A NEW FACTOR THEOREM ON ABSOLUTE MATRIX SUMMABILITY METHODS

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Abstract. The aim of this paper is to obtain a new theorem dealing with absolute matrix summability factors.

Keywords: Summability factors, absolute matrix summability, infinite series, Hölder inequality, Minkowski inequality

AMS Subject Classification: 26D15, 40D15, 40F05, 40G99

1. Introduction

Let $A = (a_{nv})$ be a normal matrix and $(s_n)$ be the sequence of the $n$th partial sums of the series $\sum a_n$, then we define

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v.$$  \hspace{1cm} (1)

Let $(\theta_n)$ be any sequence of positive constants. The series $\sum a_n$ is said to be summable $|A, \theta_n|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} \theta_n^{k-1} |\Delta A_n(s)|^k < \infty.$$  \hspace{1cm} (2)

where

$$\Delta A_n(s) = A_n(s) - A_{n-1}(s).$$  \hspace{1cm} (3)

One can also see [1] for this method. If we take $\theta_n = n$, then the $|A, \theta_n|_k$ summability reduces to $|A|_k$ summability (see [3]).

Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \hspace{0.5cm} n, v = 0, 1, \ldots \hspace{1cm} \Delta a_{nv} = a_{nv} - a_{n-1,v} \hspace{0.5cm} a_{-1,0} = 0$$  \hspace{1cm} (4)

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§ Manuscript received: March 02, 2017; accepted: May 05, 2017.

TWMS Journal of Applied and Engineering Mathematics, Vol.9, No.2 © Işık University, Department of Mathematics, 2019; all rights reserved.

351
and
\[
\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{\Delta} a_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, \ldots
\]
(5)

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have
\[
A_n(s) = \sum_{v=0}^{n} a_{nv}s^v = \sum_{v=0}^{n} \bar{a}_{nv}a^v
\]
(6)
and
\[
\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv}a^v.
\]
(7)

We say that $A$ is a normal matrix if $A$ is lower triangular and $a_{nn} \neq 0$ for all $n$.

2. The Known Result

Sulaiman [4] has proved the following theorem for matrix summability methods.

**Theorem 2.1** Let $(\lambda_n), (X_n)$ be two sequences such that $\sum_{n=1}^{\infty} n^{-1}\lambda_n X_n$ is convergent, and the conditions
\[
n\Delta \lambda_n = O(\lambda_n), \quad n \to \infty,
\]
(8)
\[
\sum_{v=1}^{n} \lambda_v = O(n\lambda_n), \quad n \to \infty,
\]
(9)
are satisfied. Let $A$ be a lower triangular with non-negative entries satisfying
\[
\bar{a}_{n0} = 1, \quad n = 0, 1, \ldots
\]
(10)
\[
a_{n-1,v} \geq a_{nv}, \quad \text{for } n \geq v + 1,
\]
(11)
\[
n\lambda_{nn} = O(1), \quad 1 = O(n\lambda_{nn})
\]
(12)
\[
\sum_{v=1}^{n-1} a_{nv}a_{n,v} = O(a_{nn}).
\]
(13)
If $t_k^v = O(1)(C, 1)$, where $t_v = \frac{1}{v+1} \sum_{r=1}^{v} r\lambda_{r\lambda}$, then the series $\sum a_n \lambda_n X_n$ is summable $|A|_k$, $k \geq 1$.

3. The Main Result

The aim of this paper is to generalize Theorem 2.1 for $|A, \theta_n|_k$ summability method in the following form.

**Theorem 3.1** Let $A$ be a positive normal matrix satisfying the conditions (10)-(13) of Theorem 2.1. Let $(\theta_n a_{nn})$ be a non-increasing sequence. If $(\theta_n)$ is any sequence of positive constants such that
\[
\sum_{n=v+1}^{\infty} (\theta_n a_{nn})^{k-1} \hat{a}_{n,v} = O\left\{ (\theta_v a_{vv})^{k-1} \right\},
\]
(14)
\[
\sum_{n=v+1}^{\infty} (\theta_n a_{nn})^{k-1} |\Delta a_{nv}| = O\left\{ (\theta_v a_{vv})^{k-1} a_{vv} \right\},
\]
(15)
and all the conditions of Theorem 2.1 are satisfied, then the series $\sum a_n \lambda_n X_n$ is summable $|A, \theta_n|_k$, $k \geq 1$, where $(\lambda_n)$ and $(X_n)$ are as in Theorem 2.1.
We need the following lemmas for the proof of Theorem 3.1.

**Lemma 3.1** [4] If \( \sum n^{-1} \lambda_n \) is convergent, then \( (\lambda_n) \) is non-negative, non-decreasing, \( \lambda_n \log n = O(1) \), and \( n \Delta \lambda_n = O(1/\log^2 n) \).

**Lemma 3.2** [4] If \( \sum n^{-1} \lambda_n X_n \) is convergent, and the conditions (8) and (9) of Theorem 2.1 are satisfied, then

\[
\begin{align*}
\sum_{n=1}^{\infty} n \lambda_n \Delta X_n &= O(1), \\
\sum_{n=1}^{m} n \lambda_n \Delta^2 X_n &= O(1),
\end{align*}
\]

**Lemma 3.3** [4] Under the conditions (10) and (11) of Theorem 2.1, we have

\[
\begin{align*}
\sum_{v=1}^{n-1} |\hat{\Delta} a_{nv}| &\leq a_{n,n}, \\
\hat{\alpha}_{n,v+1} &\geq 0, \\
\sum_{n=v+1}^{m+1} \hat{\alpha}_{n,v+1} &= O(1).
\end{align*}
\]

**Proof of Theorem 3.1**

Let \((V_n)\) denotes the A-transform of the series \( \sum_{n=1}^{\infty} a_n \lambda_n X_n \). We write \( \varphi_n = \lambda_n X_n \), so we have

\[
\Delta V_n = \sum_{v=1}^{n} \hat{\alpha}_{n,v} a_v \varphi_v = \sum_{v=1}^{n} v^{-1} \hat{\alpha}_{n,v} a_v \varphi_v
\]

Applying Abel’s transformation to this sum, we have that

\[
\begin{align*}
\hat{\Delta} V_n &= \sum_{v=1}^{n-1} \Delta_v (\hat{\alpha}_{n,v} \varphi_v v^{-1}) \sum_{r=1}^{v} r a_r + a_{n,n} \varphi_n v^{-1} \sum_{v=1}^{n} v a_v \\
&= \sum_{v=1}^{n-1} (v+1)t_v (v^{-1} (v+1)^{-1} \hat{\alpha}_{n,v} \varphi_v + (v+1)^{-1} \Delta a_{nv} \varphi_v + (v+1)^{-1} \hat{\alpha}_{n,v+1} \Delta \varphi_v + \frac{n+1}{n} a_{nn} \varphi_n t_n \\
&= \sum_{v=1}^{n-1} v^{-1} t_v \hat{\alpha}_{n,v} \varphi_v + \sum_{v=1}^{n} t_v \hat{\alpha}_{n,v} \varphi_v + \sum_{v=1}^{n-1} t_v \hat{\alpha}_{n,v+1} \Delta \varphi_v + \frac{n+1}{n} a_{nn} \varphi_n t_n \\
&= V_{n,1} + V_{n,2} + V_{n,3} + V_{n,4}.
\end{align*}
\]

To complete the proof of Theorem 3.1, by Minkowski’s inequality, it is sufficient to show that

\[
\begin{align*}
\sum_{n=1}^{\infty} \hat{a}_n^{k-1} |V_{n,r}|^k &< \infty, \quad \text{for} \quad r = 1, 2, 3, 4.
\end{align*}
\]
First, by applying Hölder’s inequality with indices $k$ and $k'$, where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, we have that

\[
\sum_{n=2}^{m+1} \theta_n^{k-1} |V_{n,1}|^k = \sum_{n=2}^{m+1} \theta_n^{k-1} \left| \sum_{v=1}^{n-1} v^{-1} \hat{a}_{n,v} \varphi_v \right|^k \\
\leq \sum_{n=2}^{m+1} \theta_n^{k-1} \sum_{v=1}^{n-1} v^{-k} t_v^{k} a_{nv} \hat{a}_{n,v}^{k-1} \left( \sum_{v=1}^{n-1} a_{nv} \hat{a}_{n,v} \right)^{k-1} \\
= O(1) \left( \theta_n a_{nn} \right)^{k-1} \sum_{v=1}^{n-1} t_v^{k} a_{nv} \hat{a}_{n,v} = O(1) \sum_{v=1}^{m} t_v^{k} a_{nv} \hat{a}_{n,v} = O(1) \sum_{v=1}^{m+1} (\theta_n a_{nn})^{k-1} \hat{a}_{n,v} \\
= O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} a_{nv} \hat{a}_{n,v} = O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} \varphi_v^{k-1} a_{nv} \varphi_v^{v-1},
\]

using $nX_n \Delta \lambda_n = O(\lambda_n X_n) = O(1)$ from Lemma 3.2 and writing $\varphi_n = \lambda_n X_n$ we have that

\[
\sum_{n=2}^{m+1} \theta_n^{k-1} |V_{n,1}|^k = O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} \varphi_v^{k} t_v^{k} \varphi_v^{v-1} = O(1) (\theta_1 a_{11})^{k-1} \sum_{v=1}^{m} \varphi_v^{k} t_v^{k} \varphi_v^{v-1} \\
= O(1) \left( \sum_{v=1}^{m} \left( \sum_{r=1}^{v} t_r^{k} \right) \Delta (v^{-1} \varphi_v) + \left( \sum_{v=1}^{m} t_v^{k} \right) m^{-1} \varphi_m \right) \\
= O(1) \left( \sum_{v=1}^{m-1} v (v^{-2} \varphi_v + (v+1)^{-1} \Delta \varphi_v) + O(1) \varphi_m \right) \\
= O(1) \left( \sum_{v=1}^{m-1} v^{-1} \varphi_v + O(1) \sum_{v=1}^{m-1} \Delta \varphi_v + O(1) \varphi_m \right) \\
= O(1) \left( \lambda_v X_v \right) + O(1) \lambda_m X_m = O(1), \quad \text{as} \quad m \to \infty,
\]

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.3. Now, using Hölder’s inequality, and by the hypotheses of Theorem 3.1 and Lemma 3.3, we have that

\[
\sum_{n=2}^{m+1} \theta_n^{k-1} |V_{n,2}|^k = \sum_{n=2}^{m+1} \theta_n^{k-1} \left| \sum_{v=1}^{n-1} \Delta a_{nv} t_v \varphi_v \right|^k \\
\leq \sum_{n=2}^{m+1} \theta_n^{k-1} \left( \sum_{v=1}^{n-1} t_v^{k} |\Delta a_{nv}| \varphi_v^{k} \left( \sum_{v=1}^{n-1} |\Delta a_{nv}| \right)^{k-1} \right) \\
= O(1) \sum_{n=2}^{m+1} (\theta_n a_{nn})^{k-1} \sum_{v=1}^{n-1} t_v^{k} |\Delta a_{nv}| \varphi_v^{k} \sum_{n=v+1}^{m+1} (\theta_n a_{nn})^{k-1} |\Delta a_{nv}| \\
= O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} a_{nv} \varphi_v^{k} t_v^{k} \varphi_v^{v-1} = \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} v^{-1} t_v^{k} \varphi_v = O(1), \quad \text{as} \quad m \to \infty,
\]
as in the case of $V_{n,1}$. Furthermore, we have that

\[
\sum_{n=2}^{m+1} \theta_n^{k-1} | V_{n,3} |^k = \sum_{n=2}^{m+1} \theta_n^{k-1} \left| \sum_{v=1}^{n-1} \hat{a}_{n,v+1} t_v \Delta \varphi_v \right|^k \\
\leq \sum_{n=2}^{m+1} \theta_n^{k-1} \sum_{v=1}^{n-1} t_v^{k-1} a_{v,v}^{1-k} \hat{a}_{n,v+1} (\Delta \varphi_v)^k \left( \sum_{v=1}^{n-1} a_{v,v} \hat{a}_{n,v+1} \right)^{k-1} \\
= O(1) \sum_{n=2}^{m+1} (\theta_n a_{nn})^{k-1} \sum_{v=1}^{n-1} t_v^{k-1} a_{v,v}^{1-k} \hat{a}_{n,v+1} (\Delta \varphi_v)^k \\
= O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} a_{v,v}^{1-k} t_v^{k-1} (\Delta \varphi_v)^k = O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} a_{v,v}^{1-k} (\Delta \varphi_v)^k = O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} a_{v,v}^{1-k} t_v^{k-1} (\Delta \varphi_v)^k = O(1) \\
= O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} a_{v,v}^{1-k} t_v^{k-1} (\Delta \varphi_v)^k = O(1)
\]

by using $n \Delta (\lambda_n X_n) = O(1)$ from Lemma 3.1 we have that

\[
\sum_{n=2}^{m+1} \theta_n^{k-1} | V_{n,3} |^k = O(1) (\theta_1 a_{11})^{k-1} \sum_{v=1}^{m} t_v^{k-1} (\Delta \varphi_v) = O(1) (\theta_1 a_{11})^{k-1} \sum_{v=1}^{m} t_v^{k-1} (\Delta \varphi_v) = O(1)
\]

as $m \to \infty$, (see [4] for detail).

Finally, as in the case of $V_{n,1}$, we have that

\[
\sum_{n=1}^{m} \theta_n^{k-1} | V_{n,4} |^k = \sum_{n=1}^{m} \theta_n^{k-1} \left| \frac{n+1}{n} \hat{a}_{nn} t_n \varphi_n \right|^k \\
= O(1) \sum_{n=1}^{m} (\theta_n a_{nn})^{k-1} a_{nn} t_n^{k-1} (\Delta \varphi_n)^k = O(1) \sum_{n=1}^{m} (\theta_n a_{nn})^{k-1} a_{nn} t_n^{k-1} (\Delta \varphi_n)^k = O(1)
\]

by the hypotheses of Theorem 3.1 and Lemma 3.3. This completes the proof of Theorem 3.1.

In the special case, if we take $\theta_n = n$ and $A$ as a lower triangular matrix in Theorem 3.1, then we obtain Theorem 2.1.

**Acknowledgment**

The author would like to express her sincerest thanks to the referee for his/her suggestions for improvement of this paper.

**References**


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