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# SOLVABILITY TO COUPLED SYSTEMS OF FUNCTIONAL EQUATIONS VIA FIXED POINT THEORY

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Abstract. The purpose of the present paper is to establish the existence and uniqueness of coupled common fixed points for a pair of mappings satisfying $F$-contraction. As a consequence of our results, we discuss the existence of a unique common solution of coupled systems of functional equations arising in dynamic programming.

Keywords: coupled common fixed point, $F$-contraction, metric space, coupled system of functional equation.

AMS Subject Classification: 47H10, 54H25

## 1. Introduction and Preliminaries

The metric fixed point theory regarded as starting with Banach contraction principle [2] in 1922 is a branch of mathematics, which is widely used not only in various mathematical theories, but also in many practical problems of natural sciences and engineering. In fact, by introducing suitable operators in different types of spaces, it is possible to find the existence and uniqueness of solutions of differential, integral or functional equations by searching the fixed points of such operators. This situation motivates researchers to study on extensions and generalizations of the Banach contraction principle [2]. One of the most interesting generalizations this phenomenon principle has been given by Wardowski [19] by introducing the following notion of $F$-contraction.

**Definition 1.1.** *A self-mapping $T$ on a metric space $(X, d)$ is said to be an $F$-contraction, if there exist $F \in \mathcal{F}$ and $\sigma \in (0, +\infty)$ such that*

$$x, y \in X, \quad d(Tx, Ty) > 0 \Longrightarrow \sigma + F(d(Tx, Ty)) \leq F(d(x, y)), \tag{1}$$

*where $\mathcal{F}$ is the set of functions $F : (0, +\infty) \to \mathbb{R}$ satisfying the following conditions:*

- $(F1)$ *$F$ is strictly increasing;*
- $(F2)$ *for each sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive numbers, $lim_{n\to\infty}\ t_n = 0$ if and only if $lim_{n\to\infty}\ F(t_n) = -\infty$;*
- $(F3)$ *there exists $\alpha \in (0, 1)$ such that $lim_{t\to 0^+}\ t^\alpha F(t) = 0$.*

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Let $T : X \to X$. The following functions $F_i : (0, +\infty) \to \mathbb{R}$ for $i \in \{1, 2, 3, 4\}$ , are the elements of $\mathcal{F}$. Furthermore, substituting in (1) these functions, we obtain the following contractions known in the literature: for all $x, y \in X$ with $Tx \neq Ty$,

$$
\begin{aligned}
&F_1(t) = \ln t, && d(Tx, Ty) \leq e^{-\sigma} d(x, y),\\
&F_2(t) = \ln t + t, && \frac{d(Tx, Ty)}{d(x, y)} e^{d(Tx,Ty)-d(x,y)} \leq e^{-\sigma},\\
&F_3(t) = -\frac{1}{\sqrt{t}}, && d(Tx, Ty) \leq \frac{1}{(1 + \sigma\sqrt{d(x, y)})^2} d(x, y),\\
&F_4(t) = \ln(t^2 + t) && \frac{d(Tx, Ty)(d(Tx, Ty) + 1)}{d(x, y)(d(x, y) + 1)} \leq e^{-\sigma}.
\end{aligned}
$$

**Remark 1.1.** *Clearly, the equation (1) implies that $T$ is a contractive mapping, that is, $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ such that $Tx \neq Ty$. Hence every $F$-contraction mapping is continuous.*

By using the concept of $F$-contraction, Wardowski [19] established a fixed point theorem which improves Banach contraction principle in a different way than in the known results from the literature. For more details about this subject, see [11, 16–18, 20] and references therein.

**Theorem 1.1** ( [19]). *Let $(X, d)$ be a complete metric space and $T : X \to X$ be an $F$-contraction. Then $T$ has a unique fixed point $x^\star$. Moreover, for each $x \in X$, the sequence $\{T^n x\}$ converges to $x^\star$.*

Another concept, coupled fixed point was introduced and studied by Opoitsev [13, 14] and then by Guo and Lakhsmikantham [7]. Bhaskar and Lakhsmikantham [4] were the first to study coupled fixed points in connection to contractive type conditions. They applied their results to prove the existence and uniqueness of solutions for a periodic boundary value problem. Since then, coupled fixed point theory have been a subject of interest by many authors regarding the application potential of it, for example see [1, 6, 8–10, 12, 15].

**Definition 1.2** ( [5–7]). *Let $X$ be a non-empty set, $f$,$g : X \to X$ and $F, G : X \times X \to X$ be given mappings.*

(1) *An element $x \in X$ is called a common fixed point of $f$ and $g$, if $x = fx = gx$.*
(2) *An element $(x, y) \in X \times X$ is said to be a coupled fixed point of $F$ if $x = F(x, y)$ and $y = F(y, x)$.*
(3) *An element $(x, y) \in X \times X$ is said to be coupled common fixed point of $F$ and $G$, if $x = F(x, y) = G(x, y)$ and $y = F(y, x) = G(y, x)$.*

In this study, we establish the existence and uniquness of coupled common fixed points for a pair of mappings satisfying $F$-contraction. As a consequence of our results, we discuss the existence of a unique common solution of the following coupled systems of functional equations arising in dynamic programming:

$$
\begin{aligned}
\alpha_1(x) &= \sup_{y\in D} \{g(x, y) + U(x, y, \alpha_1(\tau(x, y)), \beta_1(\tau(x, y)))\}\\
\beta_1(x) &= \sup_{y\in D} \{g(x, y) + U(x, y, \beta_1(\tau(x, y)), \alpha_1(\tau(x, y)))\}
\end{aligned}
\tag{2}
$$

and

$$\begin{aligned}\alpha_2(x) &= \sup_{y\in D}\{g(x,y)+V(x,y,\alpha_2(\tau(x,y)),\beta_2(\tau(x,y)))\}\\ \beta_2(x) &= \sup_{y\in D}\{g(x,y)+V(x,y,\beta_2(\tau(x,y)),\alpha_2(\tau(x,y)))\}\end{aligned} \tag{3}$$

where $x \in S$ and $S$ is a state space, $D$ is a decision space, $\tau : S \times D \to S$, $g : S \times D \to \mathbb{R}$ and $U, V : S \times D \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

## 2. Main Results

First of all, we give the following lemma which will be used efficiently in the proof of essential theorem of this study.

**Lemma 2.1.** *Let $(X, d)$ be a complete metric space, $F \in \mathcal{F}$ and $f, g : X \to X$. If there exists $\sigma > 0$ such that*

$$\sigma + F(d(fx, gy)) \leq F(d(x, y)), \tag{4}$$

*for all $x, y \in X$ satisfying $\min\{d(fx, gy), d(x, y)\} > 0$. Then $f$ and $g$ have a unique common fixed point.*

*Proof.* Notice that, by $(F1)$ and (4), we deduce that

$$d(fx, gy) \leq d(x, y), \quad \text{for all } x, y \in X. \tag{5}$$

Firstly, we prove that $u$ is a fixed point of $f$ if and only if $u$ is a fixed point of $g$. Suppose that $u$ is a fixed point of $g$, but not a fixed point of $f$. Then, considering (5), we have

$$0 < d(fu, u) = d(fu, gu) \leq d(u, u) = 0$$

which is a contradiction and this implies that $fu = u$. Similarly, it is easy to show that if $u$ is a fixed point of $f$, then $u$ is a fixed point of $g$.

Let $x_0 \in X$. Define the sequence $\{x_n\}$ in $X$ by $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $x_{2n} = x_{2n+1}$ for some $n \in \mathbb{N}$, then $x_{2n} = fx_{2n}$. Thus $x_{2n}$ is a fixed point of $f$ and so $x_{2n}$ is a fixed point of $g$, that is, $x_{2n} = fx_{2n} = gx_{2n}$. Similarly, if $x_{2n+1} = x_{2n+2}$ for some $n \in \mathbb{N}$, then it is easy to see that $x_{2n+1} = fx_{2n+1} = gx_{2n+1}$. Hence we can assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then, for $n = 2m + 1$, where $m \in \mathbb{N}_0$, using (4) we have

$$\begin{aligned}F(d(x_n, x_{n+1})) &= F(d(x_{2m+1}, x_{2m+2})) = F(d(fx_{2m}, gx_{2m+1}))\\ &\leq F(d(x_{2m}, x_{2m+1})) - \sigma\\ &\leq F(d(x_{2m-1}, x_{2m})) - 2\sigma\\ &\;\vdots\\ &\leq F(d(x_0, x_1)) - (2m+1)\sigma\\ &= F(d(x_0, x_1)) - n\sigma.\end{aligned}$$

By a similar method to above, for $n = 2m$, where $m \in \mathbb{N}_0$, we can obtain

$$F(d(x_n, x_{n+1})) \leq F(d(x_0, x_1)) - n\sigma. \tag{6}$$

Thus the inequality (6) is satisfied for all $n \in \mathbb{N}$. On taking limit of (6) as $n \to \infty$, we get $\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty$ that together with $(F2)$ gives

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{7}$$

Let $d_n := d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Thus, from (6), we have

$$F(d_n) \leq F(d_0) - n\sigma. \tag{8}$$

To prove that $\{x_n\}$ is a Cauchy sequence, let us consider condition $(F3)$. Then, there exists $\alpha \in (0,1)$ such that

$$\lim_{n\to\infty} d_n^\alpha F(d_n) = 0. \tag{9}$$

By (8), for all $n \in \mathbb{N}$, we infer that

$$d_n^\alpha F(d_n) - d_n^\alpha F(d_0) \leq -d_n^\alpha n\sigma \leq 0. \tag{10}$$

Letting $n \to \infty$ in (10) and using (9), we get

$$\lim_{n\to\infty} n d_n^\alpha = 0.$$

By the definition of limit, there exists $n_1 \in \mathbb{N}$ such that $n d_n^\alpha \leq 1$ for all $n \geq n_1$, and consequently,

$$d_n \leq \frac{1}{n^{1/\alpha}}, \qquad \text{for all } n \geq n_1. \tag{11}$$

Let $m > n \geq n_1$. Then, using the triangular inequality and (11), we have

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \\ &= \sum_{k=n}^{m-1} d_k \leq \sum_{k=n}^{m-1} \frac{1}{k^{1/\alpha}} \\ &\leq \sum_{k=n}^{\infty} \frac{1}{k^{1/\alpha}}, \end{aligned}$$

and hence $\{x_n\}$ is a Cauchy sequence in $X$. From the completeness of $(X,d)$, there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u.$$

Now, we show that $u$ is a common fixed point of $f$ and $g$. By considering (5), we deduce

$$d(x_{2n+1}, gu) = d(fx_{2n}, gu) \leq d(x_{2n}, u).$$

Passing to limit as $n \to +\infty$ in the above inequality, we obtain $d(u, gu) = 0$ and so $u = gu$. That is, $u$ is a fixed point of $g$. Taking into account the fact that $u$ is a fixed point of $f$ iff $u$ is a fixed point of $g$, we conclude that $u$ is also a fixed point of $f$, that is, $u = fu = gu$. The uniqness of common fixed point follows from (4), so we omit the details. □

Now, we are ready to present the main theorem of this section.

**Theorem 2.1.** *Let $(X,d)$ be a complete metric space, $F \in \mathcal{F}$ and $A, B : X \times X \to X$. If there exists $\sigma > 0$ such that*

$$\sigma + F(d(A(x,y), B(u,v))) \leq F(\max\{d(x,u), d(y,v)\}), \tag{12}$$

*for all $(x,y), (u,v) \in X \times X$ satisfying $\min\{d(A(x,y), B(u,v)), d(x,u), d(y,v)\} > 0$. Then $A$ and $B$ have a unique coupled common fixed point.*

*Proof.* Define $\delta : X^2 \times X^2 \to [0, +\infty)$ by

$$\delta((x,y),(u,v)) = \max\{d(x,u), d(y,v)\}, \quad \text{for all } (x,y),(u,v) \in X \times X.$$

Then, $(X \times X, \delta)$ is a complete metric space, since $(X, d)$ is complete. Consider operators $T_A, T_B : X \times X \to X \times X$ defined by

$$T_A(U) = (A(x,y), A(y,x)),$$

and

$$T_B(U) = (B(x,y), B(y,x)),$$

where $U = (x,y)$. Then, $T_A$ and $T_B$ satisfy all assumptions of Lemma 2.1. Indeed, taking account of $(F1)$ and (12), for all $U = (x,y), V = (u,v) \in X \times X$, we deduce

$$\begin{aligned} F(\delta(T_A(U), T_B(V))) &= F(\delta((A(x,y), A(y,x)), (B(u,v), B(v,u)))) \\ &= F(\max\{d(A(x,y), B(u,v)), d(A(y,x), B(v,u))\}) \\ &= \max\{F(d(A(x,y), B(u,v))), F(d(A(y,x), B(v,u)))\} \\ &\le \max\{F(\max\{d(x,u), d(y,v)\}), F(\max\{d(y,v), d(x,u)\})\} - \sigma \\ &= F(\max\{d(x,u), d(y,v)\}) - \sigma \\ &= F(\delta(U,V)) - \sigma. \end{aligned}$$

Hence, we deduce that

$$\sigma + F(\delta(T_A(U), T_B(V))) \le F(\delta(U,V)).$$

That is, $T_A$ and $T_B$ hold the inequality (4). Therefore, by Lemma 2.1, there exists a unique $U^* = (x^*, y^*) \in X \times X$ such that $T_A(U^*) = T_B(U^*) = U^*$. This means that

$$A(x^*, y^*) = B(x^*, y^*) = x^*,$$

and

$$A(y^*, x^*) = B(y^*, x^*) = y^*.$$

This finishes the proof. □

## 3. An Application

Consider the following coupled systems of functional equations

$$\begin{aligned} \alpha_1(x) &= \sup_{y \in D} \{g(x,y) + U(x, y, \alpha_1(\tau(x,y)), \beta_1(\tau(x,y)))\} \\ \beta_1(x) &= \sup_{y \in D} \{g(x,y) + U(x, y, \beta_1(\tau(x,y)), \alpha_1(\tau(x,y)))\} \end{aligned} \tag{13}$$

and

$$\begin{aligned} \alpha_2(x) &= \sup_{y \in D} \{g(x,y) + V(x, y, \alpha_2(\tau(x,y)), \beta_2(\tau(x,y)))\} \\ \beta_2(x) &= \sup_{y \in D} \{g(x,y) + V(x, y, \beta_2(\tau(x,y)), \alpha_2(\tau(x,y)))\} \end{aligned} \tag{14}$$

appear in the study of dynamic programming (see [3,8,11,16]), where $x \in S$ and $S$ is a state space, $D$ is a decision space, $\tau : S \times D \to S$, $g : S \times D \to \mathbb{R}$ and $U, V : S \times D \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Let $B(S)$ denote the set of all bounded real-valued functions on the nonempty set $S$ and, for any $h \in B(S)$, define

$$\|h\| = \sup_{x \in S} |h(x)|.$$

It is well known that $B(S)$ endowed with the sup metric

$$d(h,k)=\sup_{x\in S}|hx-kx|,$$

for all $h,k\in B(S)$, is a complete metric space.

In this section, we discuss the existence of a unique common solution to the systems of functional equations (13) and (14) that belongs to $B(S)\times B(S)$ by using the obtained results in the previous section.

**Theorem 3.1.** *Assume that the following conditions are satisfied:*

(*i*) $g:S\times D\to\mathbb{R}$ *and* $U,V:S\times D\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *are bounded functions;*

(*ii*) *there exists* $\sigma>0$ *such that for arbitrary points* $x\in S$, $y\in D$ *and* $h,k,h_1,k_1\in\mathbb{R}$,

$$|U(x,y,h,k)-V(x,y,h_1,k_1)|\le e^{-\sigma}\max\{|h-h_1|,|k-k_1|\}.$$

*Then the equations (13) and (14) have a unique bounded common solution in* $B(S)\times B(S)$.

*Proof.* Firstly, we consider the operators $P$ and $Q$ defined on $B(S)\times B(S)$ as

$$\begin{aligned}(P(u,v))(x)&=\sup_{y\in D}\{g(x,y)+U(x,y,u(\tau(x,y)),v(\tau(x,y)))\},\\(Q(u,v))(x)&=\sup_{y\in D}\{g(x,y)+V(x,y,u(\tau(x,y)),v(\tau(x,y)))\},\end{aligned}\tag{15}$$

for all $(u,v)\in B(S)\times B(S)$ and $x\in S$. Since functions $g$, $U$ and $V$ are bounded, then $P$ and $Q$ are well-defined.

Now we will show that $P$ and $Q$ satisfy the condition (12) in Theorem 2.1 with the sup metric $d$. Let $(u_1,v_1),(u_2,v_2)\in B(S)\times B(S)$. Then, by (ii), we get

$$\begin{aligned}&d(P(u_1,v_1),Q(u_2,v_2))\\&\quad=\sup_{x\in S}|P(u_1,v_1)(x)-Q(u_2,v_2)(x)|\\&\quad=\sup_{x\in S}\Big|\sup_{y\in D}\{g(x,y)+U(x,y,u_1(\tau(x,y)),v_1(\tau(x,y)))\}\\&\qquad\qquad-\sup_{y\in D}\{g(x,y)+V(x,y,u_2(\tau(x,y)),v_2(\tau(x,y)))\}\Big|\\&\quad\le\sup_{x\in S}\Big\{\sup_{y\in D}|U(x,y,u_1(\tau(x,y)),v_1(\tau(x,y)))\\&\qquad\qquad-V(x,y,u_2(\tau(x,y)),v_2(\tau(x,y)))|\Big\}\\&\quad\le\sup_{x\in S}\Big\{\sup_{y\in D}\Big(e^{-\sigma}\max\{|u_1(\tau(x,y))-u_2(\tau(x,y))|,\\&\qquad\qquad|v_1(\tau(x,y))-v_2(\tau(x,y))|\}\Big)\Big\}\\&\quad\le\sup_{x\in S}\{e^{-\sigma}\max\{\|u_1-u_2\|,\|v_1-v_2\|\}\\&\quad\le e^{-\sigma}\max\{d(u_1,u_2),d(v_1,v_2)\}.\end{aligned}\tag{16}$$

It yields that

$$d(P(u_1, v_1), Q(u_2, v_2)) \leq e^{-\sigma} \max\{d(u_1, u_2), d(v_1, v_2)\}. \tag{17}$$

By passing to logarithms and after routine calculations, we deduce that

$$\sigma + \ln(d(P(u_1, v_1), Q(u_2, v_2))) \leq \ln(\max\{d(u_1, u_2), d(v_1, v_2)\}), \tag{18}$$

for each $(u_1, v_1), (u_2, v_2) \in B(S) \times B(S)$. By setting $F \in \mathcal{F}$ by $F(t) = \ln t$ for all $t > 0$ and using (18), we infer

$$\sigma + F(d(P(u_1, v_1), Q(u_2, v_2))) \leq F(\max\{d(u_1, u_2), d(v_1, v_2)\}),$$

for all $(u_1, v_1), (u_2, v_2) \in B(S) \times B(S)$. This means that the condition (12) of Theorem 2.1 holds and consequently, $P$ and $Q$ have a unique coupled common fixed point. That is, the equations (13) and (14) have a unique bounded common solution in $B(S) \times B(S)$. □

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