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ON SOLUTION OF MODIFIED MATRIX SYLVESTER EQUATION

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Abstract. In the paper, the approach connected with the linear matrix inequalities for construction of solution of the modified Sylvester matrix equations, is used. The essence of the approach consists in replacement the initial equation with complex matrices, by two equations with real matrices. That allows to use for their solution the procedures of linear matrix inequalities. Efficiency of offered algorithm is shown on the examples.

Keywords: linear matrix inequalities (LMI), matrix Sylvester equation, modified Sylvester equation, complex matrices

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1. Introduction

The problem of construction of solutions of various variants of the modified (in one way or another) Sylvester equation continues to attract the attention of researchers (see, for example, [1, 2, 6-9]). Thus, in [1] an iterative algorithm for finding the solution $X$of the modified Sylvester matrix equations is considered:

$$AX + \bar{X}D = F, \tag{1}$$

$$AXB + C\bar{X}D = F, \tag{2}$$

The matrices in (1), (2) have the corresponding dimensions, their elements are the complex numbers, the upper bar denotes the operation of complex conjugation.

Below, for the construction of solutions (1), (2), an approach analogous to [3] with linear matrix inequalities (LMI) [4, 5] will be used. The essence of the approach is in replacing each of the equations (1), (2) with complex matrices by two equations with real matrices and the subsequent use of LMI procedures for their solution. Naturally, if necessary, we can take thus obtained solution of equation (1) or (2) as the initial approximation and use the iterative procedures for its correction [1].

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2. General relations

As noted in [4] (the relations (5)), (6)), the matrix inequality:

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} > 0, \tag{3}$$

where the matrices $Q(x) = Q^T(x)$, $R(x) = R^T(x)$, $S(x)$ linearly depend on x, equivalent to the following matrix inequalities:

$$R(x) > 0, Q(x) - S(x)R^{-1}(x)S^T(x) > 0. \tag{4}$$

Here and below, the superscript "T" means transposition.

Consider the following LMI:

$$\begin{bmatrix} Z & T \\ T^T & I \end{bmatrix} > 0, Z = Z^T, \tag{5}$$

which, according to (3), (4), can be written in the form

$$Z > TT^T.$$

Hereinafter I is the unit matrix of the corresponding size.

Relations (3) - (5) allow us to consider the following standard LMI problem on the eigenvalues of (2.9) (see [4], p.2.2.2), namely, the problem of minimizing the linear function cx (for example, $cx = tr(Z)$, where $tr(Z)$ is the trace of the matrix Z) under the conditions (5).

Relations (5) can be generalized in the form of the following LMI system:

$$\begin{bmatrix} Z_i & T_i \\ T_i^T & I \end{bmatrix} > 0, i = 1, 2, \ldots, kZ_i = Z_i^T,$$

which can be represented in the form similar to (4):

$$Z_i > T\,_i T_i^T,\ i = 1, 2, \ldots, k. \tag{6}$$

With respect to (6), we can also consider the standard LMI problem for eigenvalues, namely, the minimization problem

$$cx = \sum_{i=1}^{k} \alpha_i tr(Z_i) \tag{7}$$

under the conditions of (6). In (7) $\alpha_i$are the weight coefficients. To solve this problem, we can use the procedure mincx.m of the MATLAB package [5].

Another variant of the generalization of system (5) is possible, namely

$$\begin{bmatrix} Z & T_i \\ T_i^T & I \end{bmatrix} > 0, i = 1, 2, \ldots, k, Z = Z^T, Z > T_i T_i^T. \tag{8}$$

With respect to this system, we can also consider the minimization problem

$$cx = tr(Z). \tag{9}$$

It is obvious that for a sufficiently small value of $tr(Z)(Z \cong 0)$ we can assume, according to (6), (8), that $T_i \cong 0$.

3. Algorithm

Thus, we write the expressions for the real and imaginary parts of equations (1), (2). We represent the matrices appearing in (1), (2) as follows

$$A = A_r + iA_i, B = B_r + iB_i, C = C_r + iC_i, D = D_r + iD_i, F = F_r + iF_i, X = X_r + iX_i.$$

In these relations, the subscripts r, i denote the real and imaginary parts of the corresponding matrix. Accordingly, the relation (1) can be rewritten as follows:

$$F_r = \varphi_{1r}, F_i = \varphi_{1i},$$

$$\varphi_{1r} = A_r X_r - A_i X_i + X_r D_r + X_i D_i, \tag{10}$$

$$\varphi_{1\,i} = A_r X_i - A_i X_r + X_r D_i - X_i D_r.$$

Similarly, equation (2) can be represented as follows:

$$F_r = \varphi_{2r}, F_i = \varphi_{2i},$$

$$\varphi_{2r} = A_r X_r B_r - A_i X_r B_i - A_r X_i B_i - A_i X_i B_r + C_r X_r D_r - C_i X_r D_i + C_r X_i D_i + C_i X_i D_i, \tag{11}$$

$$\varphi_{2\,i} = A_r X_i B_r - A_i X_i B_i + A_r X_r B_i + A_i X_r B_r + C_r X_r D_i + C_i X_r D_r - C_r X_i D_r + C_i X_i D_i.$$

Further, to find the solutions of equations (1), (2), we can use the relations (8), (9). So, in the case of equation (1), in the relation (8) we have:

$$T_1 = \varphi_{1\,r} - F_r, T_2 = \varphi_{1\,i} - F_i. \tag{12}$$

Similarly, in the case of equation (2):

$$T_1 = \varphi_{2\,r} - F_r, T_2 = \varphi_{2\,i} - F_i.$$

Further, as noted in point 2, using the procedure mincx.m of the MATLAB package, one can find the solutions of equations (1) or (2). We will illustrate the procedure described above on an example.

4. Example 1.

Consider the example of [1]. The matrices appearing in (1) have the form:

$$A = \begin{bmatrix} 7+3i & 3+7i \\ 7+i & 9+10i \end{bmatrix}, D = \begin{bmatrix} 2+9i & 1+3i \\ 4+4i & 4+4i \end{bmatrix}, F = \begin{bmatrix} 25+46i & -5-7i \\ 3-7i & -29-34i \end{bmatrix}.$$

The exact value of the solution of equation (1) in such initial data is

$$X_* = \begin{bmatrix} 3+2i & -i \\ -1+i & -2+i \end{bmatrix}. \tag{13}$$

Using the relations (10), (11) to calculate the matrices T_1, T_2 and then the procedure mincx.m applied to the relations (8), (9), we find the values of the matrices X_r and X_i. The difference between these matrices and the exact values (determined by (4.1)) is characterized by the following expressions:

$$n_r = \|X_{*r} - X_r\| = 6,8 \cdot 10^{-12}, n_i = \|X_{*\,i} - X_i\| = 1,37 \cdot 10^{-11}.$$

Here $\|\cdot\|$ denotes the spectral norm of the matrix.

Thus, in this example, the algorithm described above made it possible to find a solution with a sufficiently high accuracy.

5. Algorithm for solving of equations similar to (1.1), (1.2)

Note that the algorithm described above, based on LMI procedures, with appropriate changes, allows us to find the solutions of other types of the modified Sylvester equations, different from (1), (2). We illustrate this in the case of the following equations

$$AX + X^H D = F, \tag{14}$$

$$AXB + CX^H D = F. \tag{15}$$

Hereinafter, the superscript H means the operation of transposition and complex conjugation:

$$X^H = (\bar{X})^T.$$

We note that the equation (15) can be considered as the result of the generalization to the case of complex matrices of the equation (1) of [2].

To find the solutions of equations (14), (15) with the help of LMI procedures, we can use (10), (11) with corresponding modification. These relations, in the case of equations (14), (15) will have the form:

$$\begin{gathered} F_r = \varphi_{1r}, F_i = \varphi_{1i}, \\ \varphi_{1r} = A_r X_r - A_i X_i + X_r^T D_r + X_i^T D_i, \\ \varphi_{1\,i} = A_r X_i - A_i X_r + X_r^T D_i - X_i^T D_r. \end{gathered} \tag{16}$$

Similarly, equation (2) can be represented as follows :

$$\begin{gathered} F_r = \varphi_{2r}, F_i = \varphi_{2i}, \\ \varphi_{2r} = A_r X_r B_r - A_i X_r B_i - A_r X_i B_i - A_i X_i B_r + C_r X_r^T D_r - \\ C_i X_r^T D_i + C_r X_i^T D_i + C_i X_i^T D_i. \end{gathered} \tag{17}$$

$$\varphi_{2\,i} = A_r X_i B_r - A_i X_i B_i + A_r X_r B_i + A_i X_r B_r + C_r X_r^T D_i + C_i X_r^T D_r - C_r X_i^T D_r + C_i X_i^T D_i.$$

Taking into account of these transformations (defined by (5.3), (17)), one can find the solutions of equations (14), (15) using the algorithm of §3. Let's illustrate this with an example.

6. Example 2.

The matrices in (14), except the matrix F, coincide with the corresponding matrices in Example 1. In the example under consideration, the matrix F has the form :

$$F = \begin{bmatrix} 29+34i & -1-19i \\ -13+6i & -34-29i \end{bmatrix}.$$

With these initial data, the exact solution of (14) is determined by (13). Using the relations (12), (16) to calculate the matrices T_1, T_2 and then, the procedure mincx.m, we find the values of the matrices $X_r$and X_i. The difference between these matrices and the exact values [10,11] (determined by (4.1)) characterize the following expressions

$$n_r = \|X_{*r} - X_r\| = 1,13\cdot 10^{-15}, n_i = \|X_{*\,i} - X_i\| = 9,8\cdot 10^{-16}.$$

Thus, in this example, also, the algorithm described in §3 made it possible to find the solution of (14) with a sufficiently high accuracy.

7. Conclusions

In the paper, an approach to construct the solution of the modified Sylvester matrix equations [1] similar to [3] was used,connected with LMI [4], The essence of the approach consists in replacing one initial equation with complex matrices by two equations with real matrices, which makes it possible to use the LMI apparatus for their solution.It is shown that the proposed algorithm makes it possible to find the solutions also in the case of other (different from (1.1), (2)) modified Sylvester equations. The efficiency of the proposed algorithm is demonstrated by the examples.

References

[1] Zhang, H., Yin H., (2017), New proof of the gradient-based iterative algorithm for the Sylvester conjugate matrix equation, Computers and Mathematics with Applications, 74, pp. 3260-3270.

[2] Ke, Y., Ma, C., (2017), The alternating direction methods for solving the Sylvester-type matrix equation $AXB + CX^TD = E$, J. of Comput. Math., 35(5), pp. 620-641.

[3] Larin, V.B., (2014), Algorithms for Solving a Unilateral Quadratic Matrix Equation and the Model Updating Problem, Int. Appl. Mech., 50 (3), pp. 321-334.

[4] Boyd, S., Ghaoui, L.E., Feron, E., Balakrishnan, V., (1994), Linear Matrix Inequalities in System and Control Theory, Philadelphia: SIAM.

[5] Gahinet, P., Nemirovski, A., Laub, A.J., Chilali, M., (1995), LMI control toolbox users guide, The MathWorks Inc.

[6] Aliev, F.A., Larin, V.B., (2017), A note about the solution of matrix Sylvestr equation, TWMS Journal of Pure and Applied Mathematics, 8(2), pp. 251-255.

[7] Aliev, F.A., Larin, V.B., (2017), On the construction of general solution of the generalized Sylvester equation, TWMS J. App. Eng. Math., 7, pp. 1-6.

[8] Aliev, F.A., Larin, V.B., Velieva, N.I., Gasimova, K.G., (2017), On periodic solution of generalized Sylvester matrix equation, Appl. Comput. Math., 16(1), pp. 78-84.

[9] Pourgholi, R., Esfahan, A., Houlari, T., et al. (2017), An application of Sink-Galerkin method for solving the Tszou equation, Appl.Comput.Math., 16(3), pp. 240-256.

[10] Aliev F.A., Larin V.B., On the solving of matrix equation of Sylvester type, Computational Methods For Differential Equations, 7(1), (2019), pp. 96-104.

[11] Aliev F.A., Larin V.B., (2009), About use of the Bass relations for solution of matrix equations, Appl. Comput. Math., 8(2), pp. 152-162.

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