TWMS J. App. Eng. Math. V.9, N.3, 2019, pp. 620-625

SOME INEQUALITIES FOR $\mathbb{B}^{-1}$-CONVEX FUNCTIONS VIA FRACTIONAL INTEGRAL OPERATOR

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Abstract. In this paper, $\mathbb{B}^{-1}$-convexity which is an abstract convexity type is studied. In addition, some new Hermite-Hadamard type inequalities for $\mathbb{B}^{-1}$-convex functions involving Riemann-Liouville type integral operators that are more general from classic integral operators are proven.

Keywords: Abstract convexity, $\mathbb{B}^{-1}$-convexity, fractional integral, Hermite-Hadamard inequalities.

AMS Subject Classification: 26B25, 52A40, 39B62

1. Introduction

$\mathbb{B}^{-1}$-convexity is an abstract convexity type ([7]). It is the main topic for a lot of papers ([4, 6, 7, 8, 14]). Also, it has applications in mathematical economy, optimization theory and inequality theory ([9, 20, 18]).

Integral inequalities are very significant subjects in inequality theory and mathematics. One of the most well-known of these is Hermite-Hadamard inequality ([1, 2, 3, 5, 10, 11, 12, 13, 16, 17, 19, 20]). It gives an approximation to integral mean value for a convex function. Hermite-Hadamard inequality for $\mathbb{B}^{-1}$-convex functions was studied in [20]. In this paper, the Hermite-Hadamard inequality for $\mathbb{B}^{-1}$-convex functions involving Riemann-Liouville fractional integral is proven.

In Section 2, some necessary definitions and theorems for fractional integrals and $\mathbb{B}^{-1}$-convexity are given. Additionally, the Hermite-Hadamard inequality for $\mathbb{B}^{-1}$-convex functions is recalled. In the next section, the Hermite-Hadamard inequalities for $\mathbb{B}^{-1}$-convex functions via Riemann-Liouville fractional integral operators are proven. Finaly to show that the last inequalities are more general, the conclusion section is given at the end of the paper.

2. Preliminaries

In this section, some required definition and theorems are given.

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§ Manuscript received: April 24, 2018; accepted: August 22, 2018.
TWMS Journal of Applied and Engineering Mathematics, Vol.9, No.3
The author wishes to thank Aksaray University and TUBITAK (The Scientific and Technological Research Council of Turkey). The author has been supported partially by Unit of SRP(BAP) of Aksaray University with 2017-053 project number.

2.1. **Riemann-Liouville Fractional Integral.** Let us recall the following definitions of fractional integral types. Along the paper, let $f:[a,b]\to\mathbb{R}$ be a given function, where $0\le a<b<+\infty$ and $f\in L_1[a,b]$. Also, $\Gamma(\alpha)$ is the Gamma function.

Definition 2.1. [15] *The left-sided Riemann-Liouville integral $J_{a^+}^{\alpha}f$ and the right-sided Riemann-Liouville integral $J_{b^-}^{\alpha}f$ of order $\alpha>0$ with $a\ge 0$ are defined by*

$$J_{a^+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,dt,\qquad x>a \tag{1}$$

and

$$J_{b^-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\,dt,\qquad x<b \tag{2}$$

respectively.

2.2. $\mathbb{B}^{-1}$**-convexity.** For $r\in\mathbb{Z}^-$, the map $x\to\varphi_r(x)=x^{2r+1}$ is a homeomorphism from $\mathbb{R}_*=\mathbb{R}\setminus\{0\}$ to itself; $\boldsymbol{x}=(x_1,x_2,...,x_n)\to\Phi_r(\boldsymbol{x})=(\varphi_r(x_1),\varphi_r(x_2),...,\varphi_r(x_n))$ is homeomorphism from $\mathbb{R}_*^n$ to itself.

For a finite nonempty set $A=\left\{\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)},...,\boldsymbol{x}^{(m)}\right\}\subset\mathbb{R}_*^n$ the Φ_r-convex hull (shortly r-convex hull) of A, which is denoted by $Co^r(A)$ is given via

$$Co^r(A)=\left\{\Phi_r^{-1}\left(\sum_{i=1}^{m}t_i\Phi_r(\boldsymbol{x}^{(i)})\right):t_i\ge 0,\sum_{i=1}^{m}t_i=1\right\}.$$

It is denoted by $\overset{m}{\underset{i=1}{\wedge}}\boldsymbol{x}^{(i)}$ the greatest lower bound with respect to the coordinate-wise order relation of $\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)},...,\boldsymbol{x}^{(m)}\in\mathbb{R}^n$, that is:

$$\overset{m}{\underset{i=1}{\wedge}}\boldsymbol{x}^{(i)}=\left(\min\left\{x_1^{(1)},x_1^{(2)},...,x_1^{(m)}\right\},...,\min\left\{x_n^{(1)},x_n^{(2)},...,x_n^{(m)}\right\}\right)$$

where, $x_j^{(i)}$ denotes jth coordinate of the point $\boldsymbol{x}^{(i)}$.

Thus, $\mathbb{B}^{-1}$-polytopes can be defined as follows:

Definition 2.2. [7] The Kuratowski-Painleve upper limit of the sequence of sets $\{Co^r(A)\}_{r\in\mathbb{Z}^-}$, denoted by $Co^{-\infty}(A)$ where A is a finite subset of $\mathbb{R}_*^n$, is called $\mathbb{B}^{-1}$-polytope of A.

The definition of $\mathbb{B}^{-1}$-polytope can be expressed in the following form in $\mathbb{R}_{++}^n$.

Theorem 2.1. [7] For all nonempty finite subsets $A=\left\{\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)},...,\boldsymbol{x}^{(m)}\right\}\subset\mathbb{R}_{++}^n$

$$Co^{-\infty}(A)=\lim_{r\to-\infty}Co^r(A)=\left\{\overset{m}{\underset{i=1}{\wedge}}t_i\boldsymbol{x}^{(i)}:t_i\ge 1,\min_{1\le i\le m}t_i=1\right\}.$$

Next, the definition of $\mathbb{B}^{-1}$-convex sets can be given.

Definition 2.3. [7] A subset U of $\mathbb{R}_*^n$ is called a $\mathbb{B}^{-1}$-convex if for all finite subsets $A\subset U$ the $\mathbb{B}^{-1}$-polytope $Co^{-\infty}(A)$ is contained in U.

By Theorem 2.1, the definition above for subsets of $\mathbb{R}_{++}^n$ can be reformulated:

Theorem 2.2. [7] A subset U of $\mathbb{R}_{++}^n$ is $\mathbb{B}^{-1}$-convex if and only if for all $\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)}\in U$ and all $\lambda\in[1,\infty)$ one has $\lambda\boldsymbol{x}^{(1)}\wedge\boldsymbol{x}^{(2)}\in U$.

Remark 2.1. As a result of Theorem 2.2, it can be said that $\mathbb{B}^{-1}$-convex sets in $\mathbb{R}_{++}$ are positive intervals.

Definition 2.4. [14] For $U\subset\mathbb{R}_*^n$, a function $f:U\to\mathbb{R}_*$ is called a $\mathbb{B}^{-1}$-convex function if $epi^*(f)=\{(\boldsymbol{x},\mu)\,|\boldsymbol{x}\in U,\mu\in\mathbb{R}_*,\mu\ge f(\boldsymbol{x})\}$ is a $\mathbb{B}^{-1}$-convex set.

In $\mathbb{R}^n_{++}$ the following fundamental theorem, which provides a sufficient and necessary condition for $\mathbb{B}^{-1}$-convex functions can be given[14].

Theorem 2.3. Let $U \subset \mathbb{R}^n_{++}$ and $f : U \to \mathbb{R}_{++}$. The function f is $\mathbb{B}^{-1}$-convex if and only if the set U is $\mathbb{B}^{-1}$-convex and one has the inequality

$$f(\lambda \boldsymbol{x} \wedge \boldsymbol{y}) \leq \lambda f(\boldsymbol{x}) \wedge f(\boldsymbol{y}) \tag{3}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in U$ and all $\lambda \in [1, +\infty)$.

2.3. **Hermite-Hadamard Inequality for $\mathbb{B}^{-1}$-convex Functions.** The following theorem, which gives the Hermite-Hadamard inequality involving classic integral for $\mathbb{B}^{-1}$-convex functions has been proven in [20].

Theorem 2.4. Suppose $f : [a, b] \subset \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ is a $\mathbb{B}^{-1}$-convex function. Then the following inequality holds

$$\frac{1}{b-a}\int_a^b f(t)\,dt \leq \begin{cases} \frac{f(a)(a+b)}{2a}, & \frac{b}{a} \leq \frac{f(b)}{f(a)} \\ \frac{2bf(a)f(b)-a\left[(f(a))^2+(f(b))^2\right]}{2(b-a)f(a)}, & 1 \leq \frac{f(b)}{f(a)} < \frac{b}{a} \end{cases}. \tag{4}$$

3. Hermite-Hadamard Type Inequalities Involving Riemann - Liouville Fractional Integral

Let us prove the Riemann-Liouville fractional Hermite-Hadamard inequalities for $\mathbb{B}^{-1}$-convex functions which were given in the following theorems for left-sided integral and right-sided integral, respectively.

Theorem 3.1. *Let $f : [a, b] \subset \mathbb{R}_{++} \to \mathbb{R}_{++}$ and $f \in L_1[a, b]$. If f is a $\mathbb{B}^{-1}$-convex function on $[a, b]$, then the following inequality holds:*

$$J^{\alpha}_{a^+} f(b) \leq \begin{cases} \frac{f(a)(b-a)^{\alpha}(\alpha a+b)}{a\Gamma(\alpha+2)}, & \frac{b}{a} \leq \frac{f(b)}{f(a)} \\ \frac{(f(a))^{\alpha+1}(b-a)^{\alpha}(\alpha a+b)-(bf(a)-af(b))^{\alpha+1}}{a(f(a))^{\alpha}\Gamma(\alpha+2)}, & 1 \leq \frac{f(b)}{f(a)} < \frac{b}{a} \end{cases} \tag{5}$$

with $\alpha > 0$.

Proof. Since f is a $\mathbb{B}^{-1}$-convex function, it holds the inequality (3). For desired inequality, both sides of inequality (3) can be multiplied by $\frac{(\min\{\lambda a,b\})'}{[b-\lambda a]^{1-\alpha}}$ and then integrate with respect to λ over $[1, +\infty)$. For the left side of the inequality (3), it is obtained that

$$\begin{aligned} &\int_1^{+\infty} \frac{(\min\{\lambda a, b\})'}{[b-\lambda a]^{1-\alpha}} f(\min\{\lambda a, b\})\,d\lambda \\ &= \int_1^{\frac{b}{a}} \frac{(\min\{\lambda a, b\})'}{[b-\lambda a]^{1-\alpha}} f(\min\{\lambda a, b\})\,d\lambda + \int_{\frac{b}{a}}^{+\infty} \frac{(\min\{\lambda a, b\})'}{[b-\lambda a]^{1-\alpha}} f(\min\{\lambda a, b\})\,d\lambda \\ &= \int_1^{\frac{b}{a}} \frac{a}{[b-\lambda a]^{1-\alpha}} f(\lambda a)\,d\lambda + \int_{\frac{b}{a}}^{+\infty} 0 f(b)\,d\lambda \\ &= \int_a^b [b-t]^{\alpha-1} f(t)\,dt = \Gamma(\alpha) J^{\alpha}_{a^+} f(b) \ . \end{aligned}$$

For right sided of inequality (3), two cases of $\frac{b}{a} \leq \frac{f(b)}{f(a)}$ and $1 \leq \frac{f(b)}{f(a)} < \frac{b}{a}$ have to be examined. For the case of $\frac{b}{a} \leq \frac{f(b)}{f(a)}$,

$$\begin{aligned}\int_1^{+\infty} & \frac{(\min\{\lambda a,b\})'}{[b-\lambda a]^{1-\alpha}}\min\{\lambda f(a), f(b)\}\,d\lambda \\ &= \int_1^{\frac{b}{a}} \frac{a}{[b-\lambda a]^{1-\alpha}}\lambda f(a)\,d\lambda \\ &= \frac{f(a)}{a}\int_1^{\frac{b}{a}} \lambda a\,[b-\lambda a]^{\alpha-1}\,ad\lambda = \frac{f(a)(b-a)^{\alpha}(\alpha a+b)}{a\alpha(\alpha+1)}\ .\end{aligned}$$

Hence, the inequality is

$$J_{a^+}^{\alpha} f(b) \le \frac{f(a)(b-a)^{\alpha}(\alpha a+b)}{a\Gamma(\alpha+2)}\ . \tag{6}$$

For the case of $1 \le \frac{f(b)}{f(a)} < \frac{b}{a}$, the following equality is obtained:

$$\begin{aligned}\int_1^{+\infty} & \frac{(\min\{\lambda a,b\})'}{[b-\lambda a]^{1-\alpha}}\min\{\lambda f(a), f(b)\}\,d\lambda \\ &= \int_1^{\frac{b}{a}} \frac{(\min\{\lambda a,b\})'}{[b-\lambda a]^{1-\alpha}}\min\{\lambda f(a), f(b)\}\,d\lambda + \int_{\frac{b}{a}}^{+\infty} \frac{(\min\{\lambda a,b\})'}{[b-\lambda a]^{1-\alpha}}\min\{\lambda f(a), f(b)\}\,d\lambda \\ &= \int_1^{\frac{f(b)}{f(a)}} \frac{a}{[b-\lambda a]^{1-\alpha}}\lambda f(a)\,d\lambda + \int_{\frac{f(b)}{f(a)}}^{\frac{b}{a}} \frac{a}{[b-\lambda a]^{1-\alpha}} f(b)\,d\lambda \\ &= \frac{(f(a))^{\alpha+1}(b-a)^{\alpha}(\alpha a+b) - (bf(a)-af(b))^{\alpha+1}}{a(f(a))^{\alpha}\alpha(\alpha+1)}\ .\end{aligned}$$

Thus, the inequality below is deduced.

$$J_{a^+}^{\alpha} f(b) \le \frac{(f(a))^{\alpha+1}(b-a)^{\alpha}(\alpha a+b) - (bf(a)-af(b))^{\alpha+1}}{a(f(a))^{\alpha}\Gamma(\alpha+2)}\ . \tag{7}$$

From (6) and (7), the desired inequality is obtained. □

Theorem 3.2. *Let $f:[a,b]\subset\mathbb{R}_{++}\to\mathbb{R}_{++}$ and $f\in L_1[a,b]$. If f is a $\mathbb{B}^{-1}$-convex function on $[a,b]$, then the following inequality holds:*

$$J_{b^-}^{\alpha} f(a) \le \begin{cases} \frac{f(a)(b-a)^{\alpha}(\alpha b+a)}{a\Gamma(\alpha+2)}, & \frac{b}{a}\le\frac{f(b)}{f(a)} \\ \frac{a(\alpha+1)(f(a))^{\alpha}f(b)(b-a)^{\alpha}-(af(b)-af(a))^{\alpha+1}}{a(f(a))^{\alpha}\Gamma(\alpha+2)}, & 1\le\frac{f(b)}{f(a)}<\frac{b}{a}\end{cases} \tag{8}$$

with $\alpha>0$.

Proof. Let f be a $\mathbb{B}^{-1}$-convex function. Thus, it holds the inequality (3). For inequality (8),

both sides of the inequality (3) will be multiplied by $\frac{(\min\{\lambda a,b\})'}{[\min\{\lambda a,b\}-a]^{1-\alpha}}$ and then it will be integrated with respect to λ over $[1,+\infty)$. Therefore, the followings for the left side of the inequality that is valid for $\mathbb{B}^{-1}$-convex functions is obtained.

$$\begin{aligned}\int_1^{+\infty} & \frac{(\min\{\lambda a,b\})'}{[\min\{\lambda a,b\}-a]^{1-\alpha}} f(\min\{\lambda a,b\})\,d\lambda \\ &= \int_1^{\frac{b}{a}} \frac{a}{[\lambda a-a]^{1-\alpha}} f(\lambda a)\,d\lambda \\ &= \int_a^b [t-a]^{\alpha-1} f(t)\,dt = \Gamma(\alpha)\, J_{b-}^{\alpha} f(a)\ .\end{aligned}$$

Also, for the right side of the inequality, two cases of $\frac{b}{a} \le \frac{f(b)}{f(a)}$ and $1 \le \frac{f(b)}{f(a)} < \frac{b}{a}$ have to be examined. For the first case, it is obtained that

$$\begin{aligned}\int_1^{+\infty} & \frac{(\min\{\lambda a,b\})'}{[\min\{\lambda a,b\}-a]^{1-\alpha}} \min\{\lambda f(a), f(b)\}\,d\lambda \\ &= \int_1^{\frac{b}{a}} \frac{a}{[\lambda a-a]^{1-\alpha}} \lambda f(a)\,d\lambda \\ &= \frac{f(a)(b-a)^{\alpha}(\alpha b+a)}{a\alpha(\alpha+1)}\ .\end{aligned}$$

For the second case, the following equality is obtained:

$$\begin{aligned}\int_1^{+\infty} & \frac{(\min\{\lambda a,b\})'}{[\min\{\lambda a,b\}-a]^{1-\alpha}} \min\{\lambda f(a), f(b)\}\,d\lambda \\ &= \int_1^{\frac{f(b)}{f(a)}} \frac{(\min\{\lambda a,b\})'}{[\min\{\lambda a,b\}-a]^{1-\alpha}} \min\{\lambda f(a), f(b)\}\,d\lambda + \\ &+ \int_{\frac{f(b)}{f(a)}}^{\frac{b}{a}} \frac{(\min\{\lambda a,b\})'}{[\min\{\lambda a,b\}-a]^{1-\alpha}} \min\{\lambda f(a), f(b)\}\,d\lambda \\ &= \int_1^{\frac{f(b)}{f(a)}} \frac{a}{[\lambda a-a]^{1-\alpha}} \lambda f(a)\,d\lambda + \int_{\frac{f(b)}{f(a)}}^{\frac{b}{a}} \frac{a}{[\lambda a-a]^{1-\alpha}} f(b)\,d\lambda \\ &= \frac{a(\alpha+1)(f(a))^{\alpha} f(b)(b-a)^{\alpha} - (af(b)-af(a))^{\alpha+1}}{a(f(a))^{\alpha}\alpha(\alpha+1)}\ .\end{aligned}$$

Hence, the inequality (8) can be obtained. □

Corollary 3.1. *Hermite-Hadamard inequality via Riemann-Liouville fractional integral operator for $\mathbb{B}^{-1}$-convex function is generalized form of the Hermite-Hadamard inequality.*

Indeed, the fractional integral (5) and (8) reduces to the Hermite-Hadamard inequality (4) for $\alpha = 1$.

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