 - 2(1 + 2\alpha)(1 + 7\alpha) p_1^2 + 2(1 + \alpha)^3(1 + 2\alpha) p_3}{6(1 + \alpha)^3(1 + 2\alpha)(1 + 3\alpha)}.
\]

Therefore, using Theorem 2.1 and Theorem 2.2, we obtain

\[
|a_2 a_4 - a_3^2| \leq \frac{145\alpha^5 + 625\alpha^4 + 1061\alpha^3 + 867\alpha^2 + 330\alpha + 44}{12(1 + \alpha)^4(1 + 2\alpha)^2(1 + 3\alpha)} r^4.
\]

\( \square \)

It is clear that if we take \( \alpha = 0 \) and \( \alpha = 1 \) in Theorem 2.5, we obtain the results of Theorem 2.3 and Theorem 2.4, respectively.
3. **Concluding Remarks and Observations**

In our present article, we have obtained sharp estimates for second Hankel determinants of several classes of analytic functions related to shell-like curves connected with Fibonacci numbers. Firstly, we have found a sharp bound estimate for third coefficient of a function with positive real part which is subordinate to a shell-like curve and have given a conjecture for general case. Secondly, we have studied the problem of finding the upper bounds associated with the second Hankel determinant $H_2(2)$ for these classes. We have also considered several results which are closely related to our investigation in this paper. However, we give some conjectures for sharpness of bounds.

**Acknowledgement**

This research has been supported with the grant number FEN.17.026 by DUBAP (Dicle University Coordination Committee of Scientific Research Projects). The authors would like to thank DUBAP for their supporting and the referees for the helpful suggestions.

**References**


Janusz Sokół is a professor of mathematics at University of Rzeszów (Poland) since 2016. He received his doctorate under the supervision of Prof. Jan Stankiewicz in University of Łódź, while he received his habilitation in Mariae Curie-Skłodowska University in Lublin. Long time he worked at Rzeszów University of Technology. His research focuses on geometric function theory.

Sedat İlhan is working as an associate professor in the Department of Mathematics of Faculty of Science in Dicle University, Turkey. He received his M.S. and PhD degrees from Dicle University. His area of interest includes semigroups, Fibonacci numbers and Lucas numbers.